LEONHARDI EULERI OPERA OMNIA
SUB AUSPICIIS SOCIETATIS SCIENTIARUM NATURALIUM HELVETICAE
EDENDA CURAVERUNT
F. RUDIO · A. KRAZER · A. SPEISER · L. G. DU PASQUIER

SERIES I · OPERA MATHEMATICA · VOLUMEN VIII

LEONHARDI EULERI

INTRODUCTIO IN ANALYSIN INFINITORUM

TOMUS PRIMUS

EDIDERUNT
ADOLF KRAZER ET FERDINAND RUDIO

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXXII

L. Euler

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ADIECTA EST EULERI EFFIGIES
AD IMAGINEM AB E. HANDMANN PICTAM EXPRESSA

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VORWORT DER HERAUSGEBER

Viel später, als ursprünglich erwartet war, erscheint nun auch die *Introductio*, wenigstens in ihrem ersten Teile, in der Reihe der *Opera omnia* EULERS. Die Zeitverhältnisse haben die Verspätung verschuldet; sie haben namentlich veranlaßt, daß an Stelle des im Einteilungsplane[1]) genannten Bearbeiters die Unterzeichneten die Herausgabe besorgt haben. Nachdem nun die *Institutiones calculi differentialis* (herausgegeben von G. KOWALEWSKI) und die drei Bände der *Institutiones calculi integralis* (herausgegeben von F. ENGEL und L. SCHLESINGER) schon in den Jahren 1913—1914 in unserer Eulerausgabe in neuer Bearbeitung erschienen sind, ist mit dem jetzt vorliegenden ersten Teile der *Introductio* (der zweite Teil ist der analytischen Geometrie gewidmet und kommt daher hier nicht in Betracht) das herrliche, groß angelegte Werk der Analysis abgeschlossen, das allein ausgereicht hätte, EULERS Namen unsterblich zu machen.[2])

Die *Introductio* erschien 1748[3]) zu Lausanne bei M. M. BOUSQUET. Bei diesem waren 1742 die *Opera omnia* von JOHANN BERNOULLI in vier Bänden herausgekommen. EULER schreibt darüber am 21. Mai 1743 an CHR. GOLDBACH[4]): „Der Verleger, M. BOUSQUET, hat dieselben selbst hierhergebracht und dem Könige ein magnifiq eingebundenes Exemplar praesentirt. Ich habe auch eins von dem Hn. BERNOULLI zum Praesent erhalten." Bei diesem seinem Besuche in Berlin wurde BOUSQUET durch Vermittelung von DANIEL BERNOULLI auch mit EULER bekannt.[5]) In jenem Briefe an GOLDBACH heißt es weiter: „M. BOUSQUET hat

1) *Einteilung der sämtlichen Werke LEONHARD EULERS,* Jahresbericht der Deutschen Mathematiker-Vereinigung **19**, 1910, Zweite Abt., p. 104 und 129.

2) Siehe auch das von F. ENGEL und L. SCHLESINGER verfaßte Vorwort zur neuen Ausgabe des ersten Bandes der *Institutiones calculi integralis.*

3) Ihr folgten 1755 die *Institutiones calculi differentialis* und 1768, 1769, 1770 die drei Bände der *Institutiones calculi integralis.* Beide Werke wurden auf Kosten der Petersburger Akademie gedruckt (die *Inst. calc. diff.* in Berlin).

4) Siehe *Correspondance math. et phys. publiée par P. H. FUSS,* St.-Pétersbourg 1843, t. I, p. 227; *LEONHARDI EULERI Opera omnia,* series III.

5) Siehe den Brief von DANIEL BERNOULLI an EULER vom 23. April 1743 in der soeben zitierten *Correspondance* von FUSS, t. II, p. 522; *LEONHARDI EULERI Opera omnia,* series III.

einen Contract mit mir geschlossen, kraft welches er alle meine Schriften, ausgenommen diejenigen, welche ich nach St. Petersburg zu schicken schuldig bin, drucken wird, und wird den Anfang mit dem tractatu de Isoperimetricis[1]) machen. Er hätte gern mit der Scientia navali angefangen; ich muß aber erst vernehmen, ob die Akademie noch gesinnt seyn wird, dasselbe zu drucken".[2])

EULER hatte die *Introductio* schon einige Jahre vor 1748 vollendet. Schon am 4. Juli 1744 schrieb er an GOLDBACH[3]): „Ob mein Tractat de problemate isoperimetrico in Lausanne schon völlig gedruckt ist, habe ich noch keine Nachricht erhalten. Ich habe inzwischen ein neues Werk dahin geschickt unter dem Titul *Introductio ad analysin infinitorum,* worin ich sowohl den partem sublimiorem der Algeber als der Geometrie abgehandelt und eine große Menge schwerer problematum ohne den calculum infinitesimalem resolvirt, wovon fast nichts anderswo anzutreffen. Nachdem ich mir einen Plan von einem vollständigen Tractat über die analysis infinitorum formirt hatte, so habe ich bemerkt, daß sehr viele Sachen, welche dazu eigentlich nicht gehören, und nirgend abgehandelt gefunden werden, vorhergehen müßten, und aus denselben ist dieses Werk als prodromus ad analysin infinitorum entstanden."

Auf den Inhalt der *Introductio* an dieser Stelle näher einzutreten, können wir füglich unterlassen. Eine Übersicht[4]) hat EULER selber in seiner *Praefatio* gegeben. Wir verweisen auf diese und das darauf folgende Inhaltsverzeichnis, vor allem aber auf das Werk selbst, das auch heute noch verdient, nicht nur gelesen, sondern mit Andacht studiert zu werden. Kein Mathematiker wird es ohne reichen Gewinn aus der Hand legen.

Die *Introductio* fordert um so mehr unsere Bewunderung heraus, als ihr Verfasser so gut wie gar keine Vorläufer gehabt hat, an die er sich hätte anlehnen können. Man

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1) Gemeint ist die *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti,* welches Werk in der Tat 1744 bei BOUSQUET erschien.

2) Nach einem noch nicht veröffentlichten Briefe EULERS an G. CRAMER in Genf vom 2. November 1751 wurde der allgemeine Verlagsvertrag BOUSQUET-EULER 1751 von EULER gekündigt, nachdem die *Scientia navalis* schon 1749 in Petersburg erschienen war.

3) Siehe FUSS, *Correspondance*, t. I, p. 278; siehe ferner EULERS Brief an D'ALEMBERT vom 28. September 1748, Bullett. di bibliogr. e di storia d. sc. matem. e fis. **19**, 1886, p. 145, sowie EULERS Brief an GOLDBACH vom 6. August 1748, FUSS, *Correspondance*, t. I, p. 471, und den Brief von D. BERNOULLI an EULER vom Anfang des Jahres 1745, FUSS, *Correspondance*, t. II, p. 568; *LEONHARDI EULERI Opera omnia*, series III.

4) Die Besprechungen, die der *Introductio* in den Nova acta eruditorum 1751, p. 222, und in der Nouvelle bibliothèque Germanique 7:1, 1750, p. 17, zuteil geworden sind, enthalten sachlich nichts Bemerkenswertes.

könnte höchstens an die bekannten fünf unter JAKOB BERNOULLIS Leitung in den Jahren 1689—1704 entstandenen Abhandlungen denken, die unter dem gemeinsamen Titel *Positiones arithmeticae de seriebus infinitis earumque summa finita* erschienen und später in BERNOULLIS *Opera* aufgenommen worden sind.[1]) Eine genauere Vergleichung zeigt jedoch, daß diese Abhandlungen mit ihrem verhältnismäßig eng umschriebenen Inhalt doch zu wenig Berührungspunkte mit der so viele grundlegende Untersuchungen der verschiedensten Art umspannenden *Introductio* besitzen, als daß sie ernstlich in Betracht gezogen werden dürften.[2]) EULERS Werk ist im besten Sinne des Wortes originell und unvergleichbar. Wenn er in jenem Briefe an GOLDBACH sagt, er habe darin „eine große Menge schwerer problematum ohne den calculum infinitesimalem resolvirt, wovon fast nichts anderswo anzutreffen", so hat er nicht zuviel behauptet. Wohl aber darf hinzugefügt werden, daß die *Introductio* den Beginn einer neuen Zeit bezeichnet und daß dieses Werk nicht nur durch seinen Inhalt, sondern auch durch seine Sprache maßgebend geworden ist für die ganze Entwickelung der mathematischen Wissenschaft.

BOUSQUET hat, freilich mit EULERS Erlaubnis, die *Introductio* dem Pariser Akademiker J. J. MAIRAN (1678—1771), dem Nachfolger FONTENELLES (1657—1757) im Sekretariate, gewidmet und die Widmung mit einer schwülstigen[3]) *Epistola dedicatoria* versehen. Er hat sogar das Werk mit MAIRANS Bildnis geschmückt. Wie BOUSQUET zu dieser Huldigung kam, darüber ist nichts bekannt. Offenbar hat es sich lediglich um eine Verlegerreklame gehandelt.[4]) Trotzdem haben wir geglaubt, auf den Bildschmuck, den man nun einmal bei der *Introductio* zu sehen gewohnt ist, nicht verzichten zu sollen; wir haben dafür aber dem Bande ein Bild von EULER vorangestellt. Es ist dies eine von der Firma FROBENIUS in Basel hergestellte Reproduktion des von dem Basler Maler EMANUEL HANDMANN (1718—1781) im Jahre 1753 gemalten Pastellbildes, das sich in der „Öffentlichen Kunstsammlung" zu Basel befindet.[5])

1) JACOBI BERNOULLI *Opera*, Genevae 1744, I, p. 375—402, 517—542; II, p. 745—767, 849—867, 955—975.

2) In noch höherem Maße gilt dies von NEWTONS *Arithmetica universalis* (1707), von der *Arithmetica infinitorum* (1655) von WALLIS, von desselben Verfassers *Treatise of Algebra* (1685) und von andern Werken, an die man, durch Äußerlichkeiten verleitet, etwa noch denken könnte.

3) Auch wenn man den Stil der damaligen Zeit berücksichtigt.

4) In einem noch nicht veröffentlichten Briefe an EULER vom 12. Februar 1748 entschuldigt sich MAIRAN, daß BOUSQUET ihm die *Introductio* gewidmet und er, MAIRAN, die Eitelkeit gehabt habe, die Widmung anzunehmen. Im übrigen dankt er EULER für die erteilte Erlaubnis.

5) Über die bekannteren Bilder von EULER orientiert Eneströms Artikel *Über Bildnisse von LEONHARD EULER*, Biblioth. Mathem. 7_3, 1906—1907, p. 372. Speziell von HANDMANN stammt noch das 1756 gemalte, ebenfalls in Basel befindliche Ölbild EULERS, das in Reproduktionen von CHRISTIAN VON MECHEL und FRIEDRICH WEBER die Bände I_1 (*Algebra*) und II_1 (*Mechanica*) unserer Eulerausgabe ziert.

Bei dem reichen Inhalt der *Introductio* ist es selbstverständlich, daß das Werk mannigfache Berührungspunkte mit anderen Arbeiten EULERS aufweist. Wir haben uns bemüht, in zahlreichen Anmerkungen den wünschenswerten Zusammenhang herzustellen, haben aber auch darüber hinaus, so weit als möglich, die Beziehungen der *Introductio* zu den Arbeiten anderer Mathematiker zu verfolgen und die etwas allzu spärlichen Zitate EULERS zu vervollständigen gesucht. Der *Index nominum* gibt darüber nähere Auskunft. Besondere Sorgfalt verwendeten wir auf die Ausmerzung der zahlreichen Fehler, seien es Rechenfehler, seien es andere, die der *Editio princeps* anhaften. Sämtliche Rechnungen wurden auf das genaueste kontrolliert und wo nötig verbessert. Bei dieser verantwortungsvollen und nicht ganz einfachen Arbeit hatten wir uns wiederholt der Unterstützung der Herren Dr. H. BRANDT, Karlsruhe (jetzt Aachen), und Dr. J. WILDHABER, Zürich, zu erfreuen, denen wir dafür auch an dieser Stelle unseren besten Dank aussprechen.

Zu Danke verpflichtet sind wir auch Herrn G. ENESTRÖM, auf dessen wertvolle Ratschläge in historischen Fragen wir stets rechnen durften, und sodann namentlich unseren beiden Mitredaktoren für ihre Unterstützung bei der Korrektur. Auch der Verlagsfirma B. G. TEUBNER sprechen wir gerne unsere Anerkennung und unsern Dank aus für die Sorgfalt, die sie der Drucklegung hat zuteil werden lassen, und für die Bereitwilligkeit, mit der sie stets auf unsere Wünsche eingegangen ist.

Wehmut beschleicht uns beim Abschlusse unseres Bandes — er ist der erste, an dem PAUL STÄCKEL nicht mehr hat mitarbeiten können.

Karlsruhe und Zürich, März 1922.

ADOLF KRAZER. FERDINAND RUDIO.

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   Im Jahre 1791 ließ MICHELSEN seiner Übersetzung der *Introductio* ein „Drittes Buch“ folgen, dem er auch den Nebentitel *Die Theorie der Gleichungen* beifügte. Der Haupttitel aber ist nur irreleitend, denn das Buch hat gar nichts mit der *Introductio* zu tun, sondern ist eine Sammlung von Übersetzungen algebraischer Arbeiten EULERS (der Abhandlungen 30 und 282 des ENESTRÖMSCHEN Verzeichnisses) und LAGRANGES.
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8. Einleitung in die Analysis des Unendlichen. Von LEONHARD EULER. Erster Teil. Ins Deutsche übertragen von H. MASER. X + (1) + 319 S. Berlin, SPRINGER 1885. 8⁰. — Eine Fortsetzung ist nicht erschienen.

INTRODUCTIO
IN ANALYSIN INFINITORUM

TOMUS PRIMUS

A Lausanne et Geneve, chez MARC-MICHEL BOUSQUET et Comp.e 1748.


ANALYSE
DES
INFINIMENT-
PETITS
CALCUL
DIFEREN-
TIEL
TABLE
DES
SINUS
De la Monce inv.
Soubeyran Sculp.

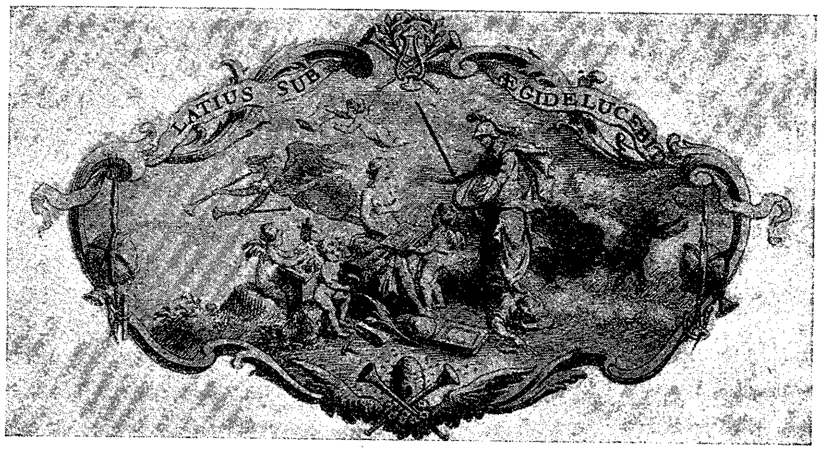
INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio*.

TOMUS PRIMUS.



LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XLVIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

NEC NON

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME

Patronos EULERIANO scripto quaerere necesse neutiquam esse Mathematicarum Disciplinarum cultoribus satis constat. Sciunt utique illi varias earum partes novis eum luminibus sic illustrasse, ut inde merito clarissimi rerum in his abstrusissimarum interpretis locum sit consequutus. Quem quin egregie tueatur, immo tollat se altius quoque opere isthoc, nemo dubitabit, certior hisce factus, indulsisse Te mihi, ut illustrissimo nomini Tuo dicatum publice prodiret. Pertinere autem hoc in me collatum beneficium ad Auctoris decus probe intelligens, Ipse, ut eo uterer, lubens concessit; et cum in rem meam faciat omnimodo, qui neglexissem?

Ab his equidem, quibus Libros inscribunt, sibi nescio quid ideo deberi, plerique tacite constituunt, acceptaque beneficia quodammodo remunerari, ut sese fere nexu liberent omni. Ego vero secus sentio. Mihi certe merum est beneficium patroni, quod scriptoris aut excusoris opera id genus honore condecorari patiantur. Hac mente utique Tibi, *Vir illustrissime*, animi gratissimi summaeque observantiae professionem hisce publicam excipias, rogo.

Paratum promtumque semper iuvandis litterarum studiis qui Te novit, et notus vel hoc nomine es cuicumque in Republica doctorum Europae totius non hospiti, plurimis officiis meae etiam conditionis homines a Te affectos fuisse statuat necesse est. Nempe, tanquam Tibi uni esset iniunctum curare, ut floreant humanum ingenium illustrantes scientiae omnes hominumque in usus adinventae artes, ad singulis inservientium artifices etiam Te demittere dignaris, vel ab illa sublimium rerum perscrutatione Caelive ipsius Tibi tam nota regione, ut, quae hucusque mentes hominum metu complebant Phaenomena minus intellecta, per Te iam grato tantum admirationis sensu contemplentur eorumque causas habeant perspectas.

Hinc ille veluti ex condicto Academiarum Orbis eruditi concursus, ut adlectum Te coetui suo consequerentur, ornamento alias carituro insigni, quo ceteras nollent prae se frui. Hinc inprimis Illustrissimae Parisiensis de Te iudicium, cum ageretur de successore sufficiendo in locum emeriti FONTENELLII, Viri, cuius ex ore calamoque fluere Scientiarum Artiumque omnium exquisitiores divitiae elegantiaeque universae perpetuo visae sunt et videbuntur, dum sani sensus quicquam humano ingenio erit. Tibi, scilicet, Commentariorum Academiae conscribendorum provincia, cui praefectus ille erat, demandabatur continuo; quam ut ornare diutius voluisses, docti omnes optabant: hoc uno minus dolentes Te aliter censuisse, quod aliis Tibi magis placituris profuturisque nihilominus litteris in universum eruditionis ingeniive thesauros impenderes. Quod ut ad ultimas usque metas hominum vitae positas incolumis, florens atque beatus praestes, omni votorum contentione precor. Vale!

Dabam Lausannae die 1. Aprilis
Anni Aerae Dionys. 1748.

PRAEFATIO

Saepenumero animadverti maximam difficultatum partem, quas Matheseos cultores in addiscenda Analysi infinitorum offendere solent, inde oriri, quod Algebra communi vix apprehensa animum ad illam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cuius notio in subsidium vocatur, sibi forment. Quanquam autem Analysis infinitorum non perfectam Algebrae communis omniumque artificiorum adhuc inventorum cognitionem requirit, tamen plurimae extant quaestiones, quarum evolutio discentium animos ad sublimiorem scientiam praeparare valet, quae tamen in communibus Algebrae elementis vel omittuntur vel non satis accurate pertractantur. Hanc ob rem non dubito, quin ea, quae in his libris congessi, hunc defectum abunde supplere queant. Non solum enim operam dedi, ut eas res, quas Analysis infinitorum absolute requirit, uberius atque distinctius exponerem, quam vulgo fieri solet, sed etiam satis multas quaestiones enodavi, quibus lectores sensim et quasi praeter expectationem ideam infiniti sibi familiarem reddent. Plures quoque quaestiones per praecepta communis Algebrae hic resolvi, quae vulgo in Analysi infinitorum tractantur, quo facilius deinceps utriusque methodi summus consensus eluceat.

Divisi hoc opus in duos libros, in quorum priori, quae ad meram Analysin pertinent, sum complexus; in posteriori vero, quae ex Geometria sunt scitu necessaria, explicavi, quoniam Analysis infinitorum ita quoque tradi solet, ut simul eius applicatio ad Geometriam ostendatur. In utroque autem prima elementa praetermisi eaque tantum exponenda duxi, quae alibi vel omnino non vel minus commode tractata vel ex diversis principiis petita reperiuntur.

In primo igitur libro, cum universa Analysis infinitorum circa quantitates variabiles earumque functiones versetur, hoc argumentum de functionibus inprimis fusius exposui atque functionum tam transformationem quam resolutionem et evolutionem per series infinitas demonstravi. Complures enumeravi functionum species, quarum in Analysi sublimiori praecipue ratio est habenda. Primum eas distinxi in algebraicas et transcendentes; quarum illae per operationes in Algebra communi usitatas ex quantitatibus variabilibus formantur, hae vero vel per alias rationes componuntur vel ex iisdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria subdivisio fit in rationales et irrationales; priores docui cum in partes simpliciores tum in factores resolvere, quae operatio in Calculo integrali maximum adiumentum affert; posteriores vero quemadmodum idoneis substitutionibus ad formam rationalem perduci queant, ostendi. Evolutio autem per series infinitas ad utrumque genus aeque pertinet atque etiam ad functiones transcendentes summa cum utilitate applicari solet; at quantopere doctrina de seriebus infinitis Analysin sublimiorem amplificaverit, nemo est, qui ignoret.

Nonnulla igitur adiunxi capita, quibus plurium serierum infinitarum proprietates atque summas sum scrutatus, quarum quaedam ita sunt comparatae, ut sine subsidio Analyseos infinitorum vix investigari posse videantur. Huiusmodi series sunt, quarum summae exprimuntur vel per logarithmos vel arcus circulares; quae quantitates cum sint transcendentes, dum per quadraturam hyperbolae et circuli exhibentur, maximam partem demum in Analysi infinitorum tractari sunt solitae. Postquam autem a potestatibus ad quantitates exponentiales essem progressus, quae nil aliud sunt nisi potestates, quarum exponentes sunt variabiles, ex earum conversione maxime naturalem ac fecundam logarithmorum ideam sum adeptus; unde non solum amplissimus eorum usus sponte est consecutus, sed etiam ex ea cunctas series infinitas, quibus vulgo istae quantitates repraesentari solent, elicere licuit; hincque adeo facillimus se prodidit modus tabulas logarithmorum construendi. Simili modo in contemplatione arcuum circularium sum versatus, quod quantitatum genus, etsi a logarithmis maxime est diversum, tamen tam arcto vinculo est connexum, ut, dum alterum imaginarium fieri videtur, in alterum transeat. Repetitis autem ex Geometria, quae de inventione sinuum et cosinuum arcuum multiplorum ac submultiplorum traduntur, ex sinu vel cosinu cuiusque arcus expressi sinum cosinumque arcus minimi et quasi evanescentis, quo ipso ad series infinitas sum deductus; unde, cum arcus evanescens sinui suo sit aequalis, cosinus vero radio, quemvis arcum cum suo sinu et cosinu ope serierum infinitarum comparavi. Tum vero tam varias expressiones cum finitas tum infinitas pro huius generis quantitatibus obtinui, ut ad earum naturam perspiciendam Calculo infinitesimali prorsus non amplius esset opus. Atque quemadmodum logarithmi peculiarem algorithmum requirunt, cuius in universa Analysi summus extat usus, ita quantitates circulares ad certam quoque algorithmi normam perduxi, ut in calculo aeque commode ac logarithmi et ipsae quantitates algebraicae tractari possent. Quantum autem hinc utilitatis ad resolutionem difficillimarum quaestionum redundet, cum nonnulla capita huius libri luculenter declarant, tum ex Analysi infinitorum plurima specimina proferri possent, nisi iam satis essent cognita et in dies magis multiplicarentur.

Maximum autem haec investigatio attulit adiumentum ad functiones fractas in factores reales resolvendas; quod argumentum, cum in Calculo integrali sit prorsus necessarium, diligentius enucleavi. Series postmodum infinitas, quae ex huiusmodi functionum evolutione nascuntur et quae recurrentium nomine innotuerunt, examini subieci; ubi earum tam summas quam terminos generales aliasque insignes proprietates exhibui, et quoniam ad haec resolutio in factores manuduxit, ita vicissim, quemadmodum producta ex pluribus, immo etiam infinitis factoribus conflata per multiplicationem in series explicentur, perpendi. Quod negotium non solum ad cognitionem innumerabilium serierum viam aperuit, sed quia hoc modo series in producta ex infinitis factoribus constantia resolvere licebat, satis commodas inveni expressiones numericas, quarum ope logarithmi sinuum, cosinuum et tangentium facillime supputari possunt. Praeterea quoque ex eodem fonte solutiones plurium quaestionum, quae circa partitionem numerorum proponi possunt, derivavi, cuiusmodi quaestiones sine hoc subsidio vires Analyseos superare videantur.

Haec tanta materiarum diversitas in plura volumina facile excrescere potuisset, sed omnia, quantum fieri potuit, tam succincte proposui, ut ubique fundamentum clarissime quidem explicaretur, uberior vero amplificatio industriae lectorum relinqueretur, quo habeant, quibus vires suas exerceant finesque Analyseos ulterius promoveant. Neque enim vereor profiteri in hoc libro non solum multa plane nova contineri, sed etiam fontes esse detectos, unde plurima insignia inventa adhuc hauriri queant.

Eodem instituto sum usus in altero[1]) libro, ubi, quae vulgo ad Geometriam sublimiorem referri solent, pertractavi. Antequam autem de sectionibus conicis, quae alias fere solae hunc locum occupant, agerem, theoriam linearum curvarum in genere ita proposui, ut ad scrutationem naturae quarumvis linearum curvarum cum utilitate adhiberi posset. Ad hoc nullum aliud subsidium affero praeter aequationem, qua cuiusque lineae curvae natura exprimitur, ex eaque cum figuram tum primarias proprietates deducere doceo; id quod potissimum in sectionibus conicis praestitisse mihi sum visus, quae antehac vel secundum solam Geometriam vel per Analysin quidem, sed nimis imperfecte ac minus naturaliter tractari sunt solitae. Ex aequatione scilicet generali pro lineis secundi ordinis primum earum proprietates generales explicavi, tum eas in genera seu species subdivisi respiciendo, utrum habeant ramos in infinitum excurrentes, an vero tota curva finito spatio includatur. Priori autem casu in-

1) *Leonhardi Euleri Opera omnia*, series I, vol. 9. A. K.

super dispiciendum erat, quot sint rami in infinitum excurrentes et cuius naturae sint singuli, an habeant lineas rectas asymptotas[1]) an minus. Sicque obtinui tres consuetas sectionum conicarum species, quarum prima est ellipsis, tota in spatio finito contenta, secunda autem hyperbola, quae quatuor habet ramos infinitos ad duas rectas asymptotas convergentes; tertia vero species prodiit parabola duos habens ramos infinitos asymptotis destitutos.

Simili porro ratione lineas tertii ordinis sum persecutus, quas post expositas earum proprietates generales divisi in sedecim genera ad eaque omnes septuaginta duas species NEUTONI[2]) revocavi. Ipsam vero methodum ita clare descripsi, ut pro quovis linearum ordine sequente divisio in genera facillime institui queat; cuius negotii periculum quoque feci in lineis quarti ordinis.

His deinde, quae ad ordines linearum pertinent, expeditis reversus sum ad generales omnium linearum affectiones eruendas. Explicavi itaque methodum definiendi tangentes curvarum, earum normales atque etiam ipsam curvaturam, quae per radium osculi aestimari solet; quae etsi nunc quidem plerumque Calculo differentiali absolvuntur, tamen idem per solam communem Algebram hic praestiti, ut deinceps transitus ab Analysi finitorum ad Analysin infinitorum eo facilior reddatur. Perpendi etiam curvarum puncta flexus contrarii, cuspides, puncta duplicia ac multiplicia; modumque exposui haec omnia ex aequationibus sine ulla difficultate definiendi. Interim tamen non nego has quaestiones multo facilius Calculi differentialis ope enodari posse. Attigi quoque controversiam de cuspide secundi ordinis, ubi ambo arcus in cuspidem coeuntes curvaturam in eandem partem vertunt, eamque ita composuisse mihi videor, ut nullum dubium amplius superesse possit. Denique adiunxi aliquot capita, in quibus lineas curvas, quae datis proprietatibus gaudeant, invenire docui pluraque tandem problemata circa singulares circuli sectiones soluta dedi.

Quae cum sint ea ex Geometria, quae ad Analysin infinitorum addiscendam maximum adminiculum afferre videntur, appendicis loco ex Stereometria theoriam solidorum eorumque superficierum per calculum proposui et, quemadmodum cuiusque superficiei natura per aequationem inter tres variabiles exponi queat, ostendi. Hinc superficiebus instar linearum in ordines digestis secundum dimensionum, quas variabiles in aequatione constituunt, numerum in primo ordine solam superficiem planam contineri ostendi. Superficies vero secundi ordinis ratione habita partium in infinitum expansarum in sex genera divisi similique modo pro ceteris ordinibus divisio institui poterit. Contemplatus sum quoque intersectiones duarum

1) EULERUS plerumque *asymtota* scripsit, sed invenitur etiam hac in ipsa praefatione scribendi modus *asymptota*. A. K.

2) I. NEWTON (1643—1727), *Enumeratio linearum tertii ordinis,* Londini 1706; *I. NEWTONI Opuscula mathematica etc.* Lausannae et Genevae 1744, t. I, p. 245. A. K.

superficierum; quae cum plerumque sint curvae non in eodem plano sitae, quemadmodum aequationibus comprehendi queant, monstravi. Tandem etiam positionem planorum tangentium atque rectarum, quae ad superficies sint normales, determinavi.

De cetero, cum non paucae res hic occurrant ab aliis iam tractatae, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem genere elaborarunt, fecerim. Cum enim mihi propositum esset omnia quam brevissime pertractare, historia cuiusque problematis magnitudinem operis non mediocriter auxisset. Interim tamen pleraeque quaestiones, quae alibi quoque solutae reperiuntur, hic solutiones ex aliis principiis sunt nactae, ita ut non exiguam partem mihi vindicare possem. Spero autem cum ista tum ea potissimum, quae prorsus nova hic proferuntur, plerisque, qui hoc studio delectantur, non ingrata esse futura.

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TOMI PRIMI

INTRODUCTIO
IN
ANALYSIN INFINITORUM.
LIBER PRIMUS,

Continens

Explicationem de Functionibus quantitatum variabilium; earum resolutione in Factores, atque evolutione per Series infinitas: una cum doctrina de Logarithmis, Arcubus circularibus, eorumque Sinubus & Tangentibus; pluribusque aliis rebus, quibus Analysis infinitorum non mediocriter adjuvatur.

LIBER PRIMUS

CAPUT I

DE FUNCTIONIBUS IN GENERE

1. *Quantitas constans est quantitas determinata perpetuo eundem valorem servans.*

Eiusmodi quantitates sunt numeri cuiusvis generis, quippe qui eundem, quem semel obtinuerunt, valorem constanter conservant; atque si huiusmodi quantitates constantes per characteres indicare convenit, adhibentur litterae alphabeti initiales a, b, c etc. In Analysi quidem communi, ubi tantum quantitates determinatae considerantur, hae litterae alphabeti priores quantitates cognitas denotare solent, posteriores vero quantitates incognitas; at in Analysi sublimiori hoc discrimen non tantopere spectatur, cum hic ad illud quantitatum discrimen praecipue respiciatur, quo aliae constantes, aliae vero variabiles statuuntur.

2. *Quantitas variabilis est quantitas indeterminata seu universalis, quae omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cuiusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur ideae specierum et generum, ita quantitas variabilis est genus, sub quo omnes quantitates determinatae continentur. Huiusmodi autem quantitates variabiles per litteras alphabeti postremas z, y, x etc. repraesentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros eius loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati eius loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales et transcendentes. Quin etiam cyphra et numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua praeter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit functio ipsius z. Sic

$$a + 3z, \quad az - 4zz, \quad az + b\sqrt{(aa - zz)}, \quad c^z \quad \text{etc.}$$

sunt functiones ipsius z.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic, etsi haec functio

$$\sqrt{(9 - zz)}$$

numeris realibus loco z substituendis nunquam valorem ternario maiorem recipere potest, tamen ipsi z valores imaginarios tribuendo, ut $5\sqrt{-1}$, nullus assignari poterit valor determinatus, quin ex formula $\sqrt{(9 - zz)}$ elici queat. Occurrunt autem nonnunquam functiones tantum apparentes, quae, utcunque quantitas variabilis varietur, tamen usque eundem valorem retinent, ut

$$z^0, \quad 1^z, \quad \frac{aa-az}{a-z},$$

quae, etsi speciem functionis mentiuntur, tamen revera sunt quantitates constantes.

6. *Praecipuum functionum discrimen in modo compositionis, quo ex quantitate variabili et quantitatibus constantibus formantur, positum est.*

Pendet ergo ab operationibus, quibus quantitates inter se componi et permisceri possunt; quae operationes sunt additio et subtractio, multiplicatio et divisio, evectio ad potestates et radicum extractio, quo etiam resolutio aequationum est referenda. Praeter has operationes, quae algebraicae vocari solent, dantur complures aliae transcendentes, ut exponentiales, logarithmicae atque innumerabiles aliae, quas Calculus integralis suppeditat.

Interim species quaedam functionum notari possunt, ut multipla

$$2z, \quad 3z, \quad \frac{3}{5}z, \quad az \text{ etc.}$$

et potestates ipsius z, ut

$$z^2, \quad z^3, \quad z^{\frac{1}{2}}, \quad z^{-1} \text{ etc.};$$

quae uti ex unica operatione sunt desumptae, ita expressiones, quae ex operationibus quibuscunque nascuntur, functionum nomine insigniuntur.

7. *Functiones dividuntur in algebraicas et transcendentes; illae sunt, quae componuntur per operationes algebraicas solas; hae vero, in quibus operationes transcendentes insunt.*

Sunt ergo multipla ac potestates ipsius z functiones algebraicae atque omnes omnino expressiones, quae per operationes algebraicas ante memoratas formantur, cuiusmodi est

$$\frac{a+bz^n-c\sqrt{(2z-zz)}}{aaz-3bz^3}.$$

Quin etiam functiones algebraicae saepenumero ne quidem explicite exhiberi possunt, cuiusmodi functio ipsius z est Z, si definiatur per huiusmodi aequationem

$$Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1.$$

Quanquam enim haec aequatio resolvi nequit, tamen constat Z aequari expressioni cuipiam ex variabili z et constantibus compositae ac propterea fore Z functionem quandam ipsius z.[1]) Ceterum de functionibus transcendentibus notandum est eas demum fore transcendentes, si operatio transcendens non solum ingrediatur, sed etiam quantitatem variabilem afficiat. Si enim operationes transcendentes tantum ad quantitates constantes pertineant, functio nihilominus algebraica est censenda; uti si c denotet circumferentiam circuli, cuius radius sit $= 1$, erit utique c quantitas transcendens, verumtamen hae expressiones

$$c + z, \quad cz^3, \quad 4z^c \quad \text{etc.}$$

erunt functiones algebraicae ipsius z. Parvi quidem est momenti dubium, quod a quibusdam movetur, utrum eiusmodi expressiones z^c functionibus algebraicis annumerari iure possint necne; quin etiam potestates ipsius z, quarum exponentes sint numeri irrationales, uti $z^{\sqrt{2}}$, nonnulli maluerunt functiones *interscendentes* quam algebraicas appellare.[2])

8. *Functiones algebraicae subdividuntur in rationales et irrationales; illae sunt, si quantitas variabilis in nulla irrationalitate involvitur; hae vero, in quibus signa radicalia quantitatem variabilem afficiunt.*

In functionibus ergo rationalibus aliae operationes praeter additionem, subtractionem, multiplicationem, divisionem et evectionem ad potestates, quarum exponentes sint numeri integri, non insunt; erunt adeo

$$a + z, \quad a - z, \quad az, \quad \frac{aa + zz}{a + z}, \quad az^3 - bz^5 \quad \text{etc.}$$

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1) Hanc opinionem, omnem radicem cuiusque aequationis expressioni cuipiam ex variabili et constantibus compositae aequari debere, EULERUS non semel enuntiavit; confer praeter paragraphos 8 et 17 huius operis Commentationes 30 et 282 (indicis ENESTROEMIANI): *De formis radicum aequationum cuiusque ordinis coniectatio,* Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 216, et *De resolutione aequationum cuiusvis gradus*, Novi comment. acad. sc. Petrop. 9 (1762/3), 1764, p. 70; *LEONHARDI EULERI Opera omnia*, series I, vol. 6. p. 1 et 170. A. K.

2) Vide epistolam a G. LEIBNIZ (1646—1716) ad J. WALLIS (1616—1703) scriptam d. 28. Maii 1697; *LEIBNIZENS Mathematische Schriften,* herausg. von C. I. GERHARDT, 1. Abt., Bd. 4, Halle 1859, p. 28. A. K.

functiones rationales ipsius z. At huiusmodi expressiones

$$\sqrt{z},\quad a+\sqrt{(aa-zz)},\quad \sqrt[3]{(a-2z+zz)},\quad \frac{aa-z\sqrt{(aa+zz)}}{a+z}\quad \text{etc.}$$

erunt functiones irrationales ipsius z.

Hae commode distinguuntur in explicitas et implicitas.

Explicitae sunt, quae per signa radicalia sunt evolutae, cuiusmodi exempla modo sunt data. *Implicitae* vero functiones irrationales sunt, quae ex resolutione aequationum ortum habent. Sic Z erit functio irrationalis implicita ipsius z, si per huiusmodi aequationem

$$Z^7 = azZ^2 - bz^5$$

definiatur, quoniam valorem explicitum pro Z admissis etiam signis radicalibus exhibere non licet, propterea quod Algebra communis nondum ad hunc perfectionis gradum est evecta.[1])

9. *Functiones rationales denuo subdividuntur in integras et fractas.*

In illis neque z usquam habet exponentes negativos neque expressiones continent fractiones, in quarum denominatores quantitas variabilis z ingrediatur; unde intelligitur functiones fractas esse, in quibus denominatores z continentes vel exponentes negativi ipsius z occurrant. Functionum integrarum haec ergo erit formula generalis

$$a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.};$$

nulla enim functio ipsius z integra excogitari potest, quae non in hac expressione contineatur. Functiones autem fractae omnes, quia plures fractiones in unam cogi possunt, continebuntur in hac formula

$$\frac{a+bz+cz^2+dz^3+ez^4+fz^5+\text{etc.}}{\alpha+\beta z+\gamma z^2+\delta z^3+\varepsilon z^4+\zeta z^5+\text{etc.}},$$

ubi notandum est quantitates constantes a, b, c, d etc., α, β, γ, δ etc., sive sint affirmativae sive negativae, sive integrae sive fractae, sive rationales sive irrationales sive etiam transcendentes, naturam functionum non mutare.

1) Vide notam 1 p. 20. A. K.

10. *Deinde potissimum tenenda est functionum divisio in uniformes ac multiformes.*

Functio autem *uniformis* est, quae, si quantitati variabili z valor determinatus quicunque tribuatur, ipsa quoque unicum valorem determinatum obtineat. Functio autem *multiformis* est, quae pro unoquoque valore determinato in locum variabilis z substituto plures valores determinatos exhibet.

Sunt igitur omnes functiones rationales, sive integrae sive fractae, functiones uniformes, quoniam eiusmodi expressiones, quicunque valor quantitati variabili tribuatur, non nisi unicum valorem praebent. Functiones autem irrationales omnes sunt multiformes, propterea quod signa radicalia sunt ambigua et geminum valorem involvunt. Dantur autem quoque inter functiones transcendentes et uniformes et multiformes; quin etiam habentur functiones infinitiformes, cuiusmodi est arcus circuli sinui z respondens; dantur enim arcus circulares innumerabiles, qui omnes eundem habeant sinum.

Denotent autem hae litterae P, Q, R, S, T etc. singulae functiones uniformes ipsius z.

11. *Functio biformis ipsius z est eiusmodi functio, quae pro quovis ipsius z valore determinato geminum valorem praebeat.*

Huiusmodi functiones radices quadratae exhibent, ut $\sqrt{(2z+zz)}$; quicunque enim valor pro z statuatur, expressio $\sqrt{(2z+zz)}$ duplicem habet significatum, vel affirmativum vel negativum. Generatim vero Z erit functio biformis ipsius z, si determinetur per aequationem quadraticam

$$Z^2 - PZ + Q = 0,$$

si quidem P et Q fuerint functiones uniformes ipsius z. Erit namque

$$Z = \frac{1}{2}P \pm \sqrt{\left(\frac{1}{4}P^2 - Q\right)};$$

ex quo patet cuique valori determinato ipsius z duplicem valorem determinatum ipsius Z respondere. Hic autem notandum est vel utrumque valorem functionis Z esse realem vel utrumque imaginarium. Tum vero erit semper, uti constat ex natura aequationum, binorum valorum ipsius Z summa $= P$ ac productum $= Q$.

12. *Functio triformis ipsius z est, quae pro quovis ipsius z valore tres valores determinatos exhibet.*

Huiusmodi functiones ex resolutione aequationum cubicarum originem trahunt. Si enim fuerint P, Q et R functiones uniformes sitque

$$Z^3 - PZ^2 + QZ - R = 0,$$

erit Z functio triformis ipsius z, quia pro quolibet valore determinato ipsius z triplicem valorem obtinet. Tres isti ipsius Z valores unicuique valori ipsius z respondentes vel erunt omnes reales vel unicus erit realis, dum bini reliqui sunt imaginarii. Ceterum constat horum trium valorum summam perpetuo esse $= P$, summam factorum ex binis esse $= Q$ et productum ex omnibus tribus esse $= R$.

13. *Functio quadriformis ipsius z est, quae pro quovis ipsius z valore quatuor valores determinatos exhibet.*

Huiusmodi functiones ex resolutione aequationum biquadraticarum nascuntur. Quodsi enim P, Q, R et S denotent functiones uniformes ipsius z fueritque

$$Z^4 - PZ^3 + QZ^2 - RZ + S = 0,$$

erit Z functio quadriformis ipsius z, eo quod cuique valori ipsius z quadruplex valor ipsius Z respondet. Quatuor horum valorum ergo vel omnes erunt reales vel duo reales duoque imaginarii, vel omnes quatuor erunt imaginarii. Ceterum perpetuo summa horum quatuor valorum ipsius Z est $= P$, summa factorum ex binis $= Q$, summa factorum ex ternis $= R$ ac productum omnium $= S$.

Simili autem modo comparata est ratio functionum quinqueformium et sequentium.

14. *Erit ergo Z functio multiformis ipsius z, quae pro quovis valore ipsius z tot exhibet valores, quot numerus n continet unitates, si Z definiatur per hanc aequationem*

$$Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \text{etc.} = 0.$$

Ubi quidem notandum est n esse oportere numerum integrum atque perpetuo, ut diiudicari possit, quam multiformis sit functio Z ipsius z, aequatio, per quam Z definitur, reduci debet ad rationalitatem; quo facto exponens maximae potestatis ipsius Z indicabit quaesitum valorum numerum cuique ipsius z valori respondentium. Deinde quoque tenendum est litteras P, Q, R, S etc. denotare debere functiones uniformes ipsius z; si enim aliqua earum iam esset functio multiformis, tum functio Z multo plures praebitura esset valores unicuique valori ipsius z respondentes, quam quidem numerus dimensionum ipsius Z indicaret. Semper autem, si qui valores ipsius Z fuerint imaginarii, eorum numerus erit par[1]); unde intelligitur, si fuerit n numerus impar, perpetuo unum ad minimum valorem ipsius Z fore realem, contra autem fieri posse, si numerus n fuerit par, ut nullus prorsus valor ipsius Z sit realis.

15. *Si Z eiusmodi fuerit functio multiformis ipsius z, ut perpetuo nonnisi unicum valorem exhibeat realem, tum Z functionem uniformem ipsius z mentietur ac plerumque loco functionis uniformis usurpari poterit.*

Eiusmodi functiones erunt

$$\sqrt[3]{P},\quad \sqrt[5]{P},\quad \sqrt[7]{P}\quad \text{etc.},$$

quippe quae perpetuo nonnisi unicum valorem realem praebent reliquis omnibus existentibus imaginariis, dummodo P fuerit functio uniformis ipsius z. Hanc ob rem huiusmodi expressio $P^{\frac{m}{n}}$, quoties n fuerit numerus impar, functionibus uniformibus annumerari poterit, sive m fuerit numerus par sive impar. Quodsi autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit valorem realem vel duos; ex quo eiusmodi expressiones $P^{\frac{m}{n}}$ existente n numero pari eodem iure functionibus biformibus accenseri poterunt, siquidem fractio $\frac{m}{n}$ ad minores terminos non fuerit reducibilis.

16. *Si fuerit y functio quaecunque ipsius z, tum vicissim z erit functio ipsius y.*

Cum enim y sit functio ipsius z, sive uniformis sive multiformis, dabitur aequatio, qua y per z et constantes quantitates definitur. Ex eadem vero aequatione vicissim z per y et constantes definiri poterit; unde, quoniam

1) Vide § 30 et imprimis notam 1 ibi adiectam. A. K.

y est quantitas variabilis, z aequabitur expressioni ex y et constantibus compositae eritque adeo functio ipsius y. Hinc quoque patebit, quam multiformis functio futura sit z ipsius y, fierique potest, ut, etiamsi y fuerit functio uniformis ipsius z, tamen z futura sit functio multiformis ipsius y. Sic, si y ex hac aequatione per z definiatur

$$y^3 = ayz - bzz,$$

erit utique y functio triformis ipsius z, contra vero z functio tantum biformis ipsius y.

17. *Si fuerint y et x functiones ipsius z, erit quoque y functio ipsius x et vicissim x functio ipsius y.*

Cum enim sit y functio ipsius z, erit quoque z functio ipsius y similique modo erit etiam z functio ipsius x. Hanc ob rem functio ipsius y aequalis erit functioni ipsius x; ex qua aequatione et y per x et viceversa x per y definiri poterit; quocirca manifestum est esse y functionem ipsius x atque x functionem ipsius y. Saepissime quidem has functiones explicite exhibere non licet ob defectum Algebrae[1]); interim tamen nihilominus, quasi omnes aequationes resolvi possent, haec functionum reciprocatio perspicitur. Ceterum per methodum in Algebra traditam ex datis binis aequationibus, quarum altera continet y et z, altera vero x et z, per eliminationem quantitatis z formabitur una aequatio relationem inter x et y exprimens.[2])

18. *Species denique quaedam functionum peculiares sunt notandae; sic functio par ipsius z est, quae eundem dat valorem, sive pro z ponatur valor determinatus $+k$ sive $-k$.*

Huiusmodi ergo functio par ipsius z erit zz; sive enim ponatur $z = +k$ sive $z = -k$, eundem valorem praebebit expressio zz, nempe

1) Vide notam 1 p. 20. A. K.

2) I. Newton, *Arithmetica universalis*, Cantabrigiae 1707, 3. ed. Lugd. Batav. 1732, p. 57—63. Vide porro L. Euleri Commentationem 310 (indicis Enestroemiani): *Nouvelle méthode d'éliminer les quantités inconnues des équations*, Mém. de l'acad. d. sc. de Berlin [20] (1764), 1766, p. 91; *Leonhardi Euleri Opera omnia*, series I, vol. 6, p. 197. A. K.

$zz = +kk$. Simili modo functiones pares ipsius z erunt hae ipsius z potestates z^4, z^6, z^8 et generatim omnis potestas z^m, si fuerit m numerus par, sive affirmativus sive negativus. Quin etiam, cum $z^{\frac{m}{n}}$ mentiatur functionem ipsius z uniformem, si n sit numerus impar, perspicuum est $z^{\frac{m}{n}}$ fore functionem parem ipsius z, si m fuerit numerus par, n vero numerus impar. Hanc ob rem expressiones ex huiusmodi potestatibus utcunque compositae praebebunt functiones pares ipsius z; sic Z erit functio par ipsius z, si fuerit

$$Z = a + bz^2 + cz^4 + dz^6 + \text{etc.},$$

item si fuerit

$$Z = \frac{a + bz^2 + cz^4 + dz^6 + \text{etc.}}{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \text{etc.}}.$$

Similique modo exponentes fractos ipsius z introducendo erit Z functio par ipsius z, si fuerit

$$Z = a + bz^{\frac{2}{3}} + cz^{\frac{2}{5}} + dz^{\frac{4}{7}} + \text{etc.}$$

vel

$$Z = a + bz^{-\frac{2}{3}} + cz^{-\frac{4}{3}} + dz^{-\frac{2}{5}} + \text{etc.}$$

vel

$$Z = \frac{a + bz^{\frac{2}{7}} + cz^{-\frac{4}{5}} + dz^{\frac{8}{3}} + \text{etc.}}{\alpha + \beta z^{\frac{2}{3}} + \gamma z^{-\frac{2}{5}} + \delta z^{\frac{4}{7}} + \text{etc.}}.$$

Cuiusmodi expressiones, cum omnes sint functiones uniformes ipsius z, appellari poterunt functiones pares uniformes ipsius z.

19. *Functio multiformis par ipsius z est, quae, etiamsi pro quovis valore ipsius z plures exhibeat valores determinatos, tamen eosdem valores praebet, sive ponatur $z = +k$ sive $z = -k$.*

Sit Z eiusmodi functio multiformis par ipsius z; quoniam natura functionis multiformis exprimitur per aequationem inter Z et z, in qua Z tot habeat dimensiones, quot varios valores complectatur, manifestum est Z fore functionem multiformem parem, si in aequatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit

$$Z^2 = az^4Z + bz^2,$$

erit Z functio biformis par ipsius z; sin autem sit

$$Z^3 - az^2Z^2 + bz^4Z - cz^8 = 0,$$

erit Z functio triformis par ipsius z; atque generatim, si P, Q, R, S etc. denotent functiones uniformes pares ipsius z, erit Z functio biformis par ipsius z, si sit

$$Z^2 - PZ + Q = 0.$$

At Z erit functio triformis par ipsius z, si sit

$$Z^3 - PZ^2 + QZ - R = 0,$$

et ita porro.

20. *Functio ergo, sive uniformis sive multiformis, par ipsius z erit eiusmodi expressio ex quantitate variabili z et constantibus conflata, in qua ubique numerus dimensionum ipsius z sit par.*

Huiusmodi ergo functiones praeter uniformes, quarum exempla ante sunt allata, erunt hae expressiones

$$a + \sqrt{(bb - zz)}, \quad azz + \sqrt[3]{(a^6z^4 - bz^2)}, \quad \text{item} \quad az^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})} \quad \text{etc.}$$

Unde patet functiones pares ita definiri posse, ut dicantur esse functiones ipsius zz.

Si enim ponatur $y = zz$ fueritque Z functio quaecunque ipsius y, restituto ubique zz loco y erit Z eiusmodi functio ipsius z, in qua z ubique parem habeat dimensionum numerum. Excipiendi tamen sunt ii casus, quibus in expressione ipsius Z occurrunt $\sqrt{y}$ ac huiusmodi aliae formae, quae facto $y = zz$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{ay}$ functio ipsius y, tamen posito $y = zz$ eadem expressio non erit functio par ipsius z, cum fiat

$$y + \sqrt{ay} = zz + z\sqrt{a}.$$

Exclusis ergo his casibus definitio ultima functionum parium erit bona atque ad eiusmodi functiones formandas idonea.

21. *Functio impar ipsius z est eiusmodi functio, cuius valor, si loco z ponatur $-z$, fit quoque negativus.*

Huiusmodi functiones ergo impares erunt omnes potestates ipsius z, quarum exponentes sunt numeri impares, ut z^1, z^3, z^5, z^7 etc., item z^{-1}, z^{-3}, z^{-5} etc.; tum vero etiam $z^{\frac{m}{n}}$ erit functio impar, si ambo numeri m et n fuerint numeri impares. Generatim vero omnis expressio ex huiusmodi potestatibus composita erit functio impar ipsius z; cuiusmodi sunt

$$az + bz^3, \quad az + az^{-1}, \quad \text{item} \quad z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{5}{3}} \quad \text{etc.}$$

Harum autem functionum natura et inventio ex functionibus paribus facilius perspicietur.

22. *Si functio par ipsius z multiplicetur per z vel per eiusdem functionem imparem quamcunque, productum erit functio impar ipsius z.*

Sit P functio par ipsius z, quae idcirco manet eadem, si loco z ponatur $-z$; quodsi ergo in producto Pz ponatur $-z$ loco z, prodibit $-Pz$, unde Pz erit functio impar ipsius z. Sit iam P functio par ipsius z et Q functio impar ipsius z atque ex definitione patet, si loco z ponatur $-z$, valorem ipsius P manere eundem, at valorem ipsius Q abire in sui negativum $-Q$, quare productum PQ posito $-z$ loco z abibit in $-PQ$, hoc est in sui negativum, eritque ideo PQ functio impar ipsius z. Sic, cum sit $a + \sqrt{(aa + zz)}$ functio par et z^3 functio impar ipsius z, erit productum

$$az^3 + z^3\sqrt{(aa + zz)}$$

functio impar ipsius z similique modo

$$z \times \frac{a + bzz}{\alpha + \beta zz} = \frac{az + bz^3}{\alpha + \beta zz}$$

functio impar ipsius z.

Ex his vero etiam intelligitur, si duarum functionum P et Q, quarum altera P est par, altera Q impar, altera per alteram dividatur, quotum fore functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ functio impar ipsius z.

23. *Si functio impar per functionem imparem vel multiplicetur vel dividatur, quod resultat erit functio par.*

Sint Q et S functiones impares ipsius z, ita ut posito $-z$ loco z Q abeat in $-Q$ et S in $-S$, atque perspicuum est tam productum QS quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur $-z$, ideoque esse utrumque functionem parem ipsius z. Manifestum itaque porro est cuiusque functionis imparis quadratum esse functionem parem, cubum vero functionem imparem, biquadratum iterum functionem parem atque ita porro.

24. *Si fuerit y functio impar ipsius z, erit vicissim z functio impar ipsius y.*

Cum enim sit y functio impar ipsius z, si ponatur $-z$ loco z, abibit y in $-y$. Quodsi ergo z per y definiatur, necesse est, ut posito $-y$ loco y quoque z abeat in $-z$, eritque ideo z functio impar ipsius y. Sic, quia posito

$$y = z^3$$

est y functio impar ipsius z, erit quoque ex aequatione

$$z^3 = y \quad \text{seu} \quad z = y^{\frac{1}{3}}$$

z functio impar ipsius y. Et quia, si fuerit

$$y = az + bz^3,$$

est y functio impar ipsius z, erit vicissim ex aequatione

$$bz^3 + az = y$$

valor ipsius z per y expressus functio impar ipsius y.

25. *Si natura functionis y per eiusmodi aequationem definiatur, in cuius singulis terminis numerus dimensionum, quas y et z occupant coniunctim, sit vel par ubique vel impar, tum erit y functio impar ipsius z.*

Quodsi enim in eiusmodi aequatione ubique loco z scribatur $-z$ simulque $-y$ loco y, omnes aequationis termini vel manebunt iidem vel fient negativi, utroque vero casu aequatio manebit eadem. Unde patet $-y$ eodem modo per $-z$ determinatum iri, quo $+y$ per $+z$ determinatur, et hanc ob rem, si loco z ponatur $-z$, valor ipsius y abibit in $-y$ seu y erit functio impar ipsius z. Sic, si fuerit vel

$$yy = ayz + bzz + c$$

vel

$$y^3 + ayyz = byzz + cy + dz,$$

ex utraque aequatione y erit functio impar ipsius z.

26. *Si Z fuerit functio ipsius z et Y functio ipsius y atque Y eodem modo definiatur per variabilem y et constantes, quo Z definitur per variabilem z et constantes, tum hae functiones Y et Z vocantur functiones similes ipsarum y et z.*

Si scilicet fuerit

$$Z = a + bz + cz^2 \quad \text{et} \quad Y = a + by + cy^2,$$

erunt Z et Y functiones similes ipsarum z et y similique modo in multiformibus, si fuerit

$$Z^3 = azzZ + b \quad \text{et} \quad Y^3 = ayyY + b,$$

erunt Z et Y functiones similes ipsarum z et y. Hinc sequitur, si Y et Z fuerint huiusmodi functiones similes ipsarum y et z, tum, si loco z scribatur y, functionem Z abituram esse in functionem Y. Solet haec similitudo etiam hoc modo verbis exprimi, ut Y talis functio dicatur ipsius y, qualis functio sit Z ipsius z. Hae locutiones perinde occurrent, sive quantitates variabiles z et y a se invicem pendeant sive secus; sic, qualis functio est

$$ay + by^3$$

ipsius y, talis functio erit

$$a(y+n) + b(y+n)^3$$

ipsius $y + n$, existente scilicet $z = y + n$; tum, qualis functio est

$$\frac{a + bz + czz}{\alpha + \beta z + \gamma zz}$$

ipsius z, talis functio erit

$$\frac{azz+bz+c}{\alpha zz+\beta z+\gamma}$$

ipsius $\frac{1}{z}$ posito $y=\frac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis functionum, cuius per universam Analysin sublimiorem uberrimus est usus.

Ceterum haec in genere de natura functionum unius variabilis sufficere possunt, cum plenior expositio in applicatione sequente tradatur.

CAPUT II

DE TRANSFORMATIONE FUNCTIONUM

27. *Functiones in alias formas transmutantur vel loco quantitatis variabilis aliam introducendo vel eandem quantitatem variabilem retinendo.*

Quodsi eadem quantitas variabilis servatur, functio proprie mutari non potest. Sed omnis transformatio consistit in alio modo eandem functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse. Huiusmodi transformationes sunt, si loco huius functionis

$$2 - 3z + zz \quad \text{ponatur} \quad (1 - z)(2 - z),$$

vel

$$(a + z)^3 \quad \text{loco} \quad a^3 + 3aaz + 3azz + z^3,$$

vel

$$\frac{a}{a - z} + \frac{a}{a + z} \quad \text{loco} \quad \frac{2aa}{aa - zz},$$

vel

$$\sqrt{(1 + zz)} + z \quad \text{loco} \quad \frac{1}{\sqrt{(1 + zz)} - z};$$

quae expressiones, etsi forma differunt, tamen revera congruunt. Saepenumero autem harum plurium formarum idem significantium una aptior est ad propositum efficiendum quam reliquae et hanc ob rem formam commodissimam eligi oportet.

Alter transformationis modus, quo loco quantitatis variabilis z alia quantitas variabilis y introducitur, quae quidem ad z datam teneat relationem, per substitutionem fieri dicitur; hocque modo ita uti convenit, ut functio proposita succinctius et commodius exprimatur; uti, si ista proposita fuerit ipsius z functio

$$a^4 - 4a^3z + 6aazz - 4az^3 + z^4,$$

si loco $a - z$ ponatur y, prodibit ista multo simplicior ipsius y functio

$$y^4,$$

et si habeatur haec functio irrationalis

$$\sqrt{(aa + zz)}$$

ipsius z, si ponatur

$$z = \frac{aa - yy}{2y},$$

ista functio per y expressa fiet rationalis

$$= \frac{aa + yy}{2y}.$$

Hunc autem transformationis modum in sequens caput differam hoc capite illum, qui sine substitutione procedit, expositurus.

28. *Functio integra ipsius z saepenumero commode in suos factores resolvitur sicque in productum transformatur.*

Quando functio integra hoc pacto in factores resolvitur, eius natura multo facilius perspicitur; casus enim statim innotescunt, quibus functionis valor fit $= 0$. Sic haec ipsius z functio

$$6 - 7z + z^3$$

transformatur in hoc productum

$$(1 - z)(2 - z)(3 + z),$$

ex quo statim liquet functionem propositam tribus casibus fieri $= 0$, scilicet si $z = 1$ et $z = 2$ et $z = -3$, quae proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Istiusmodi factores, in quibus variabilis z nulla [altior] occurrit potestas, vocantur factores *simplices*, ut distinguantur a factoribus *compositis*, in quibus ipsius z inest quadratum vel cubus vel alia potestas altior. Erit ergo in genere

$$f + gz$$

forma factorum simplicium,

$$f + gz + hzz$$

forma factorum duplicium,

$$f + gz + hzz + iz^3$$

forma factorum triplicium et ita porro. Perspicuum autem est factorem duplicem duos complecti factores simplices, factorem triplicem tres simplices et ita porro. Hinc functio ipsius z integra, in qua exponens summae potestatis ipsius z est $= n$, continebit n factores simplices; ex quo simul, si qui factores fuerint vel duplices vel triplices etc., numerus factorum cognoscetur.

29. *Factores simplices functionis cuiuscunque integrae Z ipsius z reperiuntur, si functio Z nihilo aequalis ponatur atque ex hac aequatione omnes ipsius z radices investigentur; singulae enim ipsius z radices dabunt totidem factores simplices functionis Z.*

Quodsi enim ex aequatione $Z = 0$ fuerit quaepiam radix $z = f$, erit $z - f$ divisor ac proinde factor functionis Z; sic igitur investigandis omnibus radicibus aequationis $Z = 0$, quae sint

$$z = f, \quad z = g, \quad z = h \quad \text{etc.},$$

functio Z resolvetur in suos factores simplices atque transformabitur in productum

$$Z = (z - f)(z - g)(z - h) \text{ etc.};$$

ubi quidem notandum est, si summae potestatis ipsius z in Z non fuerit coefficiens $= +1$, tum productum $(z-f)(z-g)$ etc. insuper per illum coefficientem multiplicari debere. Sic, si fuerit

$$Z = Az^n + Bz^{n-1} + Cz^{n-2} + \text{etc.},$$

erit

$$Z = A(z-f)(z-g)(z-h) \text{ etc.}$$

At si fuerit

$$Z = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

atque aequationis $Z=0$ radices z repertae sint f, g, h, i etc., erit

$$Z=A\left(1-\frac{z}{f}\right)\left(1-\frac{z}{g}\right)\left(1-\frac{z}{h}\right) \text{ etc.}$$

Ex his autem vicissim intelligitur, si functionis Z factor fuerit $z-f$ seu $1-\frac{z}{f}$, tum valorem functionis in nihilum abire, si loco z ponatur f. Facto enim $z=f$, unus factor $z-f$ seu $1-\frac{z}{f}$ functionis Z ideoque ipsa functio Z evanescere debet.

30. *Factores simplices ergo erunt vel reales vel imaginarii et, si functio Z habeat factores imaginarios, eorum numerus semper erit par.*

Cum enim factores simplices nascantur ex radicibus aequationis $Z=0$, radices reales praebebunt factores reales et imaginariae imaginarios; in omni autem aequatione numerus radicum imaginariarum semper est par, quamobrem functio Z vel nullos habebit factores imaginarios vel duos vel quatuor vel sex etc.[1]) Quodsi functio Z duos tantum habeat factores imaginarios, eorum productum erit reale ideoque praebebit factorem duplicem realem. Sit enim $P=$ producto ex omnibus factoribus realibus, erit productum duorum factorum imaginariorum $=\frac{Z}{P}$ hincque reale. Simili modo si functio Z habeat quatuor vel sex vel octo etc. factores imaginarios, erit eorum productum semper reale, nempe aequale quoto, qui oritur, si functio Z dividatur per productum omnium factorum realium.

31. *Si fuerit Q productum reale ex quatuor factoribus simplicibus imaginariis, tum idem hoc productum Q resolvi poterit in duos factores duplices reales.*[2])

Habebit enim Q eiusmodi formam

$$z^4+Az^3+Bz^2+Cz+D;$$

quae si negetur in duos factores duplices reales resolvi posse, resolubilis erit

1) Confer § 35 et 36, porro L. EULERI Commentationem 170 (indicis ENESTROEMIANI): *Recherches sur les racines imaginaires des équations*, Mém. de l'acad. d. sc. de Berlin 5 (1749), 1751, p. 222; *LEONHARDI EULERI Opera omnia*, series I, vol. 6, p. 78. A. K.

2) Vide § 9 Commentationis 170 modo laudatae, imprimis notam ibi adiectam. A. K.

statuenda in duos factores duplices imaginarios, qui huiusmodi formam habebunt

$$zz-2(p+q\sqrt{-1})z+r+s\sqrt{-1}$$

et

$$zz-2(p-q\sqrt{-1})z+r-s\sqrt{-1};$$

aliae enim formae imaginariae concipi non possunt, quarum productum fiat reale, nempe $=z^4+Az^3+Bz^2+Cz+D$. Ex his autem factoribus imaginariis duplicibus sequentes emergent quatuor factores simplices imaginarii ipsius Q:

$$\text{I. } z-(p+q\sqrt{-1})+\sqrt{(pp+2pq\sqrt{-1}-qq-r-s\sqrt{-1})},$$

$$\text{II. } z-(p+q\sqrt{-1})-\sqrt{(pp+2pq\sqrt{-1}-qq-r-s\sqrt{-1})},$$

$$\text{III. } z-(p-q\sqrt{-1})+\sqrt{(pp-2pq\sqrt{-1}-qq-r+s\sqrt{-1})},$$

$$\text{IV. } z-(p-q\sqrt{-1})-\sqrt{(pp-2pq\sqrt{-1}-qq-r+s\sqrt{-1})}.$$

Horum factorum multiplicentur primus ac tertius in se invicem posito brevitatis gratia

$$t=pp-qq-r \quad \text{et} \quad u=2pq-s$$

eritque horum factorum productum

$$=zz-\left(2p-\sqrt{2t+2\sqrt{(tt+uu)}}\right)z$$
$$+pp+qq-p\sqrt{2t+2\sqrt{(tt+uu)}}-q\sqrt{-2t+2\sqrt{(tt+uu)}}+\sqrt{(tt+uu)},$$

quod utique est reale. Simili autem modo productum ex factoribus secundo et quarto erit reale, nempe

$$=zz-\left(2p+\sqrt{2t+2\sqrt{(tt+uu)}}\right)z$$
$$+pp+qq+p\sqrt{2t+2\sqrt{(tt+uu)}}+q\sqrt{-2t+2\sqrt{(tt+uu)}}+\sqrt{(tt+uu)}.$$

Quocirca productum propositum Q, quod in duos factores duplices reales resolvi posse negabatur, nihilominus actu in duos factores duplices reales est resolutum.

32. *Si functio integra Z ipsius z quotcunque habeat factores simplices imaginarios, bini semper ita coniungi possunt, ut eorum productum fiat reale.*

Quoniam numerus radicum imaginariarum semper est par, sit is $= 2n$ ac primo quidem patet [§ 30] productum harum radicum imaginariarum omnium esse reale. Quodsi ergo duae tantum radices imaginariae habeantur, erit earum productum utique reale; sin autem quatuor habeantur factores imaginarii, tum, uti vidimus, eorum productum resolvi potest in duos factores duplices reales formae $fzz + gz + h$. Quanquam autem eundem demonstrandi modum ad altiores potestates extendere non licet, tamen extra dubium videtur esse positum eandem proprietatem in quotcunque factores imaginarios competere, ita ut semper loco $2n$ factorum simplicium imaginariorum induci queant n factores duplices reales. Hinc omnis functio integra ipsius z resolvi poterit in factores reales vel simplices vel duplices. Quod quamvis non summo rigore sit demonstratum, tamen eius veritas in sequentibus[1]) magis corroborabitur, ubi huius generis functiones

$$a + bz^n, \quad a + bz^n + cz^{2n}, \quad a + bz^n + cz^{2n} + dz^{3n} \quad \text{etc.}$$

actu in istiusmodi factores duplices reales resolventur.

33. *Si functio integra Z posito $z = a$ induat valorem A et posito $z = b$ induat valorem B, tum loco z valores medios inter a et b ponendo functio Z quosvis valores medios inter A et B accipere potest.*

Cum enim Z sit functio uniformis ipsius z, quicunque valor realis ipsi z tribuatur, functio quoque Z hinc valorem realem obtinebit. Cum igitur Z priore casu $z = a$ nanciscatur valorem A, posteriore casu $z = b$ autem valorem B, ab A ad B transire non poterit nisi per omnes valores medios transeundo. Quodsi ergo aequatio $Z - A = 0$ habeat radicem realem simulque $Z - B = 0$ radicem realem suppeditet, tum aequatio quoque $Z - C = 0$ radicem habebit realem, siquidem C intra valores A et B contineatur. Hinc si expressiones $Z - A$ et $Z - B$ habeant factorem simplicem realem, tum expressio quaecunque $Z - C$ factorem simplicem habebit realem, dummodo C intra valores A et B contineatur.

1) Vide § 150–154. A. K.

34. *Si in functione integra Z exponens maximae ipsius z potestatis fuerit numerus impar* $2n+1$, *tum ea functio Z unicum ad minimum habebit factorem simplicem realem.*

Habebit scilicet Z huiusmodi formam

$$z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} + \text{etc.};$$

in qua si ponatur $z = \infty$, quia valores singulorum terminorum prae primo evanescunt, fiet

$$Z = (\infty)^{2n+1} = \infty$$

ideoque $Z - \infty$ factorem simplicem habebit realem, nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet

$$Z = (-\infty)^{2n+1} = -\infty$$

ideoque habebit $Z + \infty$ factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$ quam $Z + \infty$ habeat factorem simplicem realem, sequitur etiam $Z - C$ habiturum esse factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ et $-\infty$, hoc est, si C fuerit numerus realis quicunque, sive affirmativus sive negativus. Hanc ob rem facto $C = 0$ habebit quoque ipsa functio Z factorem simplicem realem $z - c$ atque quantitas c continebitur intra limites $+\infty$ et $-\infty$ eritque idcirco vel quantitas affirmativa vel negativa vel nihil.

35. *Functio igitur integra Z, in qua exponens maximae potestatis ipsius z est numerus impar, vel unum habebit factorem simplicem realem vel tres vel quinque vel septem etc.*

Cum enim demonstratum sit functionem Z certo unum habere factorem simplicem realem $z - c$, ponamus eam praeterea unum factorem habere $z - d$ atque dividatur functio Z, in qua maxima ipsius z potestas sit z^{2n+1}, per $(z-c)(z-d)$; erit quoti maxima potestas $= z^{2n-1}$, cuius exponens, cum sit numerus impar, indicat denuo ipsius Z dari factorem simplicem realem. Si ergo Z plures uno habeat factores simplices reales, habebit vel tres vel (quoniam eodem modo progredi licet) quinque vel septem etc. Erit scilicet numerus factorum simplicium realium impar, et quia numerus omnium factorum simplicium est $= 2n + 1$, erit numerus factorum imaginariorum par.

36. *Functio integra Z, in qua exponens maximae potestatis ipsius z est numerus par $2n$, vel duos habebit factores simplices reales vel quatuor vel sex vel etc.*

Ponamus ipsius Z constare factorum simplicium realium numerum imparem $2m+1$; si ergo per horum omnium productum dividatur functio Z, quoti maxima potestas erit $= z^{2n-2m-1}$ eiusque ideo exponens numerus impar; habebit ergo functio Z praeterea unum certo factorem simplicem realem, ex quo numerus omnium factorum simplicium realium ad minimum erit $= 2m+2$ ideoque par ac numerus factorum imaginariorum pariter par. Omnis ergo functionis integrae factores simplices imaginarii sunt numero pares, quemadmodum quidem iam ante [§ 30] statuimus.

37. *Si in functione integra Z exponens maximae potestatis ipsius z fuerit numerus par atque terminus absolutus seu constans signo — affectus, tum functio Z ad minimum duos habet factores simplices reales.*

Functio ergo Z, de qua hic sermo est, huiusmodi formam habebit

$$z^{2n} \pm \alpha z^{2n-1} \pm \beta z^{2n-2} \pm \cdots \pm \nu z - A.$$

Si iam ponatur $z=\infty$, fiet, uti supra vidimus, $Z=\infty$ atque, si ponatur $z=0$, fiet $Z=-A$. Habebit ergo $Z-\infty$ factorem realem $z-\infty$ et $Z+A$ factorem $z-0$; unde, cum 0 contineatur intra limites $-\infty$ et $+A$, sequitur $Z+0$ habere factorem simplicem realem $z-c$, ita ut c contineatur intra limites 0 et ∞. Deinde, cum posito $z=-\infty$ fiat $Z=\infty$ ideoque $Z-\infty$ factorem habeat $z+\infty$ et $Z+A$ factorem $z+0$, sequitur quoque $Z+0$ factorem simplicem realem habere $z+d$, ita ut d intra limites 0 et ∞ contineatur; unde constat propositum. Ex his igitur perspicitur, si Z talis fuerit functio, qualis hic est descripta, aequationem $Z=0$ duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic aequatio haec

$$z^4 + \alpha z^3 + \beta z^2 + \gamma z - aa = 0$$

duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in functione fracta quantitas variabilis z tot vel plures habeat dimensiones in numeratore quam in denominatore, tum ista functio resolvi poterit in duas partes, quarum altera est functio integra, altera fracta, in cuius numeratore quantitas variabilis z pauciores habeat dimensiones quam in denominatore.*

Si enim exponens maximae potestatis ipsius z minor fuerit in denominatore quam in numeratore, tum numerator per denominatorem dividatur more solito, donec in quoto ad exponentes negativos ipsius z perveniatur; hoc ergo loco abrupta divisionis operatione quotus constabit ex parte integra atque fractione, in cuius numeratore minor erit dimensionum numerus ipsius z quam in denominatore; hic autem quotus functioni propositae est aequalis. Sic, si haec proposita fuerit functio fracta

$$\frac{1+z^4}{1+zz},$$

ea per divisionem ita resolvetur:

$$\begin{array}{rl} zz+1) & z^4+1 \quad \left(zz-1+\frac{2}{1+zz}\right. \\ & \underline{z^4+zz} \\ & \quad -zz+1 \\ & \quad \underline{-zz-1} \\ & \qquad +2 \end{array}$$

eritque

$$\frac{1+z^4}{1+zz}=zz-1+\frac{2}{1+zz}.$$

Huiusmodi functiones fractae, in quibus quantitas variabilis z tot vel plures habet dimensiones in numeratore quam in denominatore, ad similitudinem Arithmeticae vocari possunt fractiones *spuriae* vel functiones fractae spuriae, quo distinguantur a functionibus fractis *genuinis*, in quarum numeratore quantitas variabilis z pauciores habet dimensiones quam in denominatore. Functio itaque fracta spuria resolvi poterit in functionem integram et functionem fractam genuinam haecque resolutio per vulgarem divisionis operationem absolvetur.

39. *Si denominator functionis fractae duos habeat factores inter se primos, tum ipsa functio fracta resolvetur in duas fractiones, quarum denominatores sint illis binis factoribus respective aequales.*

Quanquam haec resolutio ad functiones fractas spurias aeque pertinet atque ad genuinas, tamen eam ad genuinas potissimum accommodabimus. Resoluto autem denominatore huiusmodi functionis fractae in duos factores inter se primos, ipsa functio resolvetur in duas alias functiones fractas genuinas, quarum denominatores sint illis binis factoribus respective aequales, haecque resolutio, siquidem fractiones sint genuinae, unico modo fieri potest; cuius rei veritas ex exemplo clarius quam per ratiocinium perspicietur. Sit ergo proposita haec functio fracta

$$\frac{1-2z+3zz-4z^3}{1+4z^4};$$

cuius denominator $1+4z^4$ cum sit aequalis huic producto

$$(1+2z+2zz)(1-2z+2zz),$$

fractio proposita in duas fractiones resolvetur, quarum alterius denominator erit $1+2z+2zz$, alterius $1-2z+2zz$; ad quas inveniendas, quia sunt genuinae, statuantur numeratores illius $=\alpha+\beta z$, huius $=\gamma+\delta z$ eritque per hypothesin

$$\frac{1-2z+3zz-4z^3}{1+4z^4}=\frac{\alpha+\beta z}{1+2z+2zz}+\frac{\gamma+\delta z}{1-2z+2zz};$$

addantur actu hae duae fractiones eritque summae

numerator	denominator
$+\alpha-2\alpha z+2\alpha zz$ $+\beta z-2\beta zz+2\beta z^3$ $+\gamma+2\gamma z+2\gamma zz$ $+\delta z+2\delta zz+2\delta z^3,$	$1+4z^4.$

Cum ergo denominator aequalis sit denominatori fractionis propositae, numeratores quoque aequales reddi debent; quod ob tot litteras incognitas α, β, γ, δ, quot sunt termini aequales efficiendi, utique fieri idque unico modo poterit; nanciscimur scilicet has quatuor aequationes

I. $\alpha + \gamma = 1,$ III. $2\alpha - 2\beta + 2\gamma + 2\delta = 3,$

II. $-2\alpha + \beta + 2\gamma + \delta = -2,$ IV. $2\beta + 2\delta = -4.$

Hinc ob

$$\alpha + \gamma = 1 \quad \text{et} \quad \beta + \delta = -2$$

aequationes II. et III. dabunt

$$\alpha - \gamma = 0 \quad \text{et} \quad \delta - \beta = \frac{1}{2},$$

ex quibus fit

$$\alpha = \frac{1}{2}, \quad \gamma = \frac{1}{2}, \quad \beta = -\frac{5}{4}, \quad \delta = -\frac{3}{4},$$

ideoque fractio proposita

$$\frac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$$

transformatur in has duas

$$\frac{\frac{1}{2} - \frac{5}{4}z}{1 + 2z + 2zz} + \frac{\frac{1}{2} - \frac{3}{4}z}{1 - 2z + 2zz}.$$

Simili autem modo facile perspicietur resolutionem semper succedere debere, quoniam semper tot litterae incognitae introducuntur, quot opus est ad numeratorem propositum eliciendum. Ex doctrina vero fractionum communi intelligitur hanc resolutionem succedere non posse, nisi isti denominatoris factores fuerint inter se primi.

40. *Functio igitur fracta* $\frac{M}{N}$ *in tot fractiones simplices formae* $\frac{A}{p - qz}$ *resolvi poterit, quot factores simplices habet denominator* N *inter se inaequales.*

Repraesentat hic fractio $\frac{M}{N}$ functionem quamcunque fractam genuinam, ita ut M et N sint functiones integrae ipsius z atque summa potestas ipsius z in M minor sit quam in N. Quodsi ergo denominator N in suos factores simplices resolvatur hique inter se fuerint inaequales, expressio $\frac{M}{N}$ in tot fractiones resolvetur, quot factores simplices in denominatore N continentur, propterea quod quisque factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit factor ipsius N, is erit denominator fractionis cuiusdam partialis et, cum in numeratore huius fractionis numerus dimensionum ipsius z minor esse debeat quam in denominatore $p - qz$, numerator necessario erit quantitas constans. Hinc ex unoquoque factore simplici $p - qz$ denominatoris N nascetur fractio simplex $\frac{A}{p-qz}$, ita ut summa omnium harum fractionum sit aequalis fractioni propositae $\frac{M}{N}$.

EXEMPLUM

Sit exempli causa proposita haec functio fracta

$$\frac{1+zz}{z-z^3}.$$

Quia factores simplices denominatoris sunt z, $1-z$ et $1+z$, ista functio resolvetur in has tres fractiones simplices

$$\frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z} = \frac{1+zz}{z-z^3},$$

ubi numeratores constantes A, B et C definire oportet. Reducantur hae fractiones ad communem denominatorem, qui erit $z - z^3$, atque numeratorum summa aequari debebit ipsi $1 + zz$, unde ista aequatio oritur

$$\begin{aligned} A + Bz - Azz &= 1 + zz = 1 + 0z + zz, \\ + Cz + Bzz & \\ - Czz & \end{aligned}$$

quae totidem comparationes praebet, quot sunt litterae incognitae A, B, C; erit scilicet

$$\text{I.}\quad A = 1,$$

$$\text{II.}\quad B + C = 0,$$

$$\text{III.}\quad -A + B - C = 1.$$

Hinc fit

$$B - C = 2$$

et porro

$$A = 1, \quad B = 1 \quad \text{et} \quad C = -1.$$

Functio ergo proposita

$$\frac{1+zz}{z-z^3}$$

resolvitur in hanc formam

$$\frac{1}{z}+\frac{1}{1-z}-\frac{1}{1+z}.$$

Simili autem modo intelligitur, quotcunque habuerit denominator N factores simplices inter se inaequales, semper fractionem $\frac{M}{N}$ in totidem fractiones simplices resolvi. Sin autem aliquot factores fuerint aequales inter se, tum alio modo post explicando resolutio institui debet.

41. *Cum igitur quilibet factor simplex denominatoris N suppeditet fractionem simplicem pro resolutione functionis propositae $\frac{M}{N}$, ostendendum est, quomodo ex factore simplici denominatoris N cognito fractio simplex respondens reperiatur.*[1])

Sit $p-qz$ factor simplex ipsius N, ita ut sit

$$N=(p-qz)S$$

atque S functio integra ipsius z; ponatur fractio ex factore $p-qz$ orta

$$=\frac{A}{p-qz}$$

et sit fractio ex altero factore denominatoris S oriunda

$$=\frac{P}{S},$$

ita ut secundum § 39 futurum sit

$$\frac{M}{N}=\frac{A}{p-qz}+\frac{P}{S}=\frac{M}{(p-qz)S};$$

hinc erit

$$\frac{P}{S}=\frac{M-AS}{(p-qz)S};$$

1) Confer L. Euleri Commentationes 540 et 728 (indicis Enestroemiani): *Nova methodus fractiones quascunque rationales in fractiones simplices resolvendi*, Acta acad. sc. Petrop. 1780: I, 1783, p. 32, et *De resolutione fractionum compositarum in simpliciores*, Mém. de l'acad. d. sc. de St. Pétersbourg 1 (1803/6), 1809, p. 3; *Leonhardi Euleri Opera omnia*, series I, vol. 6, p. 370 et 465. A. K.

quae fractiones cum congruere debeant, necesse est, ut $M - AS$ sit divisibile per $p - qz$, quoniam functio integra P ipsi quoto aequatur. Quando vero $p - qz$ divisor existit ipsius $M - AS$, haec expressio posito $z = \frac{p}{q}$ evanescit. Ponatur ergo ubique loco z hic valor constans $\frac{p}{q}$ in M et S; erit $M - AS = 0$, ex quo fiet

$$A = \frac{M}{S},$$

hocque ergo modo reperitur numerator A fractionis quaesitae $\frac{A}{p - qz}$; atque si ex singulis denominatoris N factoribus simplicibus, dummodo sint inter se inaequales, huiusmodi fractiones simplices formentur, harum fractionum simplicium omnium summa erit aequalis functioni propositae $\frac{M}{N}$.

EXEMPLUM

Sic, si in exemplo praecedente

$$\frac{1 + zz}{z - z^3},$$

ubi est

$$M = 1 + zz \quad \text{et} \quad N = z - z^3,$$

sumatur z pro factore simplici, erit

$$S = 1 - zz$$

atque fractionis simplicis $\frac{A}{z}$ hinc ortae erit numerator

$$A = \frac{1 + zz}{1 - zz} = 1$$

posito $z = 0$, quem valorem z obtinet, si ipse hic factor simplex z nihilo aequalis ponatur.

Simili modo si pro denominatoris factore sumatur $1 - z$, ut sit

$$S = z + zz,$$

erit

$$A = \frac{1 + zz}{z + zz}$$

facto $1 - z = 0$, unde erit

$$A = 1$$

et ex factore $1 - z$ nascitur fractio $\frac{1}{1-z}$.

Tertius denique factor $1 + z$ ob

$$S = z - zz$$

et

$$A = \frac{1+zz}{z-zz}$$

posito $1 + z = 0$ seu $z = -1$ dabit

$$A = -1$$

et fractionem simplicem $= \frac{-1}{1+z}$.

Quare per hanc regulam reperitur

$$\frac{1+zz}{z-z^3} = \frac{1}{z} + \frac{1}{1-z} - \frac{1}{1+z}$$

ut ante.

42. *Functio fracta huius formae* $\frac{P}{(p-qz)^n}$, *cuius numerator* P *non tantam ipsius* z *potestatem involvit quantam denominator* $(p-qz)^n$, *transmutari potest in huiusmodi fractiones partiales*

$$\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \frac{D}{(p-qz)^{n-3}} + \cdots + \frac{K}{p-qz},$$

quarum omnium numeratores sint quantitates constantes.

Quoniam maxima potestas ipsius z in P minor est quam z^n, erit z^{n-1} ideoque P huiusmodi habebit formam

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \cdots + \varkappa z^{n-1}$$

existente terminorum numero $= n$, cui aequari debet numerator summae omnium fractionum partialium, postquam singulae ad eundem denominatorem $(p-qz)^n$ fuerint perductae; qui numerator propterea erit

$$= A + B(p-qz) + C(p-qz)^2 + D(p-qz)^3 + \cdots + K(p-qz)^{n-1}.$$

Huius maxima ipsius z potestas est, ut ibi, z^{n-1} atque tot habentur litterae incognitae $A, B, C, D, \ldots K$ (quarum numerus est $= n$), quot sunt termini congruentes reddendi. Quamobrem litterae A, B, C etc. ita definiri poterunt, ut fiat functio fracta genuina

$$\frac{P}{(p-qz)^n} = \frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \frac{D}{(p-qz)^{n-3}} + \cdots + \frac{K}{p-qz}.$$

Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. *Si functionis fractae* $\frac{M}{N}$ *denominator* N *factorem habeat* $(p-qz)^2$, *sequenti modo fractiones partiales ex hoc factore oriundae reperientur.*

Cuiusmodi fractiones partiales ex singulis factoribus denominatoris simplicibus, qui alios sibi aequales non habeant, oriantur, ante est ostensum; nunc igitur ponamus duos factores inter se esse aequales seu iis coniunctis denominatoris N factorem esse $(p-qz)^2$. Ex hoc ergo factore per paragraphum praecedentem duae nascentur fractiones partiales hae

$$\frac{A}{(p-qz)^2} + \frac{B}{p-qz}.$$

Sit autem

$$N = (p-qz)^2 S$$

eritque

$$\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2} + \frac{B}{p-qz} + \frac{P}{S}$$

denotante $\frac{P}{S}$ omnes fractiones simplices iunctim sumptas ex denominatoris factore S ortas. Hinc erit

$$\frac{P}{S} = \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$$

et

$$P = \frac{M - AS - B(p-qz)S}{(p-qz)^2} = \text{functioni integrae.}$$

Debet ergo

$$M - AS - B(p-qz)S$$

divisibile esse per $(p-qz)^2$. Sit primum divisibile per $p-qz$ atque tota expressio $M - AS - B(p - qz)S$ evanescet posito $p - qz = 0$ seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$ loco z eritque $M - AS = 0$ ideoque

$$A = \frac{M}{S};$$

scilicet fractio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A constantem.

Hoc invento quantitas $M - AS - B(p - qz)S$ etiam per $(p - qz)^2$ divisibilis esse debet seu $\frac{M - AS}{p - qz} - BS$ denuo per $p - qz$ divisibilis esse debet. Posito ergo ubique $z = \frac{p}{q}$ erit

$$\frac{M - AS}{p - qz} = BS$$

ideoque

$$B = \frac{M - AS}{(p - qz)S} = \frac{1}{p - qz}\left(\frac{M}{S} - A\right),$$

ubi notandum est, cum $M - AS$ divisibile sit per $p - qz$, hanc divisionem prius institui debere, quam loco z substituatur $\frac{p}{q}$. Vel ponatur

$$\frac{M - AS}{p - qz} = T$$

eritque

$$B = \frac{T}{S}$$

posito $z = \frac{p}{q}$.

Inventis ergo numeratoribus A et B erunt fractiones partiales ex denominatoris N factore $(p - qz)^2$ ortae hae

$$\frac{A}{(p - qz)^2} + \frac{B}{p - qz}.$$

EXEMPLUM 1

Sit haec proposita functio fracta

$$\frac{1 - zz}{zz(1 + zz)};$$

erit ob denominatoris factorem quadratum zz

$$S = 1 + zz \quad \text{et} \quad M = 1 - zz.$$

Sint fractiones partiales ex zz ortae

$$\frac{A}{zz} + \frac{B}{z};$$

erit

$$A = \frac{M}{S} = \frac{1 - zz}{1 + zz}$$

posito factore $z = 0$ hincque

$$A = 1.$$

Tum erit $M - AS = -2zz$, quod divisum per factorem simplicem z dabit

$$T = -2z$$

hincque

$$B = \frac{T}{S} = \frac{-2z}{1 + zz}$$

posito $z = 0$; unde erit

$$B = 0$$

atque ex factore denominatoris zz orietur unica haec fractio partialis $\frac{1}{zz}$.

EXEMPLUM 2

Sit haec proposita functio fracta

$$\frac{z^3}{(1-z)^2(1+z^4)},$$

cuius ob denominatoris factorem quadratum $(1-z)^2$ fractiones partiales sint

$$\frac{A}{(1-z)^2} + \frac{B}{1-z}.$$

Erit ergo

$$M = z^3 \quad \text{et} \quad S = 1 + z^4$$

ideoque

$$A = \frac{M}{S} = \frac{z^3}{1 + z^4}$$

posito $1 - z = 0$ seu $z = 1$, unde fit

$$A = \frac{1}{2}.$$

Prodibit ergo

$$M - AS = z^3 - \frac{1}{2} - \frac{1}{2}z^4 = -\frac{1}{2} + z^3 - \frac{1}{2}z^4,$$

quod per $1 - z$ divisum dat

$$T = -\frac{1}{2} - \frac{1}{2}z - \frac{1}{2}zz + \frac{1}{2}z^3;$$

ideoque

$$B = \frac{T}{S} = \frac{-1 - z - zz + z^3}{2 + 2z^4}$$

posito $z = 1$, ita ut sit

$$B = -\frac{1}{2}.$$

Fractiones ergo partiales quaesitae sunt

$$\frac{1}{2(1-z)^2} - \frac{1}{2(1-z)}.$$

44. *Si functionis fractae* $\frac{M}{N}$ *denominator* N *factorem habeat* $(p - qz)^3$, *sequenti modo fractiones partiales ex hoc factore oriundae*

$$\frac{A}{(p-qz)^3} + \frac{B}{(p-qz)^2} + \frac{C}{p-qz}$$

reperientur.

Ponatur

$$N = (p - qz)^3 S$$

sitque fractio ex factore S orta $= \frac{P}{S}$; erit

$$P = \frac{M - AS - B(p-qz)S - C(p-qz)^2 S}{(p-qz)^3} = \text{functioni integrae.}$$

Numerator ergo

$$M - AS - B(p-qz)S - C(p-qz)^2 S$$

ante omnia divisibilis esse debet per $p - qz$, unde is posito $p - qz = 0$ seu $z = \frac{p}{q}$ evanescere debet eritque adeo $M - AS = 0$ ideoque

$$A = \frac{M}{S}$$

posito $z = \frac{p}{q}$.

Invento hoc pacto A erit $M - AS$ divisibile per $p - qz$; ponatur ergo

$$\frac{M - AS}{p - qz} = T$$

atque

$$T - BS - C(p - qz)S$$

adhuc per $(p - qz)^2$ erit divisibile; fiet ergo $= 0$ posito $p - qz = 0$, ex quo prodit

$$B = \frac{T}{S}$$

posito $z = \frac{p}{q}$.

Sic autem invento B erit $T - BS$ divisibile per $p - qz$. Hanc ob rem posito

$$\frac{T - BS}{p - qz} = V$$

superest, ut

$$V - CS$$

divisibile sit per $p - qz$, eritque ergo $V - CS = 0$ posito $p - qz = 0$ atque

$$C = \frac{V}{S}$$

posito $z = \frac{p}{q}$.

Inventis ergo hoc modo numeratoribus A, B, C fractiones partiales ex denominatoris N factore $(p - qz)^3$ ortae erunt

$$\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}.$$

EXEMPLUM

Sit proposita haec fracta functio

$$\frac{zz}{(1-z)^3(1+zz)},$$

ex cuius denominatoris factore cubico $(1-z)^3$ oriantur hae fractiones partiales

$$\frac{A}{(1-z)^3}+\frac{B}{(1-z)^2}+\frac{C}{1-z}.$$

Erit ergo

$$M=zz \quad \text{et} \quad S=1+zz;$$

unde fit primum

$$A=\frac{zz}{1+zz}$$

posito $1-z=0$ seu $z=1$, ex quo prodit

$$A=\frac{1}{2}.$$

Iam ponatur

$$T=\frac{M-AS}{1-z};$$

erit

$$T=\frac{\frac{1}{2}zz-\frac{1}{2}}{1-z}=-\frac{1}{2}-\frac{1}{2}z;$$

unde oritur

$$B=\frac{-\frac{1}{2}-\frac{1}{2}z}{1+zz}$$

posito $z=1$, ita ut sit

$$B=-\frac{1}{2}.$$

Ponatur porro

$$V=\frac{T-BS}{1-z}=\frac{T+\frac{1}{2}S}{1-z};$$

erit

$$V=\frac{-\frac{1}{2}z+\frac{1}{2}zz}{1-z}=-\frac{1}{2}z;$$

unde fit

$$C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1+zz}$$

posito $z=1$, ita ut sit

$$C = -\frac{1}{4}.$$

Quocirca fractiones partiales ex denominatoris factore $(1-z)^3$ ortae erunt

$$\frac{1}{2(1-z)^3} - \frac{1}{2(1-z)^2} - \frac{1}{4(1-z)}.$$

45. *Si functionis fractae* $\frac{M}{N}$ *denominator* N *factorem habeat* $(p-qz)^n$, *fractiones partiales hinc ortae*

$$\frac{A}{(p-qz)^n} + \frac{B}{(p-qz)^{n-1}} + \frac{C}{(p-qz)^{n-2}} + \cdots + \frac{K}{p-qz}$$

sequenti modo invenientur.

Ponatur denominator

$$N = (p-qz)^n Z$$

atque ratiocinium ut ante instituendo reperietur, ut sequitur, primo

$$A = \frac{M}{Z}$$

posito $z = \frac{p}{q}$.

Ponatur

$$P = \frac{M-AZ}{p-qz};$$

secundo

$$B = \frac{P}{Z}$$

posito $z = \frac{p}{q}$.

Ponatur

$$Q = \frac{P-BZ}{p-qz};$$

tertio

$$C = \frac{Q}{Z}$$

posito $z = \frac{p}{q}$.

Ponatur

$$R = \frac{Q - CZ}{p - qz};$$

quarto

$$D = \frac{R}{Z}$$

posito $z = \frac{p}{q}$.

Ponatur

$$S = \frac{R - DZ}{p - qz};$$

quinto

$$E = \frac{S}{Z}$$

posito $z = \frac{p}{q}$; etc.

Hoc ergo modo si definiantur singuli numeratores constantes A, B, C, D etc., invenientur omnes fractiones partiales, quae ex denominatoris N factore $(p - qz)^n$ nascuntur.

EXEMPLUM

Sit proposita ista functio fracta

$$\frac{1 + zz}{z^5(1 + z^3)},$$

ex cuius denominatoris factore z^5 nascantur hae fractiones partiales

$$\frac{A}{z^5} + \frac{B}{z^4} + \frac{C}{z^3} + \frac{D}{z^2} + \frac{E}{z}.$$

Ad quarum numeratores constantes inveniendos erit

$$M = 1 + zz \quad \text{atque} \quad Z = 1 + z^3 \quad \text{et} \quad \frac{p}{q} = 0.$$

Sequens ergo calculus ineatur.

Primum est

$$A = \frac{M}{Z} = \frac{1 + zz}{1 + z^3}$$

posito $z = 0$, ergo

$$A = 1.$$

Ponatur

$$P=\frac{M-AZ}{z}=\frac{zz-z^3}{z}=z-zz$$

eritque secundo

$$B=\frac{P}{Z}=\frac{z-zz}{1+z^3}$$

posito $z=0$, ergo

$$B=0.$$

Ponatur

$$Q=\frac{P-BZ}{z}=\frac{z-zz}{z}=1-z$$

eritque tertio

$$C=\frac{Q}{Z}=\frac{1-z}{1+z^3}$$

posito $z=0$, ergo

$$C=1.$$

Ponatur

$$R=\frac{Q-CZ}{z}=\frac{-z-z^3}{z}=-1-zz;$$

erit quarto

$$D=\frac{R}{Z}=\frac{-1-zz}{1+z^3}$$

posito $z=0$, ex quo fit

$$D=-1.$$

Ponatur

$$S=\frac{R-DZ}{z}=\frac{-zz+z^3}{z}=-z+zz;$$

erit quinto

$$E=\frac{S}{Z}=\frac{-z+zz}{1+z^3}$$

posito $z=0$, unde fit

$$E=0.$$

Quocirca fractiones partiales quaesitae erunt hae:

$$\frac{1}{z^5}+\frac{0}{z^4}+\frac{1}{z^3}-\frac{1}{z^2}+\frac{0}{z}.$$

[45a].[1]) *Quaecunque ergo proposita fuerit functio rationalis fracta* $\frac{M}{N}$, *ea sequenti modo in partes resolvetur atque in formam simplicissimam transmutabitur.*

Quaerantur denominatoris N omnes factores simplices sive reales sive imaginarii; quorum qui sibi pares non habeant, seorsim tractentur et ex unoquoque per § 41 fractio partialis eruatur. Quodsi idem factor simplex bis vel pluries occurrat, ii coniunctim sumantur atque ex eorum producto, quod erit potestas formae $(p-qz)^n$, quaerantur fractiones partiales convenientes per § 45. Hocque modo cum ex singulis factoribus simplicibus denominatoris erutae fuerint fractiones partiales, tum harum omnium aggregatum aequabitur functioni propositae $\frac{M}{N}$, nisi fuerit spuria; si enim fuerit spuria, pars integra insuper extrahi atque ad istas fractiones partiales inventas adiici debebit, quo prodeat valor functionis $\frac{M}{N}$ in forma simplicissima expressus. Perinde autem est, sive fractiones partiales ante extractionem partis integrae sive post quaerantur. Eaedem enim ex singulis denominatoris N factoribus prodeunt fractiones partiales, sive adhibeatur ipse numerator M sive idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM

Quaeratur valor functionis

$$\frac{1}{z^3(1-z)^2(1+z)}$$

in forma simplicissima expressus.

Sumatur primum factor denominatoris solitarius $1+z$, qui dat $\frac{p}{q}=-1$; erit

$$M=1 \quad \text{et} \quad Z=z^3-2z^4+z^5.$$

Hinc ad fractionem $\frac{A}{1+z}$ inveniendam erit

$$A=\frac{1}{z^3-2z^4+z^5}$$

posito $z=-1$ ideoque fit

$$A=-\frac{1}{4}$$

1) In editione principe et huic et sequenti paragrapho per errorem numerus 46 datus est. A. K.

atque ex factore $1+z$ oritur haec fractio partialis

$$-\frac{1}{4(1+z)}.$$

Iam sumatur factor quadratus $(1-z)^2$, qui dat

$$\frac{p}{q}=1, \quad M=1 \quad \text{et} \quad Z=z^3+z^4.$$

Positis ergo fractionibus partialibus hinc ortis

$$\frac{A}{(1-z)^2}+\frac{B}{1-z}$$

erit

$$A=\frac{1}{z^3+z^4}$$

posito $z=1$, ergo

$$A=\frac{1}{2}.$$

Fiat

$$P=\frac{M-\frac{1}{2}Z}{1-z}=\frac{1-\frac{1}{2}z^3-\frac{1}{2}z^4}{1-z}=1+z+zz+\frac{1}{2}z^3$$

eritque

$$B=\frac{P}{Z}=\frac{1+z+zz+\frac{1}{2}z^3}{z^3+z^4}$$

posito $z=1$, ergo

$$B=\frac{7}{4}$$

et fractiones partiales quaesitae

$$\frac{1}{2(1-z)^2}+\frac{7}{4(1-z)}.$$

Denique tertius factor cubicus z^3 dat

$$\frac{p}{q}=0, \quad M=1 \quad \text{et} \quad Z=1-z-zz+z^3.$$

Positis ergo fractionibus partialibus his

$$\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}$$

erit primum

$$A = \frac{M}{Z} = \frac{1}{1 - z - zz + z^3}$$

posito $z = 0$, ergo

$$A = 1.$$

Ponatur

$$P = \frac{M - Z}{z} = 1 + z - zz;$$

erit

$$B = \frac{P}{Z}$$

posito $z = 0$, ergo

$$B = 1.$$

Ponatur

$$Q = \frac{P - Z}{z} = 2 - zz;$$

erit

$$C = \frac{Q}{Z}$$

posito $z = 0$, ergo

$$C = 2.$$

Hanc ob rem functio proposita

$$\frac{1}{z^3(1-z)^2(1+z)}$$

in hanc formam resolvitur

$$\frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{z} + \frac{1}{2(1-z)^2} + \frac{7}{4(1-z)} - \frac{1}{4(1+z)}.$$

Nulla enim pars integra insuper accedit, quia fractio proposita non est spuria.[1])

1) De reali functionum fractarum evolutione agitur in cap. XII. A. K.

CAPUT III

DE TRANSFORMATIONE FUNCTIONUM PER SUBSTITUTIONEM

46. *Si fuerit y functio quaecunque ipsius z atque z definiatur per novam variabilem x, tum quoque y per x definiri poterit.*

Cum ergo antea y fuisset functio ipsius z, nunc nova quantitas variabilis x inducitur, per quam utraque priorum y et z definiatur. Sic, si fuerit

$$y = \frac{1-zz}{1+zz}$$

atque ponatur

$$z = \frac{1-x}{1+x},$$

hoc valore loco z substituto erit

$$y = \frac{2x}{1+xx}.$$

Sumpto ergo pro x valore quocunque determinato ex eo reperientur valores determinati pro z et y sicque invenitur valor ipsius y respondens illi valori ipsius z, qui simul prodiit. Uti, si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$ et $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$, cui expressioni y aequatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem haec novae variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius y per z data laborat, tollitur; vel quando ob aequationem altioris gradus, qua relatio inter y et z exprimitur, non licet functionem explicitam ipsius z ipsi y aequalem exhibere, nova variabilis x introducitur, ex qua utraque y et z commode definiri queat; unde insignis substitutionum usus iam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit*

$$y = \sqrt{(a + bz)},$$

nova variabilis x, per quam utraque z et y rationaliter exprimatur, sequenti modo invenietur.

Quoniam tam z quam y debet esse functio rationalis ipsius x, perspicuum est hoc obtineri, si ponatur

$$\sqrt{(a + bz)} = bx.$$

Fiet enim primo

$$y = bx \quad \text{et} \quad a + bz = bbxx$$

hincque

$$z = bxx - \frac{a}{b}.$$

Quocirca utraque quantitas y et z per functionem rationalem ipsius x exprimitur; scilicet cum sit $y = \sqrt{(a + bz)}$, fiat

$$z = bxx - \frac{a}{b};$$

erit

$$y = bx.$$

48. *Si fuerit*

$$y = (a + bz)^{m:n},$$

nova variabilis x, per quam tam y quam z rationaliter exprimatur, sic reperietur.

Ponatur

$$y = x^m$$

fietque

$$(a + bz)^{m:n} = x^m \quad \text{ideoque} \quad (a + bz)^{1:n} = x, \quad \text{ergo} \quad a + bz = x^n$$

et

$$z = \frac{x^n - a}{b}.$$

Sic ergo utraque quantitas y et z rationaliter per x definietur, ope scilicet substitutionis

$$z = \frac{x^n - a}{b},$$

quae praebet

$$y = x^m.$$

Quamvis igitur neque y per z neque vicissim z per y rationaliter exprimi possit, tamen utraque reddita est functio rationalis novae quantitatis variabilis x per substitutionem introductae, scopo substitutionis omnino convenienter.

49. *Si fuerit*

$$y = \left(\frac{a + bz}{f + gz}\right)^{m:n},$$

requiritur nova quantitas variabilis x, per quam utraque y et z rationaliter exprimatur.

Manifestum primo est, si ponatur

$$y = x^m,$$

quaesito satisfieri; erit enim

$$\left(\frac{a + bz}{f + gz}\right)^{m:n} = x^m \quad \text{ideoque} \quad \frac{a + bz}{f + gz} = x^n;$$

ex qua aequatione elicitur

$$z = \frac{a - fx^n}{gx^n - b},$$

quae substitutio praebet

$$y = x^m.$$

Hinc quoque intelligitur, si fuerit

$$\left(\frac{\alpha + \beta y}{\gamma + \delta y}\right)^n = \left(\frac{a + bz}{f + gz}\right)^m,$$

tam y quam z rationaliter per x expressum iri, si utraque formula ponatur $= x^{mn}$; reperietur enim

$$y = \frac{\alpha - \gamma x^m}{\delta x^m - \beta}$$

et

$$z = \frac{a - f x^n}{g x^n - b};$$

qui casus nil habent difficultatis.

50. *Si fuerit*

$$y = \sqrt{((a + bz)(c + dz))},$$

substitutio idonea invenietur, qua y et z rationaliter exprimuntur, hoc modo.

Ponatur

$$\sqrt{((a + bz)(c + dz))} = (a + bz)x;$$

facile enim perspicitur hinc valorem rationalem pro z esse proditurum, quia valor ipsius z per aequationem simplicem determinatur. Erit ergo

$$c + dz = (a + bz)xx$$

hincque

$$z = \frac{c - axx}{bxx - d}.$$

Quare porro fiet

$$a + bz = \frac{bc - ad}{bxx - d}$$

et ob $y = \sqrt{((a + bz)(c + dz))} = (a + bz)x$ habebitur

$$y = \frac{(bc - ad)x}{bxx - d}.$$

Functio ergo irrationalis $y = \sqrt{((a + bz)(c + dz))}$ ad rationalitatem perducitur ope substitutionis

$$z = \frac{c - axx}{bxx - d},$$

quippe quae dabit

$$y = \frac{(bc - ad)x}{bxx - d}.$$

Sic, si fuerit

$$y = \sqrt{(aa - zz)} = \sqrt{((a + z)(a - z))},$$

ob $b = +1$, $c = a$, $d = -1$ ponatur

$$z = \frac{a - axx}{1 + xx}$$

eritque

$$y = \frac{2ax}{1 + xx}.$$

Quoties ergo quantitas post signum $\sqrt{}$ habuerit duos factores simplices reales, hoc modo reductio ad rationalitatem absolvetur; sin autem factores bini simplices fuerint imaginarii, sequenti modo uti praestabit.

51. *Sit*

$$y = \sqrt{(p + qz + rzz)}$$

atque requiritur substitutio idonea pro z facienda, ut valor ipsius y fiat rationalis.

Pluribus modis hoc fieri potest, prout p et q fuerint quantitates vel affirmativae vel negativae. Sit primo p quantitas affirmativa ac ponatur aa pro p; etiamsi enim p non sit quadratum, tamen irrationalitas quantitatum constantium praesens negotium non turbat. Sit igitur

I. $y = \sqrt{(aa + bz + czz)}$ ac ponatur

$$\sqrt{(aa + bz + czz)} = a + xz;$$

erit

$$b + cz = 2ax + xxz,$$

unde fit

$$z = \frac{b - 2ax}{xx - c};$$

tum vero erit

$$y = a + xz = \frac{bx - axx - ac}{xx - c},$$

ubi z et y sunt functiones rationales ipsius x. Sit iam

II. $y = \sqrt{(aazz + bz + c)}$ ac ponatur

$$\sqrt{(aazz + bz + c)} = az + x;$$

erit

$$bz + c = 2axz + xx$$

et

$$z = \frac{xx - c}{b - 2ax}.$$

Tum autem fit

$$y = az + x = \frac{-ac + bx - axx}{b - 2ax}.$$

III. Si fuerint p et r quantitates negativae, tum, nisi sit $qq > 4pr$, valor ipsius y semper erit imaginarius. Quodsi autem fuerit $qq > 4pr$, expressio $p + qz + rzz$ in duos factores resolvi poterit, qui casus ad paragraphum praecedentem reducitur. Saepenumero autem commodius ad hanc formam reducitur

$$y = \sqrt{(aa + (b + cz)(d + ez))};$$

pro qua ad rationalitatem perducenda ponatur

$$y = a + (b + cz)x$$

eritque

$$d + ez = 2ax + bxx + cxxz,$$

unde fit

$$z = \frac{d - 2ax - bxx}{cxx - e}$$

et

$$y = \frac{-ae + (cd - be)x - acxx}{cxx - e}.$$

Interdum commodius fieri potest reductio ad hanc formam

$$y = \sqrt{(aazz + (b + cz)(d + ez))}.$$

Tum ponatur

$$y = az + (b + cz)x;$$

erit

$$d + ez = 2axz + bxx + cxxz$$

et

$$z = \frac{bxx - d}{e - 2ax - cxx}$$

atque

$$y = \frac{-ad + (be - cd)x - abxx}{e - 2ax - cxx}.$$

EXEMPLUM

Si habeatur ista ipsius z functio irrationalis

$$y = \sqrt{(-1 + 3z - zz)},$$

quae, cum reduci queat ad hanc formam

$$y = \sqrt{(1 - 2 + 3z - zz)} = \sqrt{(1 - (1 - z)(2 - z))},$$

ponatur

$$y = 1 - (1 - z)x,$$

erit

$$-2 + z = -2x + xx - xxz$$

et

$$z = \frac{2 - 2x + xx}{1 + xx}.$$

Deinde est

$$1 - z = \frac{-1 + 2x}{1 + xx}$$

et

$$y = 1 - (1 - z)x = \frac{1 + x - xx}{1 + xx}.$$

Atque hi sunt fere casus, quos Algebra indeterminata seu methodus *Diophantea* suppeditat, neque alios casus in his tractatis non comprehensos per substitutionem rationalem ad rationalitatem reducere licet. Quocirca ad alterum substitutionis usum monstrandum progredior.

52. *Si y eiusmodi fuerit functio ipsius z, ut sit*

$$ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0,$$

invenire novam variabilem x, per quam valores ipsarum y et z explicite assignari queant.

Quoniam resolutio aequationum generalis non habetur, ex aequatione proposita $ay^{\alpha} + bz^{\beta} + cy^{\gamma}z^{\delta} = 0$ neque y per z neque vicissim z per y exhiberi potest. Quo igitur huic incommodo remedium afferatur, ponatur

$$y = x^m z^n$$

eritque

$$ax^{\alpha m} z^{\alpha n} + bz^\beta + cx^{\gamma m} z^{\gamma n + \delta} = 0.$$

Determinetur nunc exponens n ita, ut ex hac aequatione valor ipsius z definiri queat, quod tribus modis praestari potest.

I. Sit

$$\alpha n = \beta \quad \text{ideoque} \quad n = \frac{\beta}{\alpha};$$

erit aequatione per $z^{\alpha n} = z^\beta$ divisa

$$ax^{\alpha m} + b + cx^{\gamma m} z^{\gamma n - \beta + \delta} = 0,$$

unde oritur

$$z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{1}{\gamma n - \beta + \delta}} \quad \text{sive} \quad z = \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\alpha}{\beta\gamma - \alpha\beta + \alpha\delta}}$$

et

$$y = x^m \left(\frac{-ax^{\alpha m} - b}{cx^{\gamma m}}\right)^{\frac{\beta}{\beta\gamma - \alpha\beta + \alpha\delta}}.$$

II. Sit

$$\beta = \gamma n + \delta \quad \text{seu} \quad n = \frac{\beta - \delta}{\gamma};$$

erit aequatione per z^β divisa

$$ax^{\alpha m} z^{\alpha n - \beta} + b + cx^{\gamma m} = 0,$$

unde oritur

$$z = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{1}{\alpha n - \beta}} = \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\gamma}{\alpha\beta - \alpha\delta - \beta\gamma}}$$

atque

$$y = x^m \left(\frac{-b - cx^{\gamma m}}{ax^{\alpha m}}\right)^{\frac{\beta - \delta}{\alpha\beta - \alpha\delta - \beta\gamma}}.$$

III. Sit

$$\alpha n = \gamma n + \delta \quad \text{seu} \quad n = \frac{\delta}{\alpha - \gamma};$$

erit aequatione per $z^{\alpha n}$ divisa

$$ax^{\alpha m} + bz^{\beta - \alpha n} + cx^{\gamma m} = 0,$$

unde oritur

$$z = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{1}{\beta - \alpha n}} = \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{\alpha - \gamma}{\alpha\beta - \beta\gamma - \alpha\delta}}$$

atque

$$y = x^m \left(\frac{-ax^{\alpha m} - cx^{\gamma m}}{b}\right)^{\frac{\delta}{\alpha\beta - \beta\gamma - \alpha\delta}}.$$

Tribus igitur diversis modis erutae sunt functiones ipsius x, quae ipsis z et y sunt aequales. Praeterea vero pro m numerum pro lubitu substituere licet cyphra excepta sicque formulae ad commodissimam expressionem reduci poterunt.

EXEMPLUM

Exprimatur natura functionis y per hanc aequationem

$$y^3 + z^3 - cyz = 0$$

atque quaerantur functiones ipsius x ipsis y et z aequales.

Erit ergo

$$a = -1, \quad b = -1, \quad \alpha = 3, \quad \beta = 3, \quad \gamma = 1 \quad \text{et} \quad \delta = 1.$$

Hinc primus modus dabit posito $m = 1$

$$z = \left(\frac{x^3 + 1}{cx}\right)^{-1} \quad \text{et} \quad y = x\left(\frac{x^3 + 1}{cx}\right)^{-1}$$

sive

$$z = \frac{cx}{1 + x^3} \quad \text{et} \quad y = \frac{cxx}{1 + x^3},$$

quarum expressionum utraque adeo est rationalis.

Secundus modus vero dabit hos valores

$$z = \left(\frac{cx - 1}{x^3}\right)^{1:3} \quad \text{et} \quad y = x\left(\frac{cx - 1}{x^3}\right)^{2:3}$$

sive

$$z = \frac{1}{x}\sqrt[3]{(cx - 1)} \quad \text{et} \quad y = \frac{1}{x}\sqrt[3]{(cx - 1)^2}.$$

Tertius modus ita rem expediet, ut sit

$$z = (cx - x^3)^{2:3} \quad \text{et} \quad y = x(cx - x^3)^{1:3}.$$

53. *Hinc a posteriori intelligitur, cuiusmodi aequationes, quibus valor functionis y per z determinatur, hoc modo novam variabilem x introducendo resolvi queant.*

Ponamus enim resolutione iam instituta prodiisse has determinationes

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{p:r}$$

atque

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{q:r}$$

eritque

$$y^p = x^p z^q$$

et hinc

$$x = yz^{-q:p}.$$

Cum igitur sit

$$z^{r:p} = \frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}},$$

si loco x eius valorem $yz^{-q:p}$ substituamus, prodibit ista aequatio

$$z^{r:p} = \frac{ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \text{etc.}}{A + By^\mu z^{-\mu q:p} + Cy^\nu z^{-\nu q:p} + \text{etc.}},$$

quae reducitur ad hanc

$$Az^{r:p} + By^\mu z^{(r-\mu q):p} + Cy^\nu z^{(r-\nu q):p} + \text{etc.} = ay^\alpha z^{-\alpha q:p} + by^\beta z^{-\beta q:p} + cy^\gamma z^{-\gamma q:p} + \text{etc.},$$

quae multiplicata per $z^{\alpha q:p}$ transibit in hanc

$$Az^{(\alpha q+r):p} + By^\mu z^{(\alpha q-\mu q+r):p} + Cy^\nu z^{(\alpha q-\nu q+r):p} + \text{etc.}$$
$$= ay^\alpha + by^\beta z^{(\alpha q-\beta q):p} + cy^\gamma z^{(\alpha q-\gamma q):p} + \text{etc.}$$

Ponatur

$$\frac{\alpha q + r}{p} = m \quad \text{et} \quad \frac{\alpha q - \beta q}{p} = n;$$

fiet

$$p = \alpha - \beta, \quad q = n \quad \text{et} \quad r = \alpha m - \beta m - \alpha n$$

atque nascetur ista aequatio

$$Az^m + By^\mu z^{m-\mu n:(\alpha-\beta)} + Cy^\nu z^{m-\nu n:(\alpha-\beta)} + \text{etc.}$$
$$= ay^\alpha + by^\beta z^n + cy^\gamma z^{(\alpha-\gamma)n:(\alpha-\beta)} + \text{etc.},$$

quae propterea ita resolvetur, ut sit

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{\frac{\alpha-\beta}{\alpha m - \beta m - \alpha n}}$$

et

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{\frac{n}{\alpha m - \beta m - \alpha n}}.$$

Vel ponatur

$$\frac{\alpha q + r}{p} = m \quad \text{et} \quad \frac{\alpha q - \mu q + r}{p} = n;$$

erit

$$m - n = \frac{\mu q}{p} \quad \text{et} \quad \frac{q}{p} = \frac{m-n}{\mu} \quad \text{atque} \quad \frac{r}{p} = m - \frac{\alpha m - \alpha n}{\mu}.$$

Hinc fit

$$p = \mu, \quad q = m - n \quad \text{et} \quad r = \mu m - \alpha m + \alpha n$$

atque haec aequatio resultabit

$$Az^m + By^\mu z^n + Cy^\nu z^{m-\nu(m-n):\mu} + \text{etc.}$$
$$= ay^\alpha + by^\beta z^{(\alpha-\beta)(m-n):\mu} + cy^\gamma z^{(\alpha-\gamma)(m-n):\mu} + \text{etc.},$$

quae ita resolvetur, ut sit

$$z = \left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{\frac{\mu}{\mu m - \alpha m + \alpha n}}$$

et

$$y = x\left(\frac{ax^\alpha + bx^\beta + cx^\gamma + \text{etc.}}{A + Bx^\mu + Cx^\nu + \text{etc.}}\right)^{\frac{m-n}{\mu m - \alpha m + \alpha n}}.$$

54. *Si y ita pendeat a z, ut sit*

$$ayy + byz + czz + dy + ez = 0,$$

sequenti modo tam y quam z rationaliter per novam variabilem x exprimetur.

Ponatur $y = xz$; erit divisione per z facta

$$axxz + bxz + cz + dx + e = 0,$$

ex qua reperitur

$$z = \frac{-dx - e}{axx + bx + c}$$

et

$$y = \frac{-dxx - ex}{axx + bx + c}.$$

At vero ad formam propositam reduci potest haec aequatio inter y et z

$$ayy + byz + czz + dy + ez + f = 0$$

diminuendo vel augendo utramque variabilem certa quadam quantitate constante, unde et haec aequatio per novam variabilem x rationaliter explicari potest.

55. *Si y ita pendeat a z, ut sit*

$$ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0,$$

sequenti modo tam y quam z rationaliter per novam variabilem x exprimi poterit.

Ponatur $y = xz$ et facta substitutione tota aequatio per zz dividi poterit; prodibit autem

$$ax^3z + bxxz + cxz + dz + exx + fx + g = 0.$$

Unde oritur

$$z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d},$$

ex quo erit

$$y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}.$$

Ex his casibus facile intelligitur, quemadmodum aequationes altiorum graduum, quibus y per z definitur, comparatae esse debeant, ut huiusmodi resolutio locum habere queat. Ceterum hi casus in superioribus formulis § 53 continentur, at, quia formulae generales non tam facile ad huiusmodi casus saepius occurrentes accommodantur, visum est horum aliquos seorsim evolvere.

56. *Si y ita pendeat a z, ut sit*

$$ayy + byz + czz = d,$$

hoc modo utraque quantitas y et z per novam variabilem x exprimetur.

Ponatur $y = xz$ eritque

$$(axx + bx + c)zz = d$$

ideoque

$$z = \sqrt{\frac{d}{axx + bx + c}}$$

et

$$y = x\sqrt{\frac{d}{axx + bx + c}}.$$

Simili modo si fuerit

$$ay^3 + by^2z + cyz^2 + dz^3 = ey + fz,$$

posito $y = xz$ tota aequatio per z divisa dabit

$$(ax^3 + bxx + cx + d)zz = ex + f,$$

unde oritur

$$z = \sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}$$

et

$$y = x\sqrt{\frac{ex + f}{ax^3 + bxx + cx + d}}.$$

Hi autem casus aliique similes resolutiones admittentes comprehenduntur in sequente paragrapho.

57. *Si y ita pendeat a z, ut sit*

$$ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \text{etc.} = \alpha y^n + \beta y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \text{etc.},$$

sequenti modo tam z quam y commode per novam variabilem x exprimetur.

Sit $y = xz$ atque facta substitutione tota aequatio dividi poterit per z^n, siquidem exponens m sit maior quam n, eritque

$$(ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \text{etc.})z^{m-n} = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.},$$

unde obtinebitur

$$z = \left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}}{a x^m + b x^{m-1} + c x^{m-2} + d x^{m-3} + \text{etc.}}\right)^{1:(m-n)}$$

et

$$y = x\left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}}{a x^m + b x^{m-1} + c x^{m-2} + d x^{m-3} + \text{etc.}}\right)^{1:(m-n)}.$$

Haec scilicet resolutio locum habet, si in aequatione naturam functionis y per z exprimente duplex tantum ubique occurrit dimensionum ab y et z sumptarum numerus, uti in casu tractato in singulis terminis numerus dimensionum vel est m vel n.

58. *Si in aequatione inter y et z triplicis generis dimensiones occurrant, quarum summa tantum superet mediam, quantum haec media infimam, ope resolutionis aequationis quadratae variabiles y et z per novam x exprimi poterunt.*

Si enim ponatur $y = xz$, divisione per minimam ipsius z potestatem facta valor ipsius z per x ope extractionis radicis quadratae exhibebitur, id quod ex sequentibus exemplis erit manifestum.

EXEMPLUM 1

Sit

$$ay^3 + byyz + cyzz + dz^3 = 2eyy + 2fyz + 2gzz + hy + iz$$

ac ponatur $y = xz$; erit divisione per z facta

$$(ax^3 + bxx + cx + d)zz = 2(exx + fx + g)z + hx + i,$$

ex qua sequens ipsius z obtinebitur valor

$$z = \frac{exx + fx + g \pm \sqrt{((exx + fx + g)^2 + (ax^3 + bxx + cx + d)(hx + i))}}{ax^3 + bxx + cx + d},$$

quo invento erit $y = xz$.

EXEMPLUM 2

Sit

$$y^5 = 2az^3 + by + cz$$

ac posito $y = xz$ erit

$$x^5 z^4 = 2azz + bx + c,$$

ex qua reperitur

$$zz = \frac{a \pm \sqrt{(aa + bx^6 + cx^5)}}{x^5}$$

et

$$z = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{xx\sqrt{x}}$$

et

$$y = \frac{\sqrt{(a \pm \sqrt{(aa + bx^6 + cx^5)})}}{x\sqrt{x}}.$$

EXEMPLUM 3

Sit

$$y^{10} = 2ayz^6 + byz^3 + cz^4;$$

in qua cum dimensiones sint 10, 7 et 4, ponatur $y = xz$ atque aequatio per z^4 divisa abibit in hanc

$$x^{10} z^6 = 2axz^3 + bx + c$$

seu

$$z^6 = \frac{2axz^3 + bx + c}{x^{10}},$$

unde invenitur

$$z^3 = \frac{ax \pm x\sqrt{(aa + bx^9 + cx^8)}}{x^{10}},$$

ideoque erit

$$z = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$$

atque

$$y = \frac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^2}.$$

Ex quibus exemplis usus huiusmodi substitutionum abunde perspicitur.

CAPUT IV

DE EXPLICATIONE FUNCTIONUM PER SERIES INFINITAS

59. Cum functiones fractae atque irrationales ipsius z non in forma integra $A + Bz + Cz^2 + Dz^3 +$ etc. contineantur, ita ut terminorum numerus sit finitus, quaeri solent huiusmodi expressiones in infinitum excurrentes, quae valorem cuiusvis functionis sive fractae sive irrationalis exhibeant; quin etiam natura functionum transcendentium melius intelligi censetur, si per eiusmodi formam etsi infinitam exprimantur. Cum enim natura functionis integrae optime perspiciatur, si secundum diversas potestates ipsius z explicetur atque adeo ad formam $A + Bz + Cz^2 + Dz^3 +$ etc. reducatur, ita eadem forma aptissima videtur ad reliquarum functionum omnium indolem menti repraesentandam, etiamsi terminorum numerus sit revera infinitus. Perspicuum autem est nullam functionem non integram ipsius z per numerum huiusmodi terminorum $A + Bz + Cz^2 + Dz^3 +$ etc. finitum exponi posse; eo ipso enim functio foret integra. Num vero per huiusmodi terminorum seriem infinitam exhiberi possit, si quis dubitet, hoc dubium per ipsam evolutionem cuiusque functionis tolletur. Quo autem haec explicatio latius pateat, praeter potestates ipsius z exponentes integros affirmativos habentes admitti debent potestates quaecunque. Sic dubium erit nullum, quin omnis functio ipsius z in huiusmodi expressionem infinitam transmutari possit

$$Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} + \text{etc.}$$

denotantibus exponentibus α, β, γ, δ etc. numeros quoscunque.

60. *Per divisionem autem continuam intelligitur fractionem*

$$\frac{a}{\alpha+\beta z}$$

resolvi in hanc seriem infinitam

$$\frac{a}{\alpha}-\frac{a\beta z}{\alpha^2}+\frac{a\beta^2 z^2}{\alpha^3}-\frac{a\beta^3 z^3}{\alpha^4}+\frac{a\beta^4 z^4}{\alpha^5}-\text{etc.},$$

quae, cum quilibet terminus ad sequentem habeat rationem constantem $1:-\frac{\beta z}{\alpha}$, *vocatur series geometrica.*

Potest vero quoque haec series ita inveniri, ut ipsa initio pro incognita habeatur; ponatur enim

$$\frac{a}{\alpha+\beta z}=A+Bz+Cz^2+Dz^3+Ez^4+\text{etc.}$$

atque ad aequalitatem producendam quaerantur coefficientes A, B, C, D etc. Erit ergo

$$a=(\alpha+\beta z)(A+Bz+Cz^2+Dz^3+\text{etc.})$$

et multiplicatione actu peracta fiet

$$\begin{aligned} a=\alpha A+\alpha Bz+\alpha Cz^2+\alpha Dz^3+\alpha Ez^4+\text{etc.}\\ +\beta Az+\beta Bz^2+\beta Cz^3+\beta Dz^4+\text{etc.}\end{aligned}$$

Quamobrem esse debet

$$a=\alpha A \quad \text{ideoque} \quad A=\frac{a}{\alpha}$$

et coefficientium uniuscuiusque potestatis ipsius z summa nihilo aequalis est ponenda, unde prodibunt hae aequationes

$$\begin{aligned} \alpha B+\beta A&=0,\\ \alpha C+\beta B&=0,\\ \alpha D+\beta C&=0,\\ \alpha E+\beta D&=0\\ &\text{etc.;}\end{aligned}$$

cognito ergo quovis coefficiente facile reperitur sequens; si enim fuerit coefficiens termini cuiusque $= P$ et sequens $= Q$, erit

$$\alpha Q + \beta P = 0 \quad \text{sive} \quad Q = -\frac{\beta P}{\alpha}.$$

Cum igitur terminus primus A sit determinatus $= \frac{a}{\alpha}$, ex eo sequentes litterae B, C, D etc. definiuntur eodem modo, quo ex divisione sunt orti. Ceterum ex inspectione perspicitur in serie infinita pro $\frac{a}{\alpha + \beta z}$ inventa potestatis z^n coefficientem fore $= \pm \frac{a\beta^n}{\alpha^{n+1}}$, ubi signum $+$ locum habet, si n sit numerus par, signum $-$ autem, si n sit numerus impar, seu coefficiens erit $= \frac{a}{\alpha}\left(\frac{-\beta}{\alpha}\right)^n$.

61. *Simili modo ope divisionis continuatae haec functio fracta*

$$\frac{a + bz}{\alpha + \beta z + \gamma zz}$$

in seriem infinitam converti potest.

Cum autem divisio sit taediosa neque tam facile naturam seriei infinitae ostendat, commodius erit seriem quaesitam fingere atque modo ante tradito determinare. Sit igitur

$$\frac{a + bz}{\alpha + \beta z + \gamma zz} = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.};$$

multiplicetur utrinque per $\alpha + \beta z + \gamma zz$ atque fiet

$$\begin{aligned} a + bz = \alpha A + \alpha Bz &+ \alpha Cz^2 + \alpha Dz^3 + \alpha Ez^4 + \text{etc.} \\ + \beta Az &+ \beta Bz^2 + \beta Cz^3 + \beta Dz^4 + \text{etc.} \\ &+ \gamma Az^2 + \gamma Bz^3 + \gamma Cz^4 + \text{etc.} \end{aligned}$$

Hinc erit

$$\alpha A = a, \quad \alpha B + \beta A = b,$$

unde reperitur

$$A = \frac{a}{\alpha} \quad \text{et} \quad B = \frac{b}{\alpha} - \frac{a\beta}{\alpha\alpha};$$

reliquae vero litterae ex sequentibus aequationibus determinabuntur

$$\alpha C + \beta B + \gamma A = 0,$$
$$\alpha D + \beta C + \gamma B = 0,$$
$$\alpha E + \beta D + \gamma C = 0,$$
$$\alpha F + \beta E + \gamma D = 0$$
etc.;

hinc ergo ex binis quibusque coefficientibus contiguis sequens reperitur. Sic, si duo coefficientes contigui fuerint P, Q et sequens R, erit

$$\alpha R + \beta Q + \gamma P = 0 \quad \text{seu} \quad R = \frac{-\beta Q - \gamma P}{\alpha}.$$

Cum igitur duae litterae primae A et B iam sint inventae, sequentes C, D, E, F etc. omnes successive ex iis invenientur sicque reperietur series infinita $A + Bz + Cz^2 + Dz^3 +$ etc. functioni fractae propositae $\frac{a+bz}{\alpha+\beta z+\gamma z^2}$ aequalis.

EXEMPLUM

Si fuerit proposita haec fractio

$$\frac{1+2z}{1-z-zz}$$

huicque aequalis statuatur series

$$A + Bz + Cz^2 + Dz^3 + \text{etc.},$$

ob

$$a = 1, \quad b = 2, \quad \alpha = 1, \quad \beta = -1, \quad \gamma = -1$$

erit

$$A = 1, \quad B = 3;$$

tum vero erit

$$C = B + A,$$
$$D = C + B,$$
$$E = D + C,$$
$$F = E + D$$
etc.

Quilibet ergo coefficiens aequalis est summae duorum praecedentium; quare si cogniti fuerint duo coefficientes contigui P et Q, erit sequens

$$R = P + Q.$$

Cum igitur duo coefficientes primi A et B sint cogniti, fractio proposita

$$\frac{1+2z}{1-z-zz}$$

in hanc seriem infinitam transmutatur

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \text{etc.},$$

quae nullo negotio, quousque libuerit, continuari potest.

62. Ex his iam satis intelligitur indoles serierum infinitarum, in quas functiones fractae transmutantur; tenent enim eiusmodi legem, ut quilibet terminus ex aliquot praecedentibus determinari possit.

Scilicet, si denominator fractionis propositae fuerit

$$\alpha + \beta z$$

atque series infinita statuatur

$$A + Bz + Cz^2 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \text{etc.},$$

quilibet coefficiens Q ex praecedente P solo ita definietur, ut sit

$$\alpha Q + \beta P = 0.$$

Sin denominator fuerit trinomium

$$\alpha + \beta z + \gamma zz,$$

quilibet coefficiens seriei R ex duobus praecedentibus Q et P ita definietur, ut sit

$$\alpha R + \beta Q + \gamma P = 0.$$

Simili modo, si denominator fuerit quadrinomium, ut

$$\alpha + \beta z + \gamma zz + \delta z^3,$$

quilibet coefficiens seriei S ex tribus antecedentibus R, Q et P ita determinabitur, ut sit

$$\alpha S + \beta R + \gamma Q + \delta P = 0,$$

sicque de ceteris.

In his ergo seriebus quilibet terminus determinatur ex aliquot antecedentibus secundum legem quandam constantem, quae lex ex denominatore fractionis hanc seriem producentis sponte apparet. Vocari autem hae series a Celeberrimo MOIVREO, qui earum naturam maxime est scrutatus, solent *recurrentes*, propterea quod ad terminos antecedentes est recurrendum, si sequentes investigare velimus.[1])

63. Ad harum porro serierum formationem requiritur, ut denominatoris terminus constans α non sit $= 0$; cum enim inventus sit terminus seriei primus $A = \frac{a}{\alpha}$, tum is tum omnes sequentes fierent infiniti, si esset $\alpha = 0$. Hoc ergo casu excluso, quem deinceps evolvam, functio fracta in seriem infinitam recurrentem transmutanda huiusmodi habebit formam

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}},$$

ubi primum denominatoris terminum pono $= 1$; huc enim semper fractio reduci potest, nisi is sit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor, ut seriei hinc formatae omnes termini fiant affirmativi. Quodsi enim series recurrens hinc orta ponatur

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.},$$

coefficientes ita determinabuntur, ut sit

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1) A. DE MOIVRE (1667—1754), *De fractionibus algebraicis radicalitate immunibus ad fractiones simpliciores reducendis, deque summandis terminis quarumdam serierum aequali intervallo a se distantibus,* Philosophical transactions (London) 32 (1722/3), 1724, numb. 373, p. 162, imprimis p. 176. Vide etiam eiusdem auctoris *Miscellanea analytica de seriebus et quadraturis,* Londini 1730, p. 27, nec non *The doctrine of chances,* London 1718, p. 127—134.

Series recurrentes ab EULERO ipso fusius pertractatae sunt in cap. XIII et XVII huius *Introductionis.* A. K.

$$A = a,$$
$$B = \alpha A + b,$$
$$C = \alpha B + \beta A + c,$$
$$D = \alpha C + \beta B + \gamma A + d,$$
$$E = \alpha D + \beta C + \gamma B + \delta A + e$$
etc.

Quilibet ergo coefficiens aequalis est aggregato ex multiplis aliquot praecedentium una cum numero quodam, quem numerator praebet. Nisi autem numerator in infinitum progrediatur, haec additio mox cessabit atque quivis terminus secundum legem constantem ex aliquot praecedentibus determinabitur. Ne ergo lex progressionis usquam turbetur, conveniet functionem fractam genuinam adhibere; si enim fractio spuria accipiatur, tum pars integra in ea contenta ad seriem accedet atque in illis terminis, quos vel auget vel minuit, legem progressionis interrumpet. Exempli gratia haec fractio spuria

$$\frac{1 + 2z - z^3}{1 - z - zz}$$

praebebit hanc seriem

$$1 + 3z + 4zz + 6z^3 + 10z^4 + 16z^5 + 26z^6 + 42z^7 + \text{etc.},$$

ubi a lege, qua quivis coefficiens est summa duorum praecedentium, terminus quartus $6z^3$ excipitur.

64. Peculiarem contemplationem series recurrentes merentur, si denominator fractionis, unde oriuntur, fuerit potestas. Sic, si ista fractio

$$\frac{a + bz}{(1 - \alpha z)^2}$$

in seriem resolvatur, prodit

$$\begin{array}{l} a + 2\alpha a z + 3\alpha^2 a z^2 + 4\alpha^3 a z^3 + 5\alpha^4 a z^4 + \text{etc.}, \\ \quad + \;\; b \;\; + \;\; 2\alpha b \;\; + 3\alpha^2 b \;\; + 4\alpha^3 b \end{array}$$

in qua coefficiens potestatis z^n erit

$$(n + 1)\,\alpha^n a + n\alpha^{n-1} b.$$

Erit tamen haec series recurrens, quia quilibet terminus ex duobus praecedentibus determinatur, cuius determinationis lex perspicitur ex denominatore evoluto $1 - 2\alpha z + \alpha\alpha zz$.

Si ponatur $\alpha = 1$ et $z = 1$, abit series in progressionem arithmeticam generalem

$$a + (2a + b) + (3a + 2b) + (4a + 3b) + \text{etc.},$$

cuius differentiae sunt constantes. Omnis ergo progressio arithmetica est series recurrens; si enim sit

$$A + B + C + D + E + F + \text{etc.}$$

progressio arithmetica, erit

$$C = 2B - A, \quad D = 2C - B, \quad E = 2D - C \quad \text{etc.}$$

65. Deinde haec fractio

$$\frac{a + bz + czz}{(1 - \alpha z)^3},$$

ob

$$\frac{1}{(1 - \alpha z)^3} = (1 - \alpha z)^{-3} = 1 + 3\alpha z + 6\alpha^2 z^2 + 10\alpha^3 z^3 + 15\alpha^4 z^4 + \text{etc.}$$

transmutabitur in hanc seriem infinitam

$$\begin{aligned} a + 3\alpha az &+ 6\alpha^2 az^2 + 10\alpha^3 az^3 + 15\alpha^4 az^4 + \text{etc.}, \\ + \quad b &+ 3\alpha b \quad + 6\alpha^2 b \quad + 10\alpha^3 b \\ &+ \quad c \quad + 3\alpha c \quad + 6\alpha^2 c \end{aligned}$$

in qua potestas z^n coefficientem habebit

$$\frac{(n+1)(n+2)}{1 \cdot 2} \alpha^n a + \frac{n(n+1)}{1 \cdot 2} \alpha^{n-1} b + \frac{(n-1)n}{1 \cdot 2} \alpha^{n-2} c.$$

Quodsi autem ponatur $\alpha = 1$ et $z = 1$, series haec abibit in progressionem generalem secundi ordinis, cuius differentiae secundae sunt constantes. Designet

$$A + B + C + D + E + F + \text{etc.}$$

huiusmodi progressionem; erit ea simul series recurrens, cuius quilibet terminus ex tribus antecedentibus ita determinatur, ut sit

$$D = 3C - 3B + A, \quad E = 3D - 3C + B, \quad F = 3E - 3D + C \quad \text{etc.}$$

Cum igitur terminorum in progressione arithmetica procedentium secundae differentiae quoque sint aequales, nempe $=0$, haec proprietas quoque ad progressiones arithmeticas extenditur.

66. Simili modo haec fractio

$$\frac{a + bz + czz + dz^3}{(1-\alpha z)^4}$$

dabit seriem infinitam, in qua potestas ipsius z quaecunque z^n hunc habebit coefficientem

$$\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\alpha^n a + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\alpha^{n-1}b + \frac{(n-1)n(n+1)}{1\cdot 2\cdot 3}\alpha^{n-2}c$$
$$+ \frac{(n-2)(n-1)n}{1\cdot 2\cdot 3}\alpha^{n-3}d.$$

Posito ergo $\alpha = 1$ et $z = 1$ haec series in se complectetur omnes progressiones algebraicas tertii ordinis, quarum differentiae tertiae sunt constantes; omnes ergo huius ordinis progressiones, cuiusmodi sit

$$A + B + C + D + E + F + \text{etc.},$$

erunt simul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortae, unde erit

$$E = 4D - 6C + 4B - A, \quad F = 4E - 6D + 4C - B \quad \text{etc.},$$

quae proprietas simul in omnes progressiones inferiorum ordinum competit.

67. Hoc modo ostendetur omnes progressiones algebraicas cuiuscunque ordinis, quae tandem ad differentias constantes deducunt, esse series recurrentes, quarum lex definiatur ex denominatore $(1-z)^n$ existente n numero maiore quam is, qui ordinem progressionis indicat. Cum igitur

$$a^m + (a+b)^m + (a+2b)^m + (a+3b)^m + \text{etc.}$$

exhibeat progressionem ordinis m, erit ob naturam serierum recurrentium

$$0 = a^m - \frac{n}{1}(a+b)^m + \frac{n(n-1)}{1\cdot 2}(a+2b)^m - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(a+3b)^m$$
$$+ \cdots \mp \frac{n}{1}(a+(n-1)b)^m \pm (a+nb)^m,$$

ubi signa superiora valent, si n sit numerus par, inferiora autem, si n sit numerus impar. Haec ergo aequatio semper est vera, si fuerit n numerus integer maior quam m. Hinc ergo intelligitur, quam late pateat doctrina de seriebus recurrentibus.

68. Si denominator fuerit potestas non binomii sed multinomii, natura seriei quoque alio modo explicari potest. Sit nempe haec fractio

$$\frac{1}{(1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.})^{m+1}}$$

proposita; erit series infinita hinc nata

$$\begin{aligned} 1 + \frac{m+1}{1}\alpha z &+ \frac{(m+1)(m+2)}{1\cdot 2}\alpha^2 z^2 + \frac{(m+1)(m+2)(m+3)}{1\cdot 2\cdot 3}\alpha^3 z^3 + \text{etc.} \\ &+ \frac{m+1}{1}\beta \quad + \quad \frac{(m+1)(m+2)}{1\cdot 2} 2\alpha\beta \\ & \qquad\qquad\quad + \quad \frac{m+1}{1}\gamma \end{aligned}$$

Ad naturam huius seriei penitius inspiciendam exponatur haec series per litteras generales hoc modo

$$1 + Az + Bz^2 + Cz^3 + \cdots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n + \text{etc.}$$

ac quilibet coefficiens N ex tot praecedentibus, quot sunt litterae $\alpha, \beta, \gamma, \delta$ etc., ita determinabitur, ut sit

$$N = \frac{m+n}{n}\alpha M + \frac{2m+n}{n}\beta L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I + \text{etc.};$$

quae lex continuationis etsi non est constans, sed ab exponente potestatis z pendet, tamen eidem seriei alia convenit lex progressionis constans, quam denominator evolutus praebet naturae serierum recurrentium consentaneam.

Illa vero lex non constans tantum locum habet, si numerator fractionis fuerit unitas seu quantitas constans; si enim quoque aliquot potestates ipsius z contineret, tum illa lex multo magis fieret complicata, id quod post tradita calculi differentialis principia facilius patebit.

69. Quoniam hactenus posuimus primum denominatoris terminum constantem non esse $= 0$ eiusque loco unitatem collocavimus, nunc videamus, cuiusmodi series oriantur, si in denominatore terminus constans evanescat. His casibus ergo functio fracta huiusmodi formam habebit

$$\frac{a + bz + czz + \text{etc.}}{z(1 - \alpha z - \beta zz - \gamma z^3 - \text{etc.})};$$

convertatur ergo neglecto denominatoris factore z reliqua fractio

$$\frac{a + bz + czz + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.}}$$

in seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

atque manifestum est fore

$$\frac{a + bz + czz + \text{etc.}}{z(1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.})} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \text{etc.}$$

Simili modo erit

$$\frac{a + bz + cz^2 + \text{etc.}}{z^2(1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.})} = \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \text{etc.}$$

atque generatim erit

$$\frac{a + bz + cz^2 + \text{etc.}}{z^m(1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.})} = \frac{A}{z^m} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + \frac{D}{z^{m-3}} + \text{etc.},$$

quicunque numerus fuerit exponens m.

70. Quoniam per substitutionem loco z alia variabilis x in functionem fractam introduci hocque pacto functio fracta quaevis in innumerabiles formas diversas transmutari potest, hoc modo eadem functio fracta infinitis modis

per series recurrentes explicari poterit. Sit scilicet proposita haec fractio

$$y = \frac{1+z}{1-z-zz}$$

et per seriem recurrentem

$$y = 1 + 2z + 3z^2 + 5z^3 + 8z^4 + \text{etc.};$$

ponatur $z = \frac{1}{x}$; erit

$$y = \frac{xx+x}{xx-x-1} = \frac{-x(1+x)}{1+x-xx}.$$

Iam

$$\frac{1+x}{1+x-xx} = 1 + 0x + xx - x^3 + 2x^4 - 3x^5 + 5x^6 - \text{etc.},$$

unde erit

$$y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 + \text{etc.}$$

Vel ponatur

$$z = \frac{1-x}{1+x};$$

erit

$$y = \frac{-2-2x}{1-4x-xx},$$

unde fit

$$y = -2 - 10x - 42xx - 178x^3 - 754x^4 - \text{etc.},$$

cuiusmodi series recurrentes pro y innumerabiles inveniri possunt.

71. Functiones irrationales ex hoc theoremate universali[1]) in series infinitas transformari solent, quod sit

$$(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n \cdot 2n} P^{\frac{m-2n}{n}} Q^2 + \frac{m(m-n)(m-2n)}{n \cdot 2n \cdot 3n} P^{\frac{m-3n}{n}} Q^3 + \text{etc.};$$

1) Vide L. Euleri *Institutiones calculi differentialis,* Petropoli 1755, partis posterioris cap. IV; *Leonhardi Euleri Opera omnia*, series I, vol. 10, p. 276. Vide porro L. Euleri Commentationem 465 (indicis Enestroemiani): *Demonstratio theorematis Neutoniani de evolutione potestatum binomii pro casibus, quibus exponentes non sunt numeri integri*, Novi comment. acad. sc. Petrop. 19 (1774), 1775, p. 103; *Leonhardi Euleri Opera omnia,* series I, vol. 15. A. K.

hi enim termini, nisi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infinitum excurrunt. Sic erit pro m et n numeros definitos scribendo

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2}P^{-\frac{1}{2}}Q - \frac{1\cdot 1}{2\cdot 4}P^{-\frac{3}{2}}Q^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}P^{-\frac{5}{2}}Q^3 - \text{etc.},$$

$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2}P^{-\frac{3}{2}}Q + \frac{1\cdot 3}{2\cdot 4}P^{-\frac{5}{2}}Q^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}P^{-\frac{7}{2}}Q^3 + \text{etc.},$$

$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3}P^{-\frac{2}{3}}Q - \frac{1\cdot 2}{3\cdot 6}P^{-\frac{5}{3}}Q^2 + \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}P^{-\frac{8}{3}}Q^3 - \text{etc.},$$

$$(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3}P^{-\frac{4}{3}}Q + \frac{1\cdot 4}{3\cdot 6}P^{-\frac{7}{3}}Q^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}P^{-\frac{10}{3}}Q^3 + \text{etc.},$$

$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3}P^{-\frac{1}{3}}Q - \frac{2\cdot 1}{3\cdot 6}P^{-\frac{4}{3}}Q^2 + \frac{2\cdot 1\cdot 4}{3\cdot 6\cdot 9}P^{-\frac{7}{3}}Q^3 - \text{etc.}$$

etc.

72. Huiusmodi ergo serierum termini ita progrediuntur, ut quilibet ex antecedente formari possit. Sit enim seriei, quae ex $(P+Q)^{\frac{m}{n}}$ nascitur, terminus quilibet

$$= MP^{\frac{m-kn}{n}}Q^k;$$

erit sequens

$$= \frac{m-kn}{(k+1)n}MP^{\frac{m-(k+1)n}{n}}Q^{k+1}.$$

Notandum autem est in quovis termino sequente exponentem ipsius P unitate decrescere, contra vero exponentem ipsius Q unitate crescere. Quo autem haec facilius ad quemvis casum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi potest $P^{\frac{m}{n}}\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$; evoluta enim formula $\left(1+\frac{Q}{P}\right)^{\frac{m}{n}}$ serieque resultante per $P^{\frac{m}{n}}$ multiplicata prodibit ipsa series ante data. Tum vero, si m non solum numeros integros denotet, sed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis si pro $\frac{Q}{P}$, quae est functio ipsius z, ponatur Z, habebitur

$$(1+Z)^m = 1 + \frac{m}{1}Z + \frac{m(m-1)}{1\cdot 2}Z^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}Z^3 + \text{etc.}$$

Ad sequentes progressionum leges autem observandas conveniet hanc formulae generalis in seriem conversionem notasse

$$(1+Z)^{m-1} = 1 + \frac{m-1}{1}Z + \frac{(m-1)(m-2)}{1\cdot 2}Z^2 + \frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}Z^3 + \text{etc.}$$

73. Sit igitur primum

$$Z = \alpha z$$

eritque

$$(1+\alpha z)^{m-1} = 1 + \frac{m-1}{1}\alpha z + \frac{(m-1)(m-2)}{1\cdot 2}\alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3 z^3 + \text{etc.}$$

Scribatur pro hac serie ista forma generalis

$$1 + Az + Bz^2 + Cz^3 + \cdots + Mz^{n-1} + Nz^n + \text{etc.}$$

atque quilibet coefficiens N ex praecedente M ita determinabitur, ut sit

$$N = \frac{m-n}{n}\alpha M.$$

Sic posito $n=1$, cum sit $M=1$, erit

$$N = A = \frac{m-1}{1}\alpha;$$

tum facto $n=2$ ob $M = A = \frac{m-1}{1}\alpha$ erit

$$N = B = \frac{m-2}{2}\alpha M = \frac{(m-1)(m-2)}{1\cdot 2}\alpha^2$$

similique modo porro

$$C = \frac{m-3}{3}\alpha B = \frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3,$$

ut series ante inventa declarat.

74. Sit

$$Z = \alpha z + \beta zz$$

eritque

$$(1+\alpha z+\beta zz)^{m-1} = 1 + \frac{m-1}{1}(\alpha z+\beta zz) + \frac{(m-1)(m-2)}{1\cdot 2}(\alpha z+\beta zz)^2 + \text{etc.}$$

Quodsi ergo termini secundum potestates ipsius z disponantur, erit

$$\begin{aligned}&(1+\alpha z+\beta zz)^{m-1}\\ &=1+\frac{m-1}{1}\alpha z+\frac{(m-1)(m-2)}{1\cdot 2}\alpha^2z^2+\frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3z^3+\text{etc.}\\ &\qquad\qquad+\frac{m-1}{1}\beta z^2\qquad+\frac{(m-1)(m-2)}{1\cdot 2}2\alpha\beta z^3\end{aligned}$$

Scribatur pro hac serie ista forma generalis

$$1+Az+Bz^2+Cz^3+\cdots+Lz^{n-2}+Mz^{n-1}+Nz^n+\text{etc.}$$

atque quilibet coefficiens ex duobus antecedentibus ita definietur, ut sit

$$N=\frac{m-n}{n}\alpha M+\frac{2m-n}{n}\beta L,$$

unde omnes termini ex primo, qui est 1, definiri poterunt. Erit nempe

$$\begin{aligned}A&=\frac{m-1}{1}\alpha,\\ B&=\frac{m-2}{2}\alpha A+\frac{2m-2}{2}\beta,\\ C&=\frac{m-3}{3}\alpha B+\frac{2m-3}{3}\beta A,\\ D&=\frac{m-4}{4}\alpha C+\frac{2m-4}{4}\beta B\\ &\text{etc.}\end{aligned}$$

75. Si fuerit

$$Z=\alpha z+\beta z^2+\gamma z^3,$$

erit

$$\begin{aligned}&(1+\alpha z+\beta z^2+\gamma z^3)^{m-1}\\ &=1+\frac{m-1}{1}(\alpha z+\beta z^2+\gamma z^3)+\frac{(m-1)(m-2)}{1\cdot 2}(\alpha z+\beta z^2+\gamma z^3)^2+\text{etc.},\end{aligned}$$

quae expressio, si omnes termini secundum potestates ipsius z ordinentur, abibit in hanc seriem

$$\begin{aligned}1&+\frac{m-1}{1}\alpha z+\frac{(m-1)(m-2)}{1\cdot 2}\alpha^2z^2+\frac{(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3}\alpha^3z^3+\text{etc.};\\ &\qquad\qquad+\frac{m-1}{1}\beta z^2\qquad+\frac{(m-1)(m-2)}{1\cdot 2}2\alpha\beta z^3\\ &\qquad\qquad\qquad\qquad\qquad\qquad+\frac{m-1}{1}\gamma z^3\end{aligned}$$

cuius lex progressionis ut melius patescat, ponatur eius loco

$$1 + Az + Bz^2 + Cz^3 + \cdots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^n + \text{etc.},$$

cuius seriei quilibet coefficiens ex tribus antecedentibus ita determinatur, ut sit

$$N = \frac{m-n}{n}\alpha M + \frac{2m-n}{n}\beta L + \frac{3m-n}{n}\gamma K.$$

Cum igitur primus terminus sit $=1$ et antecedentes nulli, erit

$$\begin{aligned}
A &= \frac{m-1}{1}\alpha,\\
B &= \frac{m-2}{2}\alpha A + \frac{2m-2}{2}\beta,\\
C &= \frac{m-3}{3}\alpha B + \frac{2m-3}{3}\beta A + \frac{3m-3}{3}\gamma,\\
D &= \frac{m-4}{4}\alpha C + \frac{2m-4}{4}\beta B + \frac{3m-4}{4}\gamma A,\\
E &= \frac{m-5}{5}\alpha D + \frac{2m-5}{5}\beta C + \frac{3m-5}{5}\gamma B
\end{aligned}$$

etc.

76. Generaliter ergo si ponatur

$$(1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc.})^{m-1} = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

huius seriei singuli termini ita ex praecedentibus definientur, ut sit

$$\begin{aligned}
A &= \frac{m-1}{1}\alpha,\\
B &= \frac{m-2}{2}\alpha A + \frac{2m-2}{2}\beta,\\
C &= \frac{m-3}{3}\alpha B + \frac{2m-3}{3}\beta A + \frac{3m-3}{3}\gamma,\\
D &= \frac{m-4}{4}\alpha C + \frac{2m-4}{4}\beta B + \frac{3m-4}{4}\gamma A + \frac{4m-4}{4}\delta,\\
E &= \frac{m-5}{5}\alpha D + \frac{2m-5}{5}\beta C + \frac{3m-5}{5}\gamma B + \frac{4m-5}{5}\delta A + \frac{5m-5}{5}\varepsilon
\end{aligned}$$

etc.;

quilibet scilicet terminus per tot praecedentes determinatur, quot habentur litterae α, β, γ, δ etc. in functione ipsius z, cuius potestas in seriem convertitur. Ceterum ratio huius legis convenit cum ea, quam supra § 68 [invenimus], ubi similem formam $(1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.})^{-m-1}$ in seriem infinitam resolvimus; si enim loco m scribatur $-m$ atque litterae α, β, γ, δ etc. negative accipiantur, series inventae prorsus congruent. Interim hoc loco non licet rationem huius progressionis legis a priori demonstrare, id quod per principia Calculi differentialis[1]) demum commode fieri poterit; interea ergo sufficiet veritatem per applicationem ad omnis generis exempla comprobasse.

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1) Confer L. EULERI *Institutiones calculi differentialis*, Petropoli 1755, partis posterioris cap. VIII; *LEONHARDI EULERI Opera omnia*, series I, vol. 10, p. 396. A. K.

CAPUT V

DE FUNCTIONIBUS DUARUM PLURIUMVE VARIABILIUM

77. Quanquam plures hactenus quantitates variabiles sumus contemplati, tamen eae ita erant comparatae, ut omnes unius essent functiones unaque determinata reliquae simul determinarentur. Nunc autem eiusmodi considerabimus quantitates variabiles, quae a se invicem non pendeant, ita ut, quamvis uni determinatus valor tribuatur, reliquae tamen nihilominus maneant indeterminatae ac variabiles. Eiusmodi ergo quantitates variabiles, cuiusmodi sint x, y, z, ratione significationis convenient, cum quaelibet omnes valores determinatos in se complectatur; at, si inter se comparentur, maxime erunt diversae, cum, licet pro una z valor quicunque determinatus substituatur, reliquae tamen x et y aeque late pateant atque ante. Discrimen ergo inter quantitates variabiles a se pendentes et non pendentes in hoc versatur, ut priori casu, si una determinetur, simul reliquae determinentur, posteriori vero determinatio unius significationes reliquarum minime restringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium x, y, z est expressio quomodocunque ex his quantitatibus composita.*

Ita erit

$$x^3 + xyz + az^2$$

functio quantitatum variabilium trium x, y, z. Haec ergo functio, si una determinetur variabilis, puta z, hoc est, eius loco constans numerus substituatur, manebit adhuc quantitas variabilis, scilicet functio ipsarum x et y. Atque si praeter z quoque y determinetur, tum erit adhuc functio ipsius x.

Huiusmodi ergo plurium variabilium functio non ante valorem determinatum obtinebit, quam singulae quantitates variabiles fuerint determinatae. Cum igitur una quantitas variabilis infinitis modis determinari possit, functio duarum variabilium, quia pro quavis determinatione unius infinitas determinationes suscipere potest, omnino infinities infinitas determinationes admittet. Atque in functione trium variabilium numerus determinationum erit adhuc infinities maior; sicque porro crescet pro pluribus variabilibus.

79. *Huiusmodi functiones plurium variabilium perinde atque functiones unius variabilis commodissime dividuntur in algebraicas ac transcendentes.*

Quarum illae sunt, in quibus ratio compositionis in solis Algebrae operationibus est posita; hae vero, in quarum formationem quoque operationes transcendentes ingrediuntur. In his denuo species notari possent, prout operationes transcendentes vel omnes quantitates variabiles implicant vel aliquot vel tantum unicam. Sic ista expressio

$$zz + y \log. z,$$

quia logarithmus ipsius z inest, erit quidem functio transcendens ipsarum y et z, verum ideo minus transcendens est putanda, quod, si variabilis z determinetur, supersit functio algebraica ipsius y. Interim tamen non expedit huiusmodi subdivisionibus tractationem amplificari.

80. *Functiones deinde algebraicae subdividuntur in rationales et irrationales, rationales autem porro in integras ac fractas.*

Ratio harum denominationum ex capite primo iam abunde intelligitur. Functio scilicet rationalis omnino est libera ab omni irrationalitate quantitates variabiles, quarum functio dicitur, afficiente; haecque erit integra, si nullis fractionibus inquinetur, contra vero fracta. Sic functionis integrae duarum variabilium y et z haec erit forma generalis

$$\alpha + \beta y + \gamma z + \delta y^2 + \varepsilon yz + \zeta z^2 + \eta y^3 + \theta y^2 z + \iota y z^2 + \varkappa z^3 + \text{etc.}$$

Quodsi ergo P et Q denotent huiusmodi functiones integras, sive duarum sive plurium variabilium, erit $\frac{P}{Q}$ forma generalis functionum fractarum.

Functio denique irrationalis est vel explicita vel implicita; illa per signa radicalia iam penitus est evoluta, haec autem per aequationem irresolubilem exhibetur. Sic V erit functio implicita irrationalis ipsarum y et z, si fuerit

$$V^5 = (ayz + z^3)V^2 + (y^4 + z^4)V + y^5 + 2ayz^3 + z^5.$$

81. *Multiformitas deinde in his functionibus aeque notari debet atque in iis, quae ex unica variabili constant.*

Sic functiones rationales erunt uniformes, quia singulis quantitatibus variabilibus determinatis unicum valorem determinatum exhibent. Denotent P, Q, R, S etc. functiones rationales seu uniformes variabilium x, y, z eritque V functio biformis earundem variabilium, si fuerit

$$V^2 - PV + Q = 0;$$

quicunque enim valores determinati quantitatibus x, y et z tribuuntur, functio V non unum sed duplicem perpetuo habebit valorem determinatum. Simili modo erit V functio triformis, si fuerit

$$V^3 - PV^2 + QV - R = 0,$$

atque functio quadriformis, si fuerit

$$V^4 - PV^3 + QV^2 - RV + S = 0;$$

hocque modo ratio functionum multiformium ulteriorum erit comparata.

82. Quemadmodum, si functio unius variabilis z nihilo aequalis ponitur, quantitas variabilis z valorem consequitur determinatum vel simplicem vel multiplicem, ita, si functio duarum variabilium y et z nihilo aequalis ponitur, tum altera variabilis per alteram definitur eiusque ideo functio evadit, cum ante a se mutuo non penderent. Simili modo si functio trium variabilium x, y, z nihilo aequalis statuatur, tum una variabilis per duas reliquas definitur earumque functio existit. Idem evenit, si functio non nihilo, sed quantitati constanti vel etiam alii functioni aequalis ponatur; ex omni enim aequatione, quotcunque variabiles involvat, semper una variabilis per reliquas definitur earumque fit functio; duae autem aequationes diversae inter easdem variabiles ortae binas per reliquas definient atque ita porro.

83. *Functionum autem duarum pluriumve variabilium divisio maxime notatu digna est in homogeneas et heterogeneas.*

Functio homogenea[1]) est, per quam ubique idem regnat variabilium numerus dimensionum; *functio* autem *heterogenea* est, in qua diversi occurrunt dimensionum numeri. Censetur vero unaquaeque variabilis unam dimensionem constituere; quadratum uniuscuiusque atque productum ex duabus duas; productum ex tribus variabilibus, sive iisdem sive diversis, tres, et ita porro; quantitates autem constantes ad dimensionum numerationem non admittuntur. Ita in his formulis

$$\alpha y, \ \beta z$$

unica dimensio inesse dicitur; in his vero

$$\alpha y^2, \ \beta y z, \ \gamma z^2$$

duae insunt dimensiones; in his

$$\alpha y^3, \ \beta y^2 z, \ \gamma y z^2, \ \delta z^3$$

tres; in his vero

$$\alpha y^4, \ \beta y^3 z, \ \gamma y^2 z^2, \ \delta y z^3, \ \varepsilon z^4$$

quatuor sicque porro.

84. Applicemus primum hanc distinctionem ad functiones integras atque duas tantum variabiles inesse ponamus, quoniam plurium par est ratio.

Functio igitur integra erit homogenea, in cuius singulis terminis idem existit dimensionum numerus.

Subdividentur ergo huiusmodi functiones commodissime secundum numerum dimensionum, quem variabiles in ipsis ubique constituunt. Sic erit

$$\alpha y + \beta z$$

1) Terminus technicus *homogeneus* invenitur iam apud FR. VIETA (1540—1603), *Opera mathematica* (ed. FR. À SCHOOTEN), Lugd. Batav. 1646, p. 4. Eum respiciens deinde G. LEIBNIZ distincte definivit expressiones secundum *legem homogeneorum* compositas (id est *functiones homogeneas*); vide *LEIBNIZENS Mathematische Schriften*, herausg. von C. I. GERHARDT, 2. Abt., Bd. 3, Halle 1853, p. 65. Concedendum quidem est hunc ipsum terminum *functionis homogeneae* primum inveniri apud IOH. BERNOULLI (1667—1748), *De integrationibus aequationum differentialium etc.*, Comment. acad. sc. Petrop. 1 (1726), 1728, p. 167; *Opera omnia*, Lausannae et Genevae 1742, t. III, p. 108. A. K.

forma generalis functionum integrarum unius dimensionis; haec vero expressio

$$\alpha y^2 + \beta yz + \gamma z^2$$

erit forma generalis functionum duarum dimensionum; tum forma generalis functionum trium dimensionum erit

$$\alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3;$$

quatuor dimensionum vero haec

$$\alpha y^4 + \beta y^3 z + \gamma y^2 z^2 + \delta y z^3 + \varepsilon z^4$$

et ita porro. Ad analogiam igitur erit quantitas constans sola α functio nullius dimensionis.

85. *Functio porro fracta erit homogenea, si eius numerator ac denominator fuerint functiones homogeneae.*

Sic haec fractio

$$\frac{ayz + bzz}{\alpha y + \beta z}$$

erit functio homogenea ipsarum y et z; numerus dimensionum autem habebitur, si a numero dimensionum numeratoris subtrahatur numerus dimensionum denominatoris, atque ob hanc rationem fractio allata erit functio unius dimensionis. Haec vero fractio

$$\frac{y^5 + z^5}{yy + zz}$$

erit functio trium dimensionum. Quando ergo in numeratore ac denominatore idem dimensionum numerus inest, tum fractio erit functio nullius dimensionis, uti evenit in hac fractione

$$\frac{y^3 + z^3}{yyz}$$

vel etiam in his

$$\frac{y}{z}, \quad \frac{\alpha zz}{yy}, \quad \frac{\beta y^3}{z^3}.$$

Quodsi igitur in denominatore plures sint dimensiones quam in numeratore, numerus dimensionum fractionis erit negativus; sic

$$\frac{y}{zz}$$

erit functio -1 dimensionis,

$$\frac{y+z}{y^4+z^4}$$

erit functio -3 dimensionum,

$$\frac{1}{y^5+ayz^4}$$

erit functio -5 dimensionum, quia in numeratore nulla inest dimensio. Ceterum sponte intelligitur plures functiones homogeneas, in quibus singulis idem regnat dimensionum numerus, sive additas sive subtractas praebere functionem quoque homogeneam eiusdem dimensionum numeri. Sic haec expressio

$$\alpha y + \frac{\beta zz}{y} + \frac{\gamma y^4 - \delta z^4}{yyz + yzz}$$

erit functio unius dimensionis; haec autem

$$\alpha + \frac{\beta y}{z} + \frac{\gamma zz}{yy} + \frac{yy+zz}{yy-zz}$$

erit functio nullius dimensionis.

86. Natura functionum homogenearum quoque ad expressiones irrationales extenditur. Si enim fuerit P functio quaecunque homogenea, puta n dimensionum, tum $\sqrt{P}$ erit functio $\frac{1}{2}n$ dimensionum, $\sqrt[3]{P}$ erit functio $\frac{1}{3}n$ dimensionum et generatim $P^{\frac{\mu}{\nu}}$ erit functio $\frac{\mu}{\nu}n$ dimensionum. Sic $\sqrt{(yy+zz)}$ erit functio unius dimensionis, $\sqrt[3]{(y^9+z^9)}$ erit functio trium dimensionum, $(yz+zz)^{\frac{3}{4}}$ erit functio $\frac{3}{2}$ dimensionum atque $\frac{yy+zz}{\sqrt{(y^4+z^4)}}$ erit functio nullius dimensionis. His ergo cum praecedentibus coniunctis intelligetur haec expressio

$$\frac{1}{y} + \frac{y\sqrt{(yy+zz)}}{z^3} - \frac{y}{\sqrt[3]{(y^6-z^6)}} + \frac{y\sqrt{z}}{zz\sqrt{y}+\sqrt{(y^5+z^5)}}$$

esse functio homogenea -1 dimensionis.

87. Utrum functio irrationalis implicita sit homogenea necne, ex his facile colligi potest. Sit V huiusmodi functio implicita ac

$$V^3 + PV^2 + QV + R = 0$$

existentibus P, Q et R functionibus ipsarum y et z. Primum igitur patet V functionem homogeneam esse non posse, nisi P, Q et R sint functiones homogeneae. Praeterea vero, si ponamus V esse functionem n dimensionum, erit V^2 functio $2n$ et V^3 functio $3n$ dimensionum; cum igitur ubique idem debeat esse numerus dimensionum, oportet, ut P sit functio n dimensionum, Q functio $2n$ dimensionum et R functio $3n$ dimensionum. Si ergo vicissim litterae P, Q, R [sint] functiones homogeneae respective n, $2n$, $3n$ dimensionum, hinc concludetur fore V functionem n dimensionum. Ita si fuerit

$$V^5 + (y^4 + z^4)V^3 + \alpha y^8 V - z^{10} = 0,$$

erit V functio homogenea duarum dimensionum ipsarum y et z.

88. *Si fuerit V functio homogenea n dimensionum ipsarum y et z in eaque ponatur ubique $y = uz$, functio V abibit in productum ex potestate z^n in functionem quandam variabilis u.*

Per hanc enim substitutionem $y = uz$ in singulos terminos tantae inducentur potestates ipsius z, quantae ante inerant ipsius y. Cum igitur in singulis terminis dimensiones ipsarum y et z coniunctim aequassent numerum n, nunc sola variabilis z ubique habebit n dimensiones ideoque ubique inerit eius potestas z^n. Per hanc ergo potestatem functio V fiet divisibilis et quotus erit functio variabilem tantum u involvens.

Hoc primum patebit in functionibus integris. Si enim sit

$$V = \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3,$$

posito $y = uz$ fiet

$$V = z^3(\alpha u^3 + \beta u^2 + \gamma u + \delta).$$

Deinde vero idem manifestum est in fractis. Sit enim

$$V = \frac{\alpha y + \beta z}{yy + zz},$$

nempe functio -1 dimensionis; facto $y = uz$ fiet

$$V = z^{-1} \cdot \frac{\alpha u + \beta}{uu + 1}.$$

Neque etiam functiones irrationales hinc excipiuntur. Si enim sit

$$V = \frac{y + \sqrt{(yy + zz)}}{z\sqrt{(y^3 + z^3)}},$$

quae est functio $-\frac{3}{2}$ dimensionum, posito $y = uz$ prodibit

$$V = z^{-\frac{3}{2}} \cdot \frac{u + \sqrt{(uu + 1)}}{\sqrt{(u^3 + 1)}}.$$

Hoc itaque modo functiones homogeneae duarum tantum variabilium reducentur ad functiones unius variabilis; neque enim potestas ipsius z, quia est factor, functionem illam ipsius u inquinat.

89. *Functio ergo homogenea V duarum variabilium y et z nullius dimensionis posito $y = uz$ transmutabitur in functionem unicae variabilis u puram.*

Cum enim numerus dimensionum sit nullus, potestas ipsius z, quae functionem ipsius u multiplicabit, erit $z^0 = 1$ hocque casu variabilis z prorsus ex computo egredietur. Ita si fuerit

$$V = \frac{y + z}{y - z},$$

facto $y = uz$ orietur

$$V = \frac{u + 1}{u - 1},$$

atque in irrationalibus si sit

$$V = \frac{y - \sqrt{(yy - zz)}}{z},$$

posito $y = uz$ erit

$$V = u - \sqrt{(uu - 1)}.$$

90. *Functio integra homogenea duarum variabilium y et z resolvi poterit in tot factores simplices formae $\alpha y + \beta z$, quot habuerit dimensiones.*

Cum enim functio sit homogenea, posito $y = uz$ transibit in productum ex z^n in functionem quandam ipsius u integram, quae functio propterea in factores simplices formae $\alpha u + \beta$ resolvi poterit. Multiplicentur singuli factores hi per z eritque uniuscuiusque forma $\alpha uz + \beta z = \alpha y + \beta z$ ob $uz = y$. Propter multiplicatorem autem z^n, tot huiusmodi factores nascentur, quot exponens n contineat unitates; factores autem hi simplices erunt vel reales vel imaginarii, hoc est, coefficientes α et β erunt vel reales vel imaginarii.

Ex hoc itaque sequitur functionem duarum dimensionum

$$ayy + byz + czz$$

duos habere factores simplices formae $\alpha y + \beta z$; functio autem

$$ay^3 + by^2z + cyz^2 + dz^3$$

habebit tres factores simplices formae $\alpha y + \beta z$; sicque porro functionum homogenearum integrarum, quae plures habent dimensiones, natura erit comparata.

91. Quemadmodum ergo haec expressio $\alpha y + \beta z$ continet formam generalem functionum integrarum unius dimensionis, ita

$$(\alpha y + \beta z)(\gamma y + \delta z)$$

erit forma generalis functionum integrarum duarum dimensionum; atque in hac forma

$$(\alpha y + \beta z)(\gamma y + \delta z)(\varepsilon y + \zeta z)$$

continebuntur omnes functiones integrae trium dimensionum sicque omnes functiones integrae homogeneae per producta ex tot huiusmodi factoribus $\alpha y + \beta z$ exhiberi poterunt, quot functiones illae contineant dimensiones. Isti autem factores eodem modo per resolutionem aequationum reperiuntur, quo supra [§ 29] factores simplices functionum integrarum unius variabilis invenire docuimus. Ceterum haec proprietas functionum homogenearum duarum variabilium non extenditur ad functiones homogeneas trium pluriumve variabilium;

forma enim generalis huiusmodi functionum duarum tantum dimensionum, quae est

$$ayy + byz + cyx + dxz + exx + fzz,$$

generaliter non reduci potest ad huiusmodi productum

$$(\alpha y + \beta z + \gamma x)(\delta y + \varepsilon z + \zeta x)$$

multoque minus functiones plurium dimensionum ad huiusmodi producta revocari possunt.

92. Ex his, quae de functionibus homogeneis sunt dicta, simul intelligitur, quid sit functio heterogenea; in cuius scilicet terminis non ubique idem dimensionum numerus deprehenditur. Possunt autem functiones heterogeneae subdividi pro multiplicitate dimensionum, quae in ipsis occurrunt. Sic functio *bifida* erit, in qua duplex dimensionum numerus occurrit, eritque adeo aggregatum duarum functionum homogenearum, quarum numeri dimensionum differunt; ita

$$y^5 + 2y^3z^2 + yy + zz$$

erit functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem *trifida* est, in qua tres diversi dimensionum numeri insunt seu quae in tres functiones homogeneas distribui possunt, uti

$$y^6 + y^2z^2 + z^4 + y - z.$$

Praeterea autem dantur functiones heterogeneae fractae vel irrationales tantopere permixtae, quae in functiones homogeneas resolvi non possunt, cuiusmodi sunt

$$\frac{y^3 + ayz}{by + zz}, \quad \frac{a + \sqrt{(yy + zz)}}{yy - bz}.$$

93. Interdum functio heterogenea ope substitutionis idoneae, vel loco unius vel utriusque variabilis factae, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quaedam attulisse, quibus eiusmodi reductio locum habet. Si scilicet haec proposita sit functio

$$y^5 + zzy + y^3z + \frac{z^3}{y},$$

post levem attentionem apparebit eam ad homogeneitatem perduci posito $z = xx$; prodibit enim

$$y^5 + x^4 y + y^3 xx + \frac{x^6}{y},$$

functio homogenea 5 dimensionum ipsarum x et y. Deinde haec functio

$$y + y^2 x + y^3 xx + y^5 x^4 + \frac{a}{x}$$

ad homogeneitatem reducitur ponendo $x = \frac{1}{z}$; prodit enim functio unius dimensionis

$$y + \frac{yy}{z} + \frac{y^3}{zz} + \frac{y^5}{z^4} + az.$$

Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero, qui in functione inest. Sic

$$xx + yy + zz + ay - aa$$

est functio secundi ordinis, quia duae dimensiones occurrunt. Et

$$y^4 + yz^3 - ay^2 z + abyz - aayy + b^4$$

pertinet ad functiones quarti ordinis. Ad hanc divisionem potissimum in doctrina de lineis curvis respici solet, unde adhuc una functionum integrarum divisio commemoranda venit.

95. Superest scilicet divisio functionum integrarum in *complexas* atque *incomplexas*. Functio autem complexa est, quae in factores rationales resolvi potest seu quae est productum ex duabus functionibus pluribusve rationalibus; cuiusmodi est

$$y^4 - z^4 + 2az^3 - 2byzz - aazz + 2abzy - bbyy,$$

quae est productum ex his duabus functionibus

$$(yy + zz - az + by)(yy - zz + az - by).$$

Ita vidimus omnem functionem integram homogeneam, quae tantum duas variabiles complectatur, esse functionem complexam, quoniam tot factores simplices formae $\alpha y + \beta z$ habet, quot continet dimensiones. Functio igitur integra erit incomplexa, si in factores rationales resolvi omnino nequeat, uti

$$yy + zz - aa,$$

cuius nullos dari factores rationales facile intelligitur. Ex inquisitione divisorum patebit, utrum functio proposita sit complexa an incomplexa.

CAPUT VI

DE QUANTITATIBUS EXPONENTIALIBUS AC LOGARITHMIS

96. Quanquam notio functionum transcendentium in Calculo integrali demum perpendetur, tamen, antequam eo perveniamus, quasdam species magis obvias atque ad plures investigationes aditum aperientes evolvere conveniet. Primum ergo considerandae sunt quantitates exponentiales seu potestates, quarum exponens ipse est quantitas variabilis. Perspicuum enim est huiusmodi quantitates ad functiones algebraicas referri non posse, cum in his exponentes non nisi constantes locum habeant. Multiplices autem sunt quantitates exponentiales, prout vel solus exponens est quantitas variabilis vel praeterea etiam ipsa quantitas elevata. Prioris generis est a^z, huius vero y^z; quin etiam ipse exponens potest esse quantitas exponentialis, uti in his formis a^{a^z}, a^{y^z}, y^{a^z}, x^{y^z}. Huiusmodi autem quantitatum non plura constituemus genera, cum earum natura satis clare intelligi queat, si primam tantum speciem a^z evolverimus.

97. Sit igitur proposita huiusmodi quantitas exponentialis a^z, quae est potestas quantitatis constantis a exponentem habens variabilem z. Cum igitur iste exponens z omnes numeros determinatos in se complectatur, primum patet, si loco z omnes numeri integri affirmativi successive substituantur, loco a^z hos prodituros esse valores determinatos

$$a^1, \quad a^2, \quad a^3, \quad a^4, \quad a^5, \quad a^6 \quad \text{etc.}$$

Sin autem pro z ponantur successive numeri negativi -1, -2, -3, -4 etc., prodibunt

$$\frac{1}{a}, \ \frac{1}{a^2}, \ \frac{1}{a^3}, \ \frac{1}{a^4} \ \text{etc.}$$

ac, si fuerit $z = 0$, habebitur semper

$$a^0 = 1.$$

Quodsi loco z numeri fracti ponantur, ut $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$ etc., orientur isti valores

$$\sqrt{a}, \ \sqrt[3]{a}, \ \sqrt[3]{aa}, \ \sqrt[4]{a}, \ \sqrt[4]{a^3} \ \text{etc.},$$

qui in se spectati geminos pluresve induunt valores, cum radicum extractio semper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales scilicet atque affirmativi, admitti solent, quia quantitas a^z tanquam functio uniformis ipsius z spectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit locum inter a^2 et a^3 eritque ideo quantitas eiusdem generis; et quamvis valor $a^{\frac{5}{2}}$ sit aeque $= -aa\sqrt{a}$ ac $= +aa\sqrt{a}$, tamen posterior tantum in censum venit. Eodem modo res se habet, si exponens z valores irrationales accipiat, quibus casibus, cum difficile sit numerum valorum involutorum concipere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 et a^3 comprehensus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri constantis a pendebunt. Si enim fuerit $a = 1$, semper erit $a^z = 1$, quicunque valores exponenti z tribuantur. Sin autem fuerit $a > 1$, tum valores ipsius a^z eo erunt maiores, quo maior numerus loco z substituatur, atque adeo posito $z = \infty$ in infinitum excrescunt; si fuerit $z = 0$, fiet $a^z = 1$, et si sit $z < 0$, valores a^z fient unitate minores, quoad posito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit, si sit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipsius a^z decrescent crescente z supra 0; crescent vero, si pro z numeri negativi substituantur. Cum enim sit $a < 1$, erit $\frac{1}{a} > 1$; posito ergo $\frac{1}{a} = b$ erit $a^z = b^{-z}$, unde posterior casus ex priori diiudicari poterit.

99. Si sit $a = 0$, ingens saltus in valoribus ipsius a^z deprehenditur. Quamdiu enim fuerit z numerus affirmativus seu maior nihilo, erit perpetuo

$a^z = 0$; si sit $z = 0$, erit $a^0 = 1$; sin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit

$$a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$$

ideoque infinitum.

Multo maiores autem saltus occurrent, si quantitas constans a habeat valorem negativum, puta -2. Tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi et negativi, ut ex hac serie intelligitur

$$\begin{array}{cccccccccc} a^{-4}, & a^{-3}, & a^{-2}, & a^{-1}, & a^0, & a^1, & a^2, & a^3, & a^4 & \text{etc.} \\ +\frac{1}{16}, & -\frac{1}{8}, & +\frac{1}{4}, & -\frac{1}{2}, & 1, & -2, & +4, & -8, & +16 & \text{etc.} \end{array}$$

Praeterea vero, si exponenti z valores tribuantur fracti, potestas $a^z = (-2)^z$ mox reales mox imaginarios induet valores; erit enim $a^{\frac{1}{2}} = \sqrt{-2}$ imaginarium, at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale; utrum autem, si exponenti z tribuantur valores irrationales, potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a substituendorum commemoratis statuamus a esse numerum affirmativum et unitate quidem maiorem, quia huc quoque illi casus, quibus a est numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z substituendo omnes numeros reales, qui intra limites $+\infty$ et $-\infty$ continentur, y adipiscetur omnes valores affirmativos intra limites $+\infty$ et 0 contentos. Si enim sit $z = \infty$, erit $y = \infty$; si $z = 0$, erit $y = 1$, et si $z = -\infty$, fiet $y = 0$. Vicissim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis respondens pro z, ita ut sit $a^z = y$; sin autem ipsi y tribueretur valor negativus, exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y functio quaedam ipsius z; et quemadmodum y a z pendeat, ex natura potestatum facile intelligitur; hinc enim, quicunque valor ipsi z tribuatur, valor ipsius y determinatur. Erit autem

$$yy = a^{2z}, \quad y^3 = a^{3z}$$

et generaliter erit

$$y^n = a^{nz};$$

unde sequitur fore

$$\sqrt{y} = a^{\frac{1}{2}z}, \quad \sqrt[3]{y} = a^{\frac{1}{3}z} \quad \text{et} \quad \frac{1}{y} = a^{-z}, \quad \frac{1}{yy} = a^{-2z} \quad \text{et} \quad \frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$$

et ita porro. Praeterea, si fuerit $v = a^x$, erit

$$vy = a^{x+z} \quad \text{et} \quad \frac{v}{y} = a^{x-z},$$

quorum subsidiorum beneficio eo facilius valor ipsius y ex dato valore ipsius z inveniri potest.

EXEMPLUM

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promptu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim

$$10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10000 \quad \text{et} \quad 10^0 = 1;$$

item

$$10^{-1} = \frac{1}{10} = 0{,}1, \quad 10^{-2} = \frac{1}{100} = 0{,}01, \quad 10^{-3} = \frac{1}{1000} = 0{,}001;$$

sin autem pro z fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt; sic erit

$$10^{\frac{1}{2}} = \sqrt{10} = 3{,}162277\,^{1)} \quad \text{etc.}$$

102. Quemadmodum autem dato numero a ex quovis valore ipsius z reperiri potest valor ipsius y, ita vicissim dato valore quocunque affirmativo ipsius y conveniens dabitur valor ipsius z, ut sit $a^z = y$; iste autem valor ipsius z, quatenus tanquam functio ipsius y spectatur, vocari solet *Logarithmus* ipsius y. Supponit ergo doctrina logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* logarithmorum; qua assumpta erit logarithmus cuiusque numeri y exponens potestatis a^z, ita ut

1) Valor accuratior est 3,162278, nempe 3,16227766. A. K.

ipsa potestas a^z aequalis sit numero illi y; indicari autem logarithmus numeri y solet hoc modo ly. Quodsi ergo fuerit

$$a^z = y,$$

erit

$$z = ly,$$

ex quo intelligitur basin logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate maiorem hincque nonnisi numerorum affirmativorum logarithmos realiter exhiberi posse.

103. Quicunque ergo numerus pro basi logarithmica a accipiatur, erit semper

$$l1 = 0;$$

si enim in aequatione $a^z = y$, quae convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$.

Deinde numerorum unitate maiorum logarithmi erunt affirmativi, pendentes a valore basis a; sic erit

$$la = 1, \quad laa = 2, \quad la^3 = 3, \quad la^4 = 4 \quad \text{etc.},$$

unde a posteriori intelligi potest, quantus numerus pro basi sit assumptus; scilicet ille numerus, cuius logarithmus est $= 1$, erit basis logarithmica.

Numerorum autem unitate minorum, affirmativorum tamen, logarithmi erunt negativi; erit enim

$$l\frac{1}{a} = -1, \quad l\frac{1}{aa} = -2, \quad l\frac{1}{a^3} = -3 \quad \text{etc.}$$

Numerorum autem negativorum logarithmi non erunt reales, sed imaginarii, uti iam notavimus.

104. Simili modo si fuerit $ly = z$, erit

$$lyy = 2z, \quad ly^3 = 3z$$

et generaliter

$$ly^n = nz \quad \text{seu} \quad ly^n = nly$$

ob $z = ly$. Logarithmus igitur cuiusque potestatis ipsius y aequatur logarithmo ipsius y per exponentem potestatis multiplicato; sic erit

$$l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly, \quad l\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$$

et ita porro; unde ex dato logarithmo cuiusque numeri inveniri possunt logarithmi quarumcunque ipsius potestatum.

Sin autem iam inventi sint duo logarithmi, nempe

$$ly = z \quad \text{et} \quad lv = x,$$

cum sit $y = a^z$ et $v = a^x$, erit

$$lvy = x + z = lv + ly;$$

hinc logarithmus producti duorum numerorum aequatur summae logarithmorum factorum; simili vero modo erit

$$l\frac{y}{v} = z - x = ly - lv;$$

hincque logarithmus fractionis aequatur logarithmo numeratoris dempto logarithmo denominatoris; quae regulae inserviunt logarithmis plurium numerorum inveniendis ex cognitis iam aliquot logarithmis.

105. Ex his autem patet aliorum numerorum non dari logarithmos rationales nisi potestatum basis a; nisi enim numerus alius b fuerit potestas basis a, eius logarithmus numero rationali exprimi non poterit. Neque vero etiam logarithmus ipsius b erit numerus irrationalis. Si enim foret $lb = \sqrt{n}$, tum esset $a^{\sqrt{n}} = b$; id quod fieri nequit, si quidem numeri a et b rationales statuantur. Solent autem imprimis numerorum rationalium et integrorum logarithmi desiderari, quia ex his logarithmi fractionum ac numerorum surdorum inveniri possunt. Cum igitur logarithmi numerorum, qui non sunt potestates basis a, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates transcendentes referuntur hincque logarithmi quantitatibus transcendentibus annumerari solent.

106. Hanc ob rem logarithmi numerorum vero tantum proxime per fractiones decimales exprimi solent, qui eo minus a veritate discrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per solam radicis quadratae extractionem cuiusque numeri logarithmus vero proxime determinari poterit. Cum enim posito

$$ly = z \quad \text{et} \quad lv = x$$

sit

$$l\sqrt{vy} = \frac{x+z}{2},$$

si numerus propositus b contineatur intra limites a^2 et a^3, quorum logarithmi sunt 2 et 3, quaeratur valor ipsius $a^{2\frac{1}{2}}$ seu $a^2\sqrt{a}$ atque b vel intra limites a et $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ et a^3 continebitur; utrumvis accidat, sumendo medio proportionali denuo limites propiores prodibunt hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat et quibuscum numerus propositus b sine errore confundi possit. Quoniam vero horum singulorum limitum logarithmi dantur, tandem logarithmus numeri b reperietur.

EXEMPLUM

Ponatur basis logarithmica $a = 10$, quod in tabulis usu receptis fieri solet, et quaeratur vero tantum proxime logarithmus numeri 5; quia hic continetur intra limites 1 et 10, quorum logarithmi sunt 0 et 1, sequenti modo radicum extractio continua instituatur, quoad ad limites a numero proposito 5 non amplius discrepantes perveniatur.

$A = 1,000000$,	$lA = 0,0000000$,	sit
$B = 10,000000$,	$lB = 1,0000000$,	$C = \sqrt{AB}$,
$C = 3,162277$,	$lC = 0,5000000$,	$D = \sqrt{BC}$,
$D = 5,623413$,	$lD = 0,7500000$,	$E = \sqrt{CD}$,
$E = 4,216964$,	$lE = 0,6250000$,	$F = \sqrt{DE}$,
$F = 4,869674$,	$lF = 0,6875000$,	$G = \sqrt{DF}$,
$G = 5,232991$,	$lG = 0,7187500$,	$H = \sqrt{FG}$,
$H = 5,048065$,	$lH = 0,7031250$,	$I = \sqrt{FH}$,
$I = 4,958069$,	$lI = 0,6953125$,	$K = \sqrt{HI}$,
$K = 5,002865$,	$lK = 0,6992187$,	$L = \sqrt{IK}$,
$L = 4,980416$,	$lL = 0,6972656$,	$M = \sqrt{KL}$,
$M = 4,991627$,	$lM = 0,6982421$,[1])	$N = \sqrt{KM}$,
$N = 4,997242$,	$lN = 0,6987304$,	$O = \sqrt{KN}$,
$O = 5,000052$,	$lO = 0,6989745$,	$P = \sqrt{NO}$,
$P = 4,998647$,	$lP = 0,6988525$,	$Q = \sqrt{OP}$,
$Q = 4,999350$,	$lQ = 0,6989135$,	$R = \sqrt{OQ}$,
$R = 4,999701$,	$lR = 0,6989440$,	$S = \sqrt{OR}$,
$S = 4,999876$,	$lS = 0,6989592$,	$T = \sqrt{OS}$,
$T = 4,999963$,	$lT = 0,6989668$,	$V = \sqrt{OT}$,
$V = 5,000008$,	$lV = 0,6989707$,	$W = \sqrt{TV}$,
$W = 4,999984$,	$lW = 0,6989687$,	$X = \sqrt{VW}$,
$X = 4,999997$,	$lX = 0,6989697$,	$Y = \sqrt{VX}$,
$Y = 5,000003$,	$lY = 0,6989702$,	$Z = \sqrt{XY}$,
$Z = 5,000000$,	$lZ = 0,6989700$.	

1) Quod valores accuratiores

$$lK = 0,6992187500 \quad \text{et} \quad lL = 0,6972656250$$

sunt, est accuratius

$$lM = 0,6982421875,$$

ergo, si septem figurae scribantur,

$$lM = 0,6982422.$$

Qua de causa supra et valor lM et plurimi sequentium valorum erant corrigendi; sed cum isti errores sint parvi momenti, eos corrigere negleximus. A. K.

Sic ergo mediis proportionalibus sumendis tandem perventum est ad $Z = 5{,}000000$, ex quo logarithmus numeri 5 quaesitus est 0,698970 posita basi logarithmica $= 10$. Quare erit proxime

$$10^{\frac{69897}{100000}} = 5.$$

Hoc autem modo computatus est canon logarithmorum vulgaris a BRIGGIO et VLACQUIO[1]), quamquam postea eximia inventa sunt compendia, quorum ope multo expeditius logarithmi supputari possunt.

107. Dantur ergo tot diversa logarithmorum systemata, quot varii numeri pro basi a accipi possunt, atque ideo numerus systematum logarithmicorum erit infinitus. Perpetuo autem in duobus systematis logarithmi eiusdem numeri eandem inter se servant rationem. Sit basis unius systematis $= a$, alterius $= b$ atque numeri n logarithmus in priori systemate $= p$, in posteriori $= q$; erit

$$a^p = n \text{ et } b^q = n,$$

unde

$$a^p = b^q \quad \text{ideoque} \quad a = b^{\frac{q}{p}}.$$

Oportet ergo, ut fractio $\frac{q}{p}$ constantem obtineat valorem, quicunque numerus pro n fuerit assumptus. Quodsi ergo pro uno systemate logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream logarithmi pro quovis alio systemate reperiri possunt. Sic, cum dentur logarithmi pro basi 10, hinc logarithmi pro quavis alia basi, puta 2, inveniri possunt; quaeratur enim logarithmus numeri n pro basi 2, qui sit $= q$, cum eiusdem numeri n logarithmus sit $= p$ pro basi 10. Quoniam pro basi 10 est $l2 = 0{,}3010300$ et pro basi 2 est $l2 = 1$, erit

$$0{,}3010300 : 1 = p : q$$

ideoque

1) H. BRIGGS (1556—1630), *Arithmetica logarithmica sive logarithmorum chiliades triginta etc.*, Londini 1624; A. VLACQ (1600?—1667?), *Arithmetica logarithmica sive logarithmorum chiliades centum etc.* Editio secunda aucta, Goudae 1628. Confer etiam J. NEPER (1550—1617), *Mirifici logarithmorum canonis constructio. Appendix*, Edinburgii 1619. A. K.

$$q = \frac{p}{0,30100300} = 3,32199280 \cdot p;^{1)}$$

si ergo omnes logarithmi communes multiplicentur per numerum 3,3219280[1]) prodibit tabula logarithmorum pro basi 2.

108. *Hinc sequitur duorum numerorum logarithmos in quocunque systemate eandem tenere rationem.*

Sint enim duo numeri M et N, quorum pro basi a logarithmi sint m et n; erit $M = a^m$ et $N = a^n$; hinc fiet $a^{mn} = M^n = N^m$ ideoque

$$M = N^{\frac{m}{n}};$$

in qua aequatione cum basis a non amplius insit, perspicuum est fractionem $\frac{m}{n}$ habere valorem a basi a non pendentem. Sint enim pro alia basi b numerorum eorundem M et N logarithmi μ et ν, pari modo colligetur fore

$$M = N^{\frac{\mu}{\nu}}.$$

Erit ergo

$$N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$$

hincque

$$\frac{m}{n} = \frac{\mu}{\nu} \quad \text{seu} \quad m : n = \mu : \nu.$$

Ita iam vidimus in omni logarithmorum systemate logarithmos diversarum eiusdem numeri potestatum ut y^m et y^n tenere rationem exponentium $m : n$.

109. Ad canonem ergo logarithmorum pro basi quacunque a condendum sufficit numerorum tantum primorum logarithmos methodo ante tradita vel alia commodiori supputasse. Cum enim logarithmi numerorum compositorum sint aequales summis logarithmorum singulorum factorum, logarithmi nume-

1) Editio princeps: $3,3219277 \cdot p$. Correxit A. K.

rorum compositorum per solam additionem reperientur. Sic, si habeantur logarithmi numerorum 3 et 5, erit

$$l15 = l3 + l5, \quad l45 = 2l3 + l5.$$

Atque, cum supra pro basi $a = 10$ inventus sit

$$l5 = 0{,}6989700,$$

praeterea autem sit $l10 = 1$, erit

$$l\frac{10}{5} = l2 = l10 - l5$$

ideoque orietur

$$l2 = 1 - 0{,}6989700 = 0{,}3010300;$$

ex his autem numerorum primorum 2 et 5 logarithmis inventis reperientur logarithmi omnium numerorum ex his 2 et 5 compositorum, cuiusmodi sunt isti 4, 8, 16, 32, 64 etc., 20, 40, 80, 25, 50 etc.

110. Tabularum autem logarithmicarum amplissimus est usus in contrahendis calculis numericis, propterea quod ex eiusmodi tabulis non solum dati cuiusque numeri logarithmus, sed etiam cuiusque logarithmi propositi numerus conveniens reperiri potest. Sic, si c, d, e, f, g, h denotent numeros quoscunque, citra multiplicationem reperiri poterit valor istius expressionis

$$\frac{ccd\sqrt{e}}{f\sqrt[3]{gh}};$$

erit enim huius expressionis logarithmus

$$= 2lc + ld + \frac{1}{2}le - lf - \frac{1}{3}lg - \frac{1}{3}lh;$$

cui logarithmo si quaeratur numerus respondens, habebitur valor quaesitus. Inprimis autem inserviunt tabulae logarithmicae dignitatibus atque radicibus intricatissimis inveniendis, quarum operationum loco in logarithmis tantum multiplicatio et divisio adhibetur.

EXEMPLUM 1

Quaeratur valor huius potestatis $2^{\frac{7}{12}}$.

Quoniam eius logarithmus est $= \frac{7}{12} l 2$, multiplicetur logarithmus binarii ex tabulis, qui est 0,3010300, per $\frac{7}{12}$, hoc est per $\frac{1}{2} + \frac{1}{12}$; erit

$$l 2^{\frac{7}{12}} = 0{,}1756008,$$

cui logarithmo respondet numerus

$$1{,}498307,$$

qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM 2

Si numerus incolarum cuiuspiam provinciae quotannis sui parte trigesima augeatur, initio autem in provincia habitaverint 100000 hominum, quaeritur post 100 annos incolarum numerus.

Sit brevitatis gratia initio incolarum numerus $= n$, ita ut sit

$$n = 100000;$$

anno elapso uno erit incolarum numerus

$$= \left(1 + \frac{1}{30}\right) n = \frac{31}{30} n,$$

post duos annos $= \left(\frac{31}{30}\right)^2 n$, post tres annos $= \left(\frac{31}{30}\right)^3 n$ hincque post centum annos

$$= \left(\frac{31}{30}\right)^{100} n = \left(\frac{31}{30}\right)^{100} 100000,$$

cuius logarithmus est

$$= 100\, l \frac{31}{30} + l\, 100000.$$

At est

$$l \frac{31}{30} = l 31 - l 30 = 0{,}014240439,$$

unde

$$100\, l \frac{31}{30} = 1{,}4240439;$$

ad quem si addatur $l100000 = 5$, erit logarithmus numeri incolarum quaesiti

$$= 6{,}4240439,$$

cui respondet numerus

$$= 2654874.$$

Post centum ergo annos numerus incolarum fit plus quam vicies sexies cum semisse maior.

EXEMPLUM 3

Cum post diluvium a sex hominibus genus humanum sit propagatum, si ponamus ducentis annis post numerum hominum iam ad 1000000 excrevisse, quaeritur, quanta sui parte numerus hominum quotannis augeri debuerit.

Ponamus hoc tempore numerum hominum parte sua $\frac{1}{x}$ quotannis increvisse atque post ducentos annos prodierit necesse est numerus hominum

$$= \left(\frac{1+x}{x}\right)^{200} 6 = 1000000,$$

unde fit

$$\frac{1+x}{x} = \left(\frac{1000000}{6}\right)^{\frac{1}{200}}.$$

Erit ergo

$$l\frac{1+x}{x} = \frac{1}{200}\, l\frac{1000000}{6} = \frac{1}{200}\cdot 5{,}2218487 = 0{,}0261092$$

ideoque

$$\frac{1+x}{x} = \frac{1061963}{1000000} \quad \text{et} \quad 1000000 = 61963x,$$

unde fit

$$x = 16 \text{ circiter.}$$

Ad tantam ergo hominum multiplicationem suffecisset, si quotannis decima sexta sui parte increverint; quae multiplicatio ob longaevam vitam non nimis magna censeri potest. Quodsi autem eadem ratione per intervallum 400 annorum numerus hominum crescere perrexisset, tum numerus hominum ad

$$1000000 \cdot \frac{1000000}{6} = 166666666666$$

ascendere debuisset, quibus sustentandis universus orbis terrarum nequaquam par fuisset.

EXEMPLUM 4

Si singulis seculis numerus hominum duplicetur, quaeritur incrementum annuum.

Si quotannis hominum numerum parte sua $\frac{1}{x}$ crescere ponamus et initio numerus hominum fuerit $= n$, erit is post centum annos $= \left(\frac{1+x}{x}\right)^{100} n$; qui cum esse debeat $= 2n$, erit

$$\frac{1+x}{x} = 2^{\frac{1}{100}}$$

et

$$l\frac{1+x}{x} = \frac{1}{100}\, l2 = 0{,}0030103;$$

hinc

$$\frac{1+x}{x} = \frac{10069555}{10000000},$$

ergo

$$x = \frac{10000000}{69555} = 144 \text{ circiter.}$$

Sufficit ergo, si numerus hominum quotannis parte sua $\frac{1}{144}$ augeatur. Quam ob causam maxime ridiculae sunt eorum incredulorum hominum obiectiones, qui negant tam brevi temporis spatio ab uno homine universam terram incolis impleri potuisse.

111. Potissimum autem logarithmorum usus requiritur ad eiusmodi aequationes resolvendas, in quibus quantitas incognita in exponentem ingreditur. Sic, si ad huiusmodi perveniatur aequationem

$$a^x = b,$$

ex qua incognitae x valorem erui oporteat, hoc non nisi per logarithmos effici poterit. Cum enim sit $a^x = b$, erit

$$la^x = xla = lb$$

ideoque

$$x = \frac{lb}{la},$$

ubi quidem perinde est, quonam systemate logarithmico utatur, cum in omni systemate logarithmi numerorum a et b eandem inter se teneant rationem.

EXEMPLUM 1

Si numerus hominum quotannis centesima sui parte augeatur, quaeritur, post quot annos numerus hominum fiat decuplo maior.

Ponamus hoc evenire post x annos et initio hominum numerum fuisse $= n$; erit is ergo elapsis x annis $= \left(\frac{101}{100}\right)^x n$, qui cum aequalis sit $10n$, fiet

$$\left(\frac{101}{100}\right)^x = 10$$

ideoque

$$x\,l\frac{101}{100} = l\,10$$

et

$$x = \frac{l\,10}{l\,101 - l\,100}.$$

Prodibit itaque

$$x = \frac{10\,000\,000}{43\,214} = 231 \text{ [circiter]}.$$

Post annos ergo 231 fiet hominum numerus, quorum incrementum annuum tantum centesimam partem efficit, decuplo maior; hinc post 462 annos fiet centies et post 693 annos millies maior.

EXEMPLUM 2

Quidam debet 400000 florenos hac conditione, ut quotannis usuram 5 de centenis solvere teneatur; exsolvit autem singulis annis 25000 florenos. Quaeritur, post quot annos debitum penitus extinguatur.

Scribamus a pro debita summa 400000 fl. et b pro summa 25000 fl. quotannis soluta; debebit ergo elapso uno anno

$$\frac{105}{100}a - b,$$

elapsis duobus annis

$$\left(\frac{105}{100}\right)^2 a - \frac{105}{100}b - b,$$

elapsis tribus annis

$$\left(\frac{105}{100}\right)^3 a - \left(\frac{105}{100}\right)^2 b - \frac{105}{100}b - b;$$

hinc posito brevitatis causa n pro $\frac{105}{100}$ elapsis x annis adhuc debebit

$$n^x a - n^{x-1} b - n^{x-2} b - n^{x-3} b - \cdots - b = n^x a - b(1 + n + n^2 + \cdots + n^{x-1}).$$

Cum igitur sit ex natura progressionum geometricarum

$$1 + n + n^2 + \cdots + n^{x-1} = \frac{n^x - 1}{n - 1},$$

post x annos debitor adhuc debebit

$$n^x a - \frac{n^x b - b}{n - 1} \text{ flor.},$$

quod debitum nihilo aequale positum dabit hanc aequationem

$$n^x a = \frac{n^x b - b}{n - 1}$$

seu

$$(n - 1) n^x a = n^x b - b \quad \text{ideoque} \quad (b - na + a) n^x = b$$

et

$$n^x = \frac{b}{b - (n - 1) a},$$

unde fit

$$x = \frac{lb - l(b - (n - 1) a)}{ln}.$$

Cum iam sit

$$a = 400000, \quad b = 25000, \quad n = \frac{105}{100},$$

erit

$$(n - 1) a = 20000 \quad \text{et} \quad b - (n - 1) a = 5000$$

atque annorum, quibus debitum penitus extinguitur, numerus

$$x = \frac{l\,25000 - l\,5000}{l\frac{105}{100}} = \frac{l\,5}{l\frac{21}{20}} = \frac{6\,989\,700}{211\,893};$$

erit ergo x aliquanto minor quam 33. Scilicet elapsis annis 33 non solum debitum extinguetur, sed creditor debitori reddere tenebitur

$$\frac{(n^{33} - 1) b}{n - 1} - n^{33} a = \frac{\left(\frac{21}{20}\right)^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000 \left(\frac{21}{20}\right)^{33} - 500000 \text{ flor.}$$

Quia vero est

$$l\frac{21}{20} = 0{,}0211892991,$$

erit

$$l\left(\frac{21}{20}\right)^{33} = 0{,}69924687 \quad \text{et} \quad l\,100000\left(\frac{21}{20}\right)^{33} = 5{,}6992469,$$

cui respondet hic numerus 500318,8; unde creditor debitori post 33 annos restituere debet $318\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares super basi $=10$ extracti praeter hunc usum, quem logarithmi in genere praestant, in Arithmetica decimali usu recepta singulari gaudent commodo atque ob hanc causam prae aliis systematibus insignem afferunt utilitatem. Cum enim logarithmi omnium numerorum praeter denarii potestates in fractionibus decimalibus exhibeantur, numerorum inter 1 et 10 contentorum logarithmi intra limites 0 et 1, numerorum autem inter 10 et 100 contentorum logarithmi inter limites 1 et 2, et ita porro, continebuntur. Constat ergo logarithmus quisque ex numero integro et fractione decimali et ille numerus integer vocari solet *characteristica*, fractio decimalis autem *mantissa*. Characteristica itaque unitate deficiet a numero notarum, quibus numerus constat; ita logarithmi numeri 78509 characteristica erit 4, quia is ex quinque notis seu figuris constat. Hinc ex logarithmo cuiusvis numeri statim intelligitur, ex quot figuris numerus sit compositus. Sic numerus logarithmo 7,5804631 respondens ex 8 figuris constabit.

113. Si ergo duorum logarithmorum mantissae conveniant, characteristicae vero tantum discrepent, tum numeri his logarithmis respondentes rationem habebunt ut potestas denarii ad unitatem ideoque ratione figurarum, quibus constant, convenient. Ita horum logarithmorum 4,9130187 et 6,9130187 numeri erunt 81850 et 8185000; logarithmo autem 3,9130187 conveniet 8185 et logarithmo huic 0,9130187 convenit 8,185. Sola ergo mantissa indicabit figuras numerum componentes; quibus inventis ex characteristica patebit, quot figurae a sinistra ad integra referri debeant, reliquae ad dextram vero dabunt fractiones decimales. Sic, si hic logarithmus fuerit inventus 2,7603429, mantissa indicabit has figuras 5758945, characteristica 2 autem numerum illi logarithmo

determinat, ut sit 575,8945; si characteristica esset 0, foret numerus 5,758945; sin denuo unitate minuatur, ut sit -1, erit numerus respondens decies minor, nempe 0,5758945, et characteristicae -2 respondebit 0,05758945 etc. Loco characteristicarum autem huiusmodi negativarum -1, -2, -3 etc. scribi solent 9, 8, 7 etc. atque subintelligitur hos logarithmos denario minui debere. Haec vero in manductionibus ad tabulas logarithmorum fusius exponi solent.

EXEMPLUM

Si haec progressio 2, 4, 16, 256 etc., cuius quisque terminus est quadratum praecedentis, continuetur usque ad terminum vigesimum quintum, quaeritur magnitudo huius termini ultimi.

Termini huius progressionis per exponentes ita commodius exprimuntur

$$2^1, \quad 2^2, \quad 2^4, \quad 2^8 \quad \text{etc.},$$

ubi patet exponentes progressionem geometricam constituere atque termini vigesimi quinti exponentem fore

$$2^{24} = 16777216,$$

ita ut ipse terminus quaesitus sit

$$= 2^{16777216};$$

huius ergo logarithmus erit

$$= 16777216\, l2.$$

Cum ergo sit

$$l2 = 0,30102\,99956\,63981\,1952^{1)},$$

1) In editione principe ultima figura 2 deest; ad mantissam autem 259733675932 satis accurate computandam haec figura necessaria est. Ceterum valorem ipsius $l\,2$ (et valores pro k et $\frac{1}{k}$ in § 124 expositos) EULERUS depromere potuit ex dissertatione Celeberrimi E. HALLEY (1656—1742), *A most compendious and facile method for constructing the logarithms*, Philosophical transactions (London) **19**, 1695, numb. 216, p. 58, ubi sexaginta figurae horum et aliorum valorum sunt communicatae, quas A. SHARP (1651—1742) computaverat.

Exemplum hic pertractatum invenitur etiam in EULERI libro, qui inscribitur *Institutiones calculi differentialis*, Petropoli 1755, partis posterioris cap. IV, § 82; *LEONHARDI EULERI Opera omnia*, series I, vol. 10, p. 288. A. K.

erit numeri quaesiti logarithmus

$$= 5050445{,}259\,73367,$$

ex cuius characteristica patet numerum quaesitum more solito expressum constare ex

$$5050446$$

figuris. Mantissa autem 259733675932 in tabula logarithmorum quaesita dabit figuras initiales numeri quaesiti, quae erunt 181858. Quanquam ergo iste numerus nullo modo exhiberi potest, tamen affirmari potest eum omnino ex 5050446 figuris constare atque figuras initiales sex esse 181858, quas dextrorsum adhuc 5050440 figurae sequantur, quarum insuper nonnullae ex maiori logarithmorum canone definiri possent; undecim scilicet figurae initiales erunt 18185852986.

CAPUT VII

DE QUANTITATUM EXPONENTIALIUM AC LOGARITHMORUM PER SERIES EXPLICATIONE

114. Quia est $a^0 = 1$ atque crescente exponente ipsius a simul valor potestatis augetur, si quidem a est numerus unitate maior, sequitur, si exponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus seu fractio tam exigua, ut tantum non nihilo sit aequalis; erit

$$a^\omega = 1 + \psi$$

existente ψ quoque numero infinite parvo. Ex praecedente enim capite constat, nisi ψ esset numerus infinite parvus, neque ω talem esse posse. Erit ergo vel $\psi = \omega$ vel $\psi > \omega$ vel $\psi < \omega$, quae ratio utique a quantitate litterae a pendebit; quae cum adhuc sit incognita, ponatur $\psi = k\omega$, ita ut sit

$$a^\omega = 1 + k\omega,$$

et sumpta a pro basi logarithmica erit

$$\omega = l(1 + k\omega).$$

EXEMPLUM

Quo clarius appareat, quemadmodum numerus k pendeat a basi a, ponamus esse $a = 10$ atque ex tabulis vulgaribus quaeramus logarithmum numeri quam minime unitatem superantis, puta $1 + \frac{1}{1000000}$, ita ut sit $k\omega = \frac{1}{1000000}$;

erit

$$l\left(1+\frac{1}{1000000}\right)=l\,\frac{1000001}{1000000}=0{,}00000043429=\omega.$$

Hinc ob $k\omega = 0{,}00000100000$ erit

$$\frac{1}{k}=\frac{43429}{100000}$$

et

$$k=\frac{100000}{43429}=2{,}30258;$$

unde patet k esse numerum finitum pendentem a valore basis a. Si enim alius numerus pro basi a statuatur, tum logarithmus eiusdem numeri $1+k\omega$ ad priorem datam tenebit rationem, unde simul alius valor litterae k prodiret.

115. Cum sit $a^{\omega}=1+k\omega$, erit

$$a^{i\omega}=(1+k\omega)^{i},$$

quicunque numerus loco i substituatur. Erit ergo

$$a^{i\omega}=1+\frac{i}{1}k\omega+\frac{i(i-1)}{1\cdot 2}k^{2}\omega^{2}+\frac{i(i-1)(i-2)}{1\cdot 2\cdot 3}k^{3}\omega^{3}+\text{etc.}$$

Quodsi ergo statuatur $i=\frac{z}{\omega}$ et z denotet numerum quemcunque finitum, ob ω numerum infinite parvum fiet i numerus infinite magnus hincque $\omega=\frac{z}{i}$, ita ut sit ω fractio denominatorem habens infinitum adeoque infinite parva, qualis est assumpta. Substituatur ergo $\frac{z}{i}$ loco ω eritque

$$a^{z}=\left(1+\frac{kz}{i}\right)^{i}=1+\frac{1}{1}kz+\frac{1(i-1)}{1\cdot 2i}k^{2}z^{2}+\frac{1(i-1)(i-2)}{1\cdot 2i\cdot 3i}k^{3}z^{3}$$
$$+\frac{1(i-1)(i-2)(i-3)}{1\cdot 2i\cdot 3i\cdot 4i}k^{4}z^{4}+\text{etc.},$$

quae aequatio erit vera, si pro i numerus infinite magnus substituatur. Tum vero est k numerus finitus ab a pendens, uti modo vidimus.

116. Cum autem i sit numerus infinite magnus, erit

$$\frac{i-1}{i}=1;$$

patet enim, quo maior numerus loco i substituatur, eo propius valorem fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum; hinc, si i sit numerus omni assignabili maior, fractio quoque $\frac{i-1}{i}$ ipsam unitatem adaequabit. Ob similem autem rationem erit

$$\frac{i-2}{i}=1,\quad \frac{i-3}{i}=1$$

et ita porro; hinc sequitur fore

$$\frac{i-1}{2i}=\frac{1}{2},\quad \frac{i-2}{3i}=\frac{1}{3},\quad \frac{i-3}{4i}=\frac{1}{4}$$

et ita porro. His igitur valoribus substitutis erit

$$a^z=1+\frac{kz}{1}+\frac{k^2z^2}{1\cdot 2}+\frac{k^3z^3}{1\cdot 2\cdot 3}+\frac{k^4z^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc. in infinitum.}$$

Haec autem aequatio simul relationem inter numeros a et k ostendit; posito enim $z=1$ erit

$$a=1+\frac{k}{1}+\frac{k^2}{1\cdot 2}+\frac{k^3}{1\cdot 2\cdot 3}+\frac{k^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

atque, ut a sit $=10$, necesse est, ut sit circiter $k=2{,}30258$, uti ante invenimus.

117. Ponamus esse

$$b=a^n;$$

erit sumpto numero a pro basi logarithmica $lb=n$. Hinc, cum sit $b^z=a^{nz}$, erit per seriem infinitam

$$b^z=1+\frac{knz}{1}+\frac{k^2n^2z^2}{1\cdot 2}+\frac{k^3n^3z^3}{1\cdot 2\cdot 3}+\frac{k^4n^4z^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.};$$

posito vero lb pro n erit

$$b^z = 1 + \frac{kz}{1} lb + \frac{k^2 z^2}{1\cdot 2}(lb)^2 + \frac{k^3 z^3}{1\cdot 2\cdot 3}(lb)^3 + \frac{k^4 z^4}{1\cdot 2\cdot 3\cdot 4}(lb)^4 + \text{etc.}$$

Cognito ergo valore litterae k ex dato valore basis a quantitas exponentialis quaecunque b^z per seriem infinitam exprimi poterit, cuius termini secundum potestates ipsius z procedant. His expositis ostendamus quoque, quomodo logarithmi per series infinitas explicari possint.

118. Cum sit $a^\omega = 1 + k\omega$ existente ω fractione infinite parva atque ratio inter a et k definiatur per hanc aequationem

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \text{etc.},$$

si a sumatur pro basi logarithmica, erit

$$\omega = l(1 + k\omega) \quad \text{et} \quad i\omega = l(1 + k\omega)^i.$$

Manifestum autem est, quo maior numerus pro i sumatur, eo magis potestatem $(1 + k\omega)^i$ unitatem esse superaturam atque statuendo $i =$ numero infinito valorem potestatis $(1 + k\omega)^i$ ad quemvis numerum unitate maiorem ascendere. Quodsi ergo ponatur

$$(1 + k\omega)^i = 1 + x,$$

erit

$$l(1 + x) = i\omega,$$

unde, cum sit $i\omega$ numerus finitus, logarithmus scilicet numeri $1 + x$, perspicuum est i esse debere numerum infinite magnum; alioquin enim $i\omega$ valorem finitum habere non posset.

119. Cum autem positum sit

$$(1 + k\omega)^i = 1 + x,$$

erit

$$1 + k\omega = (1 + x)^{\frac{1}{i}} \quad \text{et} \quad k\omega = (1 + x)^{\frac{1}{i}} - 1,$$

unde fit

$$i\omega = \frac{i}{k}\left((1 + x)^{\frac{1}{i}} - 1\right).$$

Quia vero est $i\omega = l(1+x)$, erit

$$l(1+x) = \frac{i}{k}(1+x)^{\frac{1}{i}} - \frac{i}{k}$$

posito i numero infinite magno. Est autem

$$(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i\cdot 2i}x^2 + \frac{1(i-1)(2i-1)}{i\cdot 2i\cdot 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i\cdot 2i\cdot 3i\cdot 4i}x^4 + \text{etc.}$$

Ob i autem numerum infinitum erit

$$\frac{i-1}{2i} = \frac{1}{2}, \quad \frac{2i-1}{3i} = \frac{2}{3}, \quad \frac{3i-1}{4i} = \frac{3}{4} \quad \text{etc.};$$

hinc erit

$$i(1+x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}$$

et consequenter

$$l(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right)$$

posita basi logarithmica $= a$ ac denotante k numerum huic basi convenientem, ut scilicet sit

$$a = 1 + \frac{k}{1} + \frac{k^2}{1\cdot 2} + \frac{k^3}{1\cdot 2\cdot 3} + \text{etc.}$$

120. Cum igitur habeamus seriem logarithmo numeri $1+x$ aequalem, eius ope ex data basi a definire poterimus valorem numeri k. Si enim ponamus $1+x = a$, ob $la = 1$ erit

$$1 = \frac{1}{k}\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{etc.}\right)$$

hincque habebitur

$$k = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{etc.},$$

cuius ideo seriei infinitae valor, si ponatur $a = 10$, circiter esse debebit $= 2{,}30258$, quanquam difficulter intelligi potest esse

$$2{,}30258 = \frac{9}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \text{etc.},$$

quoniam huius seriei termini continuo fiunt maiores neque adeo aliquot terminis sumendis summa vero propinqua haberi potest; cui incommodo mox remedium afferetur.

121. Quoniam igitur est

$$l(1+x) = \frac{1}{k}\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right),$$

erit posito x negativo

$$l(1-x) = -\frac{1}{k}\left(\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.}\right).$$

Subtrahatur series posterior a priori; erit

$$l(1+x) - l(1-x) = l\frac{1+x}{1-x} = \frac{2}{k}\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \text{etc.}\right).$$

Nunc ponatur

$$\frac{1+x}{1-x} = a,$$

ut sit

$$x = \frac{a-1}{a+1};$$

ob $la = 1$ erit

$$k = 2\left(\frac{a-1}{a+1} + \frac{(a-1)^3}{3(a+1)^3} + \frac{(a-1)^5}{5(a+1)^5} + \text{etc.}\right),$$

ex qua aequatione valor numeri k ex basi a inveniri poterit. Si ergo basis a ponatur $= 10$, erit

$$k = 2\left(\frac{9}{11} + \frac{9^3}{3\cdot 11^3} + \frac{9^5}{5\cdot 11^5} + \frac{9^7}{7\cdot 11^7} + \text{etc.}\right),$$

cuius seriei termini sensibiliter decrescunt ideoque mox valorem pro k satis propinquum exhibent.

122. Quoniam ad systema logarithmorum condendum basin a pro lubitu accipere licet, ea ita assumi poterit, ut fiat $k = 1$. Ponamus ergo esse $k = 1$ eritque per seriem supra (§ 116) inventam

$$a = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

qui termini, si in fractiones decimales convertantur atque actu addantur, praebebunt hunc valorem pro a

$$2{,}71828\,18284\,59045\,23536\,028,$$

cuius ultima adhuc nota veritati est consentanea.

Quodsi iam ex hac basi logarithmi construantur, ii vocari solent logarithmi *naturales* seu *hyperbolici*, quoniam quadratura hyperbolae per istiusmodi logarithmos exprimi potest. Ponamus autem brevitatis gratia pro numero hoc 2,71828 18284 59 etc. constanter litteram

$$e,$$

quae ergo denotabit basin logarithmorum naturalium seu hyperbolicorum [1]), cui respondet valor litterae $k = 1$; sive haec littera e quoque exprimet summam huius seriei

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{etc. in infinitum.}$$

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem, ut numeri $1+\omega$ logarithmus sit $=\omega$ denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k=1$ innotescat, omnium numerorum logarithmi hyperbolici exhiberi poterunt. Erit ergo posita e pro numero supra invento perpetuo

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Ipsi vero logarithmi hyperbolici ex his seriebus invenientur, quibus est

$$l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \text{etc.}$$

et

$$l\,\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \text{etc.},$$

---

1) Hac littera e EULERUS iam a. 1728 basin logarithmorum naturalium designaverat; confer G. ENESTRÖM, Biblioth. Mathem. 14_3, p. 81, et 5_3, p. 310. Haec eadem significatio occurrit constanter in libro, qui inscribitur *Mechanica sive motus scientia analytice exposita,* Petropoli 1736; *LEONHARDI EULERI Opera omnia*, series II, vol. 1 et 2. In Commentatione quidem 28 (indicis ENESTROEMIANI): *Specimen de constructione aequationum differentialium etc.,* Comment. acad. sc. Petrop. **6** (1732/3), 1738, *LEONHARDI EULERI Opera omnia*, series I, vol. 20, p. 1, invenitur (uti etiam antea) loco litterae e littera c, haec autem dissertatio iam a. 1733 scripta est. A. K.

quae series vehementer convergunt, si pro x statuatur fractio valde parva. Ita ex serie posteriori facili negotio inveniuntur logarithmi numerorum unitate non multo maiorum. Posito namque $x = \frac{1}{5}$ erit

$$l\frac{6}{4} = l\frac{3}{2} = \frac{2}{1\cdot 5} + \frac{2}{3\cdot 5^3} + \frac{2}{5\cdot 5^5} + \frac{2}{7\cdot 5^7} + \text{etc.}$$

et facto $x = \frac{1}{7}$ erit

$$l\frac{4}{3} = \frac{2}{1\cdot 7} + \frac{2}{3\cdot 7^3} + \frac{2}{5\cdot 7^5} + \frac{2}{7\cdot 7^7} + \text{etc.}$$

et facto $x = \frac{1}{9}$ erit

$$l\frac{5}{4} = \frac{2}{1\cdot 9} + \frac{2}{3\cdot 9^3} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} + \text{etc.}$$

Ex logarithmis vero harum fractionum reperientur logarithmi numerorum integrorum; erit enim ex natura logarithmorum

$$l\frac{3}{2} + l\frac{4}{3} = l2,$$

tum

$$l\frac{3}{2} + l2 = l3 \quad \text{et} \quad 2l2 = l4,$$

porro

$$l\frac{5}{4} + l4 = l5, \quad l2 + l3 = l6, \quad 3l2 = l8, \quad 2l3 = l9 \quad \text{et} \quad l2 + l5 = l10.$$

EXEMPLUM

Hinc logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

$l1 = 0{,}00000\,00000\,00000\,00000\,00000$

$l2 = 0{,}69314\,71805\,59945\,30941\,72321$

$l3 = 1{,}09861\,22886\,68109\,69139\,52452$

$l4 = 1{,}38629\,43611\,19890\,61883\,44642$

$l5 = 1{,}60943\,79124\,34100\,37460\,07593$

$$l\,6 = 1{,}79175\,94692\,28055\,00081\,24774$$ [1])

$$l\,7 = 1{,}94591\,01490\,55313\,30510\,53527$$ [2])

$$l\,8 = 2{,}07944\,15416\,79835\,92825\,16964$$

$$l\,9 = 2{,}19722\,45773\,36219\,38279\,04905$$

$$l\,10 = 2{,}30258\,50929\,94045\,68401\,79915$$ [3])

Hi scilicet logarithmi omnes ex superioribus tribus seriebus sunt deducti praeter $l\,7$, quem hoc compendio sum assecutus. Posui nimirum in serie posteriori $x = \frac{1}{99}$ sicque obtinui

$$l\,\frac{100}{98} = l\,\frac{50}{49} = 0{,}02020\,27073\,17519\,44840\,80453,$$ [4])

qui subtractus a

$$l\,50 = 2\,l\,5 + l\,2 = 3{,}91202\,30054\,28146\,05861\,87508$$

relinquit $l\,49$, cuius semissis dat $l\,7$.

124. Ponatur logarithmus hyperbolicus ipsius $1 + x$ seu $l(1 + x) = y$; erit

$$y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}$$

Sumpto autem numero a pro basi logarithmica sit numeri eiusdem $1 + x$ logarithmus $= v$; erit, ut vidimus,

$$v = \frac{1}{k}\left(x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}\right) = \frac{y}{k}$$

hincque

$$k = \frac{y}{v};$$

1) In editione principe ultimae figurae sunt 00081 24773. Correxit A. K.

2) In editione principe ultimae figurae sunt 30510 54639. Correxit A. K.

3) In editione principe ultimae figurae sunt 68401 79914. Correxit A. K.

4) In editione principe ultimae figurae sunt 44840 78230. Correxit A. K.

ex quo commodissime valor ipsius k basi a respondens ita definitur, ut sit aequalis cuiusvis numeri logarithmo hyperbolico diviso per logarithmum eiusdem numeri ex basi a formati. Posito ergo numero hoc $= a$ erit $v = 1$ hincque fit $k =$ logarithmo hyperbolico basis a. In systemate ergo logarithmorum communium, ubi est $a = 10$, erit $k =$ logarithmo hyperbolico ipsius 10, unde fit

$$k = 2{,}30258\,50929\,94045\,68401\,79915\ ^{1)},$$

quem valorem iam supra satis prope collegimus. Si ergo singuli logarithmi hyperbolici per hunc numerum k dividantur vel, quod eodem redit, multiplicentur per hanc fractionem decimalem

$$0{,}43429\,44819\,03251\,82765\,11289\ ^{1)},$$

prodibunt logarithmi vulgares basi $a = 10$ convenientes.

125. Cum sit

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

si ponatur $a^y = e^z$, erit sumptis logarithmis hyperbolicis $yla = z$, quia est $le = 1$; quo valore loco z substituto erit

$$a^y = 1 + \frac{yla}{1} + \frac{y^2(la)^2}{1 \cdot 2} + \frac{y^3(la)^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

unde quaelibet quantitas exponentialis ope logarithmorum hyperbolicorum per seriem infinitam explicari potest.

Tum vero denotante i numerum infinite magnum tam quantitates exponentiales quam logarithmi per potestates exponi possunt. Erit enim

1) Vide notam 3 p. 130 atque notam 1 p. 120. A. K.

$$e^z = \left(1 + \frac{z}{i}\right)^i$$

hincque

$$a^y = \left(1 + \frac{yla}{i}\right)^i,$$

deinde pro logarithmis hyperbolicis habetur

$$l(1+x) = i((1+x)^{\frac{1}{i}} - 1).$$

De cetero logarithmorum hyperbolicorum usus in Calculo integrali fusius demonstrabitur.

CAPUT VIII

DE QUANTITATIBUS TRANSCENDENTIBUS EX CIRCULO ORTIS

126. Post logarithmos et quantitates exponentiales considerari debent arcus circulares eorumque sinus et cosinus, quia non solum aliud quantitatum transcendentium genus constituunt, sed etiam ex ipsis logarithmis et exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo radium circuli seu sinum totum esse $=1$ atque satis liquet peripheriam huius circuli in numeris rationalibus exacte exprimi non posse; per approximationes autem inventa est semicircumferentia huius circuli esse[1])

$$= 3{,}14159\,26535\,89793\,23846\,26433\,83279\,50288\,41971\,69399\,37510$$
$$58209\,74944\,59230\,78164\,06286\,20899\,86280\,34825\,34211\,70679$$
$$82148\,08651\,32823\,06647\,09384\,46 +,$$

---

1) Hunc valorem EULERUS deprompsit e dissertatione, quam scripsit TH. F. DE LAGNY (1660—1734): *Mémoire sur la quadrature du cercle, et sur la mesure de tout arc, tout secteur et tout segment donné*, Mém. de l'acad. d. sc. de Paris (1719), 1721, p. 135. Confer L. EULERI Commentationem 74 (indicis ENESTROEMIANI): *De variis modis circuli quadraturam numeris proxime exprimendi*, Comment. acad. sc. Petrop. 9 (1737), 1744, p. 222; *LEONHARDI EULERI Opera omnia*, series I, vol. 14.

Figuram centesimam decimam tertiam, qualem DE LAGNY dederat et EULERUS tradiderat, falsam esse, quippe quae 8, non 7, esse debeat, adnotavit G. DE VEGA (1756—1802) in libro, qui inscribitur *Thesaurus logarithmorum completus*, Lipsiae 1794, p. 633. A. K.

pro quo numero brevitatis ergo scribam

$$\pi,$$

ita ut sit $\pi =$ semicircumferentiae circuli, cuius radius $= 1$, seu π erit longitudo arcus 180 graduum.[1])

127. Denotante z arcum huius circuli quemcunque, cuius radium perpetuo assumo $= 1$, huius arcus z considerari potissimum solent sinus et cosinus. Sinum autem arcus z in posterum hoc modo indicabo

$$\sin. A.\, z \text{ seu tantum } \sin. z,$$

cosinum vero hoc modo

$$\cos. A.\, z \text{ seu tantum } \cos. z.$$

Ita, cum π sit arcus 180^0, erit

$$\sin. 0\pi = 0, \quad \cos. 0\pi = 1$$

1) Ante EULERUM huiusmodi rationes non breviter ut nunc signo singulari, id est una littera, designabantur, sed copiose pluribus verbis circumscribebantur. Exstant quidem notabiles exceptiones. Sic apud J. CHR. STURM (1635—1703) in libro, qui inscribitur *Mathesis enucleata*, Norimbergae 1689, legitur p. 81: *Promptum autem hinc est inferre, si diameter alicuius circuli ponatur a, circumferentiam appellari posse e a (quaecumque enim inter eas fuerit ratio, illius nomen potest designari littera e).* Quin etiam W. JONES (1675—1749) rationem circumferentiae ad diametrum a. 1706 ipsa littera π designaverat; vide W. JONES, *Synopsis palmariorum matheseos or new introduction to the mathematics,* London 1706, p. 243. Verumtamen EULERUS primus hunc designandi modum reddidit generalem. Iam in libro, qui inscribitur *Mechanica sive motus scientia analytice exposita*, Petropoli 1736, *LEONHARDI EULERI Opera omnia*, series II, vol. 1 et 2, saepenumero sed non constanter rationem circumferentiae ad diametrum littera π denotaverat. Littera π invenitur praeterea in Commentatione 72 (indicis ENESTROEMIANI): *Variae observationes circa series infinitas*, Comment. acad. sc. Petrop. 9 (1737), 1744, p. 160; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. In Commentatione quidem 74 nota praecedenti laudata uti etiam in nonnullis dissertationibus prioribus EULERUS p loco π scripsit, sed inde ab eo tempore usus litterae π praevalebat et mox fiebat omnino generalis, praesertim cum haec *Introductio* edita esset.

De historia numeri π et omnino de historia quadraturae circuli vide F. RUDIO, *ARCHIMEDES, HUYGENS, LAMBERT, LEGENDRE, Vier Abhandlungen über die Kreismessung etc.,* Leipzig 1892, et E. W. HOBSON, *Squaring the circle, a history of the problem,* Cambridge 1913. A. K.

et

$$\sin.\frac{1}{2}\pi = 1, \qquad \cos.\frac{1}{2}\pi = 0,$$

$$\sin.\pi = 0, \qquad \cos.\pi = -1,$$

$$\sin.\frac{3}{2}\pi = -1, \qquad \cos.\frac{3}{2}\pi = 0,$$

$$\sin.2\pi = 0, \qquad \cos.2\pi = 1.$$

Omnes ergo sinus et cosinus intra limites $+1$ et -1 continentur. Erit autem porro

$$\cos.z = \sin.\left(\frac{1}{2}\pi - z\right) \quad \text{et} \quad \sin.z = \cos.\left(\frac{1}{2}\pi - z\right)$$

atque

$$(\sin.z)^2 + (\cos.z)^2 = 1.$$

Praeter has denominationes notandae sunt quoque hae:

$$\text{tang.}\,z,$$

quae denotat tangentem arcus z,

$$\cot.z$$

cotangentem arcus z, constatque esse

$$\text{tang.}\,z = \frac{\sin.z}{\cos.z}$$

et

$$\cot.z = \frac{\cos.z}{\sin.z} = \frac{1}{\text{tang.}\,z},$$

quae omnia ex trigonometria sunt nota.

128. Hinc vero etiam constat, si habeantur duo arcus y et z, fore

$$\sin.(y+z) = \sin.y\cos.z + \cos.y\sin.z$$

et

$$\cos.(y+z) = \cos.y\cos.z - \sin.y\sin.z$$

itemque

$$\sin.(y-z) = \sin.y\cos.z - \cos.y\sin.z$$

et

$$\cos.(y-z) = \cos.y\cos.z + \sin.y\sin.z.$$

Hinc loco y substituendo arcus $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$ etc. erit

$\sin.\left(\frac{1}{2}\pi+z\right) = +\cos. z$	$\sin.\left(\frac{1}{2}\pi-z\right) = +\cos. z$
$\cos.\left(\frac{1}{2}\pi+z\right) = -\sin. z$	$\cos.\left(\frac{1}{2}\pi-z\right) = +\sin. z$
$\sin. (\pi+z) = -\sin. z$	$\sin. (\pi-z) = +\sin. z$
$\cos. (\pi+z) = -\cos. z$	$\cos. (\pi-z) = -\cos. z$
$\sin.\left(\frac{3}{2}\pi+z\right) = -\cos. z$	$\sin.\left(\frac{3}{2}\pi-z\right) = -\cos. z$
$\cos.\left(\frac{3}{2}\pi+z\right) = +\sin. z$	$\cos.\left(\frac{3}{2}\pi-z\right) = -\sin. z$
$\sin. (2\pi+z) = +\sin. z$	$\sin. (2\pi-z) = -\sin. z$
$\cos. (2\pi+z) = +\cos. z$	$\cos. (2\pi-z) = +\cos. z$

Si ergo n denotet numerum integrum quemcunque, erit

$\sin.\left(\frac{4n+1}{2}\pi+z\right) = +\cos. z$	$\sin.\left(\frac{4n+1}{2}\pi-z\right) = +\cos. z$
$\cos.\left(\frac{4n+1}{2}\pi+z\right) = -\sin. z$	$\cos.\left(\frac{4n+1}{2}\pi-z\right) = +\sin. z$
$\sin.\left(\frac{4n+2}{2}\pi+z\right) = -\sin. z$	$\sin.\left(\frac{4n+2}{2}\pi-z\right) = +\sin. z$
$\cos.\left(\frac{4n+2}{2}\pi+z\right) = -\cos. z$	$\cos.\left(\frac{4n+2}{2}\pi-z\right) = -\cos. z$
$\sin.\left(\frac{4n+3}{2}\pi+z\right) = -\cos. z$	$\sin.\left(\frac{4n+3}{2}\pi-z\right) = -\cos. z$
$\cos.\left(\frac{4n+3}{2}\pi+z\right) = +\sin. z$	$\cos.\left(\frac{4n+3}{2}\pi-z\right) = -\sin. z$
$\sin.\left(\frac{4n+4}{2}\pi+z\right) = +\sin. z$	$\sin.\left(\frac{4n+4}{2}\pi-z\right) = -\sin. z$
$\cos.\left(\frac{4n+4}{2}\pi+z\right) = +\cos. z$	$\cos.\left(\frac{4n+4}{2}\pi-z\right) = +\cos. z$

Quae formulae verae sunt, sive n sit numerus affirmativus sive negativus integer.

129. Sit

$$\sin. z = p \quad \text{et} \quad \cos. z = q;$$

erit

$$pp + qq = 1;$$

et

$$\sin. y = m, \quad \cos. y = n,$$

ut sit quoque

$$mm + nn = 1;$$

arcuum ex his compositorum sinus et cosinus ita se habebunt:

$\sin. z = p$	$\cos. z = q$
$\sin. (y+z) = mq + np$	$\cos. (y+z) = nq - mp$
$\sin. (2y+z) = 2mnq + (nn-mm)p$	$\cos. (2y+z) = (nn-mm)q - 2mnp$
$\sin. (3y+z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$	$\cos. (3y+z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$
etc.	etc.

Arcus isti

$$z, \quad y+z, \quad 2y+z, \quad 3y+z \quad \text{etc.}$$

in arithmetica progressione progrediuntur, eorum vero tam sinus quam cosinus progressionem recurrentem constituunt, qualis ex denominatore

$$1 - 2nx + (mm + nn)xx$$

oritur; est enim

$$\sin. (2y+z) = 2n \sin. (y+z) - (mm+nn) \sin. z$$

sive

$$\sin. (2y+z) = 2\cos. y \sin. (y+z) - \sin. z$$

atque simili modo

$$\cos. (2y+z) = 2\cos. y \cos. (y+z) - \cos. z.$$

Eodem modo erit porro

$$\sin. (3y+z) = 2\cos. y \sin. (2y+z) - \sin. (y+z)$$

et

$$\cos. (3y+z) = 2\cos. y \cos. (2y+z) - \cos. (y+z)$$

itemque

$$\sin.(4y+z) = 2\cos.y\,\sin.(3y+z) - \sin.(2y+z)$$

et

$$\cos.(4y+z) = 2\cos.y\,\cos.(3y+z) - \cos.(2y+z)$$

etc.

Cuius legis beneficio arcuum in progressione arithmetica progredientium tam sinus quam cosinus, quousque libuerit, expedite formari possunt.

130. Cum sit

$$\sin.(y+z) = \sin.y\,\cos.z + \cos.y\,\sin.z$$

atque

$$\sin.(y-z) = \sin.y\,\cos.z - \cos.y\,\sin.z,$$

erit his expressionibus vel addendis vel subtrahendis

$$\sin.y\,\cos.z = \frac{\sin.(y+z)+\sin.(y-z)}{2},$$

$$\cos.y\,\sin.z = \frac{\sin.(y+z)-\sin.(y-z)}{2}.$$

Quia porro est

$$\cos.(y+z) = \cos.y\,\cos.z - \sin.y\,\sin.z$$

atque

$$\cos.(y-z) = \cos.y\,\cos.z + \sin.y\,\sin.z,$$

erit pari modo

$$\cos.y\,\cos.z = \frac{\cos.(y-z)+\cos.(y+z)}{2},$$

$$\sin.y\,\sin.z = \frac{\cos.(y-z)-\cos.(y+z)}{2}.$$

Sit

$$y = z = \frac{1}{2}v;$$

erit ex his postremis formulis

$$\left(\cos.\frac{1}{2}v\right)^2 = \frac{1+\cos.v}{2} \quad \text{et} \quad \cos.\frac{1}{2}v = \sqrt{\frac{1+\cos.v}{2}},$$

$$\left(\sin.\frac{1}{2}v\right)^2 = \frac{1-\cos.v}{2} \quad \text{et} \quad \sin.\frac{1}{2}v = \sqrt{\frac{1-\cos.v}{2}},$$

unde ex dato cosinu cuiusque anguli reperiuntur eius semissis sinus et cosinus.

131. Ponatur arcus

$$y + z = a \quad \text{et} \quad y - z = b;$$

erit

$$y = \frac{a+b}{2} \quad \text{et} \quad z = \frac{a-b}{2},$$

quibus in superioribus formulis substitutis habebuntur hae aequationes, tanquam totidem theoremata:

$$\sin. a + \sin. b = 2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2},$$

$$\sin. a - \sin. b = 2 \cos. \frac{a+b}{2} \sin. \frac{a-b}{2},$$

$$\cos. a + \cos. b = 2 \cos. \frac{a+b}{2} \cos. \frac{a-b}{2},$$

$$\cos. b - \cos. a = 2 \sin. \frac{a+b}{2} \sin. \frac{a-b}{2}.$$

Ex his porro nascuntur ope divisionis haec theoremata

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \operatorname{tang.} \frac{a+b}{2} \cot. \frac{a-b}{2} = \frac{\operatorname{tang.} \frac{a+b}{2}}{\operatorname{tang.} \frac{a-b}{2}},$$

$$\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \operatorname{tang.} \frac{a+b}{2},$$

$$\frac{\sin. a + \sin. b}{\cos. b - \cos. a} = \cot. \frac{a-b}{2},$$

$$\frac{\sin. a - \sin. b}{\cos. a + \cos. b} = \operatorname{tang.} \frac{a-b}{2},$$

$$\frac{\sin. a - \sin. b}{\cos. b - \cos. a} = \cot. \frac{a+b}{2},$$

$$\frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \cot. \frac{a+b}{2} \cot. \frac{a-b}{2}.$$

Ex his denique deducuntur ista theoremata

$$\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \frac{\cos. b - \cos. a}{\sin. a - \sin. b},$$

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} \times \frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \left(\cot. \frac{a-b}{2}\right)^2,$$

$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} \times \frac{\cos. b - \cos. a}{\cos. a + \cos. b} = \left(\operatorname{tang.} \frac{a+b}{2}\right)^2.$$

132. Cum sit
$$(\sin. z)^2 + (\cos. z)^2 = 1,$$
erit factoribus sumendis
$$(\cos. z + \sqrt{-1} \cdot \sin. z)(\cos. z - \sqrt{-1} \cdot \sin. z) = 1,$$
qui factores, etsi imaginarii, tamen ingentem praestant usum in arcubus combinandis et multiplicandis. Quaeratur enim productum horum factorum
$$(\cos. z + \sqrt{-1} \cdot \sin. z)(\cos. y + \sqrt{-1} \cdot \sin. y)$$
ac reperietur
$$\cos. y \cos. z - \sin. y \sin. z + \sqrt{-1} \cdot (\cos. y \sin. z + \sin. y \cos. z).$$
Cum autem sit
$$\cos. y \cos. z - \sin. y \sin. z = \cos. (y + z)$$
et
$$\cos. y \sin. z + \sin. y \cos. z = \sin. (y + z),$$
erit hoc productum
$$(\cos. y + \sqrt{-1} \cdot \sin. y)(\cos. z + \sqrt{-1} \cdot \sin. z) = \cos. (y + z) + \sqrt{-1} \cdot \sin. (y + z)$$
et simili modo
$$(\cos. y - \sqrt{-1} \cdot \sin. y)(\cos. z - \sqrt{-1} \cdot \sin. z) = \cos. (y + z) - \sqrt{-1} \cdot \sin. (y + z),$$
item
$$(\cos. x \pm \sqrt{-1} \cdot \sin. x)(\cos. y \pm \sqrt{-1} \cdot \sin. y)(\cos. z \pm \sqrt{-1} \cdot \sin. z)$$
$$= \cos. (x + y + z) \pm \sqrt{-1} \cdot \sin. (x + y + z).$$

133. Hinc itaque sequitur fore
$$(\cos. z \pm \sqrt{-1} \cdot \sin. z)^2 = \cos. 2z \pm \sqrt{-1} \cdot \sin. 2z$$
et
$$(\cos. z \pm \sqrt{-1} \cdot \sin. z)^3 = \cos. 3z \pm \sqrt{-1} \cdot \sin. 3z$$
ideoque generaliter erit[1])
$$(\cos. z \pm \sqrt{-1} \cdot \sin. z)^n = \cos. nz \pm \sqrt{-1} \cdot \sin. nz.$$

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1) Vide A. DE MOIVRE, *Miscellanea analytica de seriebus et quadraturis*, Londini 1730, p. 1.
A. K.

Unde ob signorum ambiguitatem erit

$$\cos. nz = \frac{(\cos. z + \sqrt{-1}\cdot\sin. z)^n + (\cos. z - \sqrt{-1}\cdot\sin. z)^n}{2}$$

et

$$\sin. nz = \frac{(\cos. z + \sqrt{-1}\cdot\sin. z)^n - (\cos. z - \sqrt{-1}\cdot\sin. z)^n}{2\sqrt{-1}}.$$

Evolutis ergo binomiis hisce erit per series

$$\begin{aligned}\cos. nz &= (\cos. z)^n - \frac{n(n-1)}{1\cdot 2}(\cos. z)^{n-2}(\sin. z)^2 \\ &+ \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(\cos. z)^{n-4}(\sin. z)^4 \\ &- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(\cos. z)^{n-6}(\sin. z)^6 \\ &+ \text{etc.}\end{aligned}$$

et

$$\begin{aligned}\sin. nz &= \frac{n}{1}(\cos. z)^{n-1}\sin. z - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\cos. z)^{n-3}(\sin. z)^3 \\ &+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}(\cos. z)^{n-5}(\sin. z)^5 \\ &- \text{etc.}\end{aligned}$$

134. Sit arcus z infinite parvus; erit $\sin. z = z$ et $\cos. z = 1$; sit autem n numerus infinite magnus, ut sit arcus nz finitae magnitudinis, puta $nz = v$; ob $\sin. z = z = \frac{v}{n}$ erit

$$\cos. v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

et

$$\sin. v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{etc.}$$

Dato ergo arcu v ope harum serierum eius sinus et cosinus inveniri poterunt; quarum formularum usus quo magis pateat, ponamus arcum v esse ad quadrantem seu 90^0 ut m ad n seu esse $v = \frac{m}{n} \cdot \frac{\pi}{2}$. Quia nunc valor ipsius π constat, si is ubique substituatur, prodibit[1])

$$\sin. A. \frac{m}{n} 90^0$$

$$\begin{aligned}
= &+ \frac{m}{n} \cdot 1{,}57079\,63267\,94896\,61923\,13216\,916 \\
&- \frac{m^3}{n^3} \cdot 0{,}64596\,40975\,06246\,25365\,57565\,639 \\
&+ \frac{m^5}{n^5} \cdot 0{,}07969\,26262\,46167\,04512\,05055\,495 \\
&- \frac{m^7}{n^7} \cdot 0{,}00468\,17541\,35318\,68810\,06854\,639 \\
&+ \frac{m^9}{n^9} \cdot 0{,}00016\,04411\,84787\,35982\,18726\,609 \\
&- \frac{m^{11}}{n^{11}} \cdot 0{,}00000\,35988\,43235\,21208\,53404\,585 \\
&+ \frac{m^{13}}{n^{13}} \cdot 0{,}00000\,00569\,21729\,21967\,92681\,178 \\
&- \frac{m^{15}}{n^{15}} \cdot 0{,}00000\,00006\,68803\,51098\,11467\,232 \\
&+ \frac{m^{17}}{n^{17}} \cdot 0{,}00000\,00000\,06066\,93573\,11061\,957 \\
&- \frac{m^{19}}{n^{19}} \cdot 0{,}00000\,00000\,00043\,77065\,46731\,374 \\
&+ \frac{m^{21}}{n^{21}} \cdot 0{,}00000\,00000\,00000\,25714\,22892\,860 \\
&- \frac{m^{23}}{n^{23}} \cdot 0{,}00000\,00000\,00000\,00125\,38995\,405 \\
&+ \frac{m^{25}}{n^{25}} \cdot 0{,}00000\,00000\,00000\,00000\,51564\,552 \\
&- \frac{m^{27}}{n^{27}} \cdot 0{,}00000\,00000\,00000\,00000\,00181\,240 \\
&+ \frac{m^{29}}{n^{29}} \cdot 0{,}00000\,00000\,00000\,00000\,00000\,551
\end{aligned}$$

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1) In editione principe ultimae figurae fractionum decimalium, quae ad $\frac{m^3}{n^3}, \frac{m^5}{n^5}, \frac{m^7}{n^7}, \ldots \frac{m^{25}}{n^{25}}, \frac{m^{27}}{n^{27}}, \frac{m^{29}}{n^{29}}$ pertinent, ita se habent:

... 636, ... 488, ... 632, ... 605, ... 580, ... 171, ... 224,
... 950, ... 370, ... 856, ... 403, ... 550, ... 239, ... 549.

Correxit A. K.

atque [1])

$$\cos. A. \frac{m}{n} 90^0$$

$$
\begin{aligned}
&= + \quad 1{,}00000\,00000\,00000\,00000\,00000\,000 \\
&- \frac{m^2}{n^2} \cdot 1{,}23370\,05501\,36169\,82735\,43113\,750 \\
&+ \frac{m^4}{n^4} \cdot 0{,}25366\,95079\,01048\,01363\,65633\,664 \\
&- \frac{m^6}{n^6} \cdot 0{,}02086\,34807\,63352\,96087\,30516\,372 \\
&+ \frac{m^8}{n^8} \cdot 0{,}00091\,92602\,74839\,42658\,02417\,162 \\
&- \frac{m^{10}}{n^{10}} \cdot 0{,}00002\,52020\,42373\,06060\,54810\,530 \\
&+ \frac{m^{12}}{n^{12}} \cdot 0{,}00000\,04710\,87477\,88181\,71503\,670 \\
&- \frac{m^{14}}{n^{14}} \cdot 0{,}00000\,00063\,86603\,08379\,18522\,411 \\
&+ \frac{m^{16}}{n^{16}} \cdot 0{,}00000\,00000\,65659\,63114\,97947\,236 \\
&- \frac{m^{18}}{n^{18}} \cdot 0{,}00000\,00000\,00529\,44002\,00734\,624 \\
&+ \frac{m^{20}}{n^{20}} \cdot 0{,}00000\,00000\,00003\,43773\,91790\,986 \\
&- \frac{m^{22}}{n^{22}} \cdot 0{,}00000\,00000\,00000\,01835\,99165\,216 \\
&+ \frac{m^{24}}{n^{24}} \cdot 0{,}00000\,00000\,00000\,00008\,20675\,330 \\
&- \frac{m^{26}}{n^{26}} \cdot 0{,}00000\,00000\,00000\,00000\,03115\,285 \\
&+ \frac{m^{28}}{n^{28}} \cdot 0{,}00000\,00000\,00000\,00000\,00010\,168 \\
&- \frac{m^{30}}{n^{30}} \cdot 0{,}00000\,00000\,00000\,00000\,00000\,029.
\end{aligned}
$$

1) In editione principe ultimae figurae fractionum decimalium, quae ad $\frac{m^2}{n^2}, \frac{m^4}{n^4}, \frac{m^6}{n^6}, \ldots \frac{m^{26}}{n^{26}}, \frac{m^{28}}{n^{28}}, \frac{m^{30}}{n^{30}}$ pertinent, ita se habent:

... 745, ... 659, ... 364, ... 158, ... 526, ... 665, ... 408, ... 230, ... 620, ... 981, ... 212, ... 327, ... 285 (quae ergo figurae accuratae sunt) ... 165, ... 026.

Correxit A. K.

Cum igitur sufficiat sinus et cosinus angulorum ad 45^0 nosse, fractio $\frac{m}{n}$ semper minor erit quam $\frac{1}{2}$ hincque etiam ob potestates fractionis $\frac{m}{n}$ series exhibitae maxime convergent, ita ut plerumque aliquot tantum termini sufficiant, praecipue si sinus et cosinus non ad tot figuras desiderentur.

135. Inventis sinibus et cosinibus inveniri quidem possunt tangentes et cotangentes per analogias consuetas; at quia in huiusmodi ingentibus numeris multiplicatio et divisio vehementer est molesta, peculiari modo eas exprimere convenit. Erit ergo

$$\operatorname{tang.} v = \frac{\sin. v}{\cos. v} = \frac{v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}}{1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}}$$

et

$$\cot. v = \frac{\cos. v}{\sin. v} = \frac{1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \frac{v^6}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}}{v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{v^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}}.$$

Si iam sit arcus $v = \frac{m}{n} 90^0$, erit eodem modo quo ante

$\text{tang. A. } \frac{m}{n} 90^0$

$= + \frac{2mn}{nn - mm} \cdot 0,6366197723676$ [1]

$+ \frac{m}{n} \cdot 0,2975567820597$

$+ \frac{m^3}{n^3} \cdot 0,0186886502773$

$+ \frac{m^5}{n^5} \cdot 0,0018424752034$

$+ \frac{m^7}{n^7} \cdot 0,0001975800715$ [1]

$+ \frac{m^9}{n^9} \cdot 0,0000216977373$ [2]

$+ \frac{m^{11}}{n^{11}} \cdot 0,0000024011370$

$+ \frac{m^{13}}{n^{13}} \cdot 0,0000002664133$ [1]

$+ \frac{m^{15}}{n^{15}} \cdot 0,0000000295865$ [1]

$+ \frac{m^{17}}{n^{17}} \cdot 0,0000000032868$ [1]

$+ \frac{m^{19}}{n^{19}} \cdot 0,0000000003652$ [1]

$+ \frac{m^{21}}{n^{21}} \cdot 0,0000000000406$ [1]

$+ \frac{m^{23}}{n^{23}} \cdot 0,0000000000045$

$+ \frac{m^{25}}{n^{25}} \cdot 0,0000000000005$

$\text{cot. A. } \frac{m}{n} 90^0$

$= + \frac{n}{m} \cdot 0,6366197723676$ [1]

$- \frac{4mn}{4nn - mm} \cdot 0,3183098861838$ [1]

$- \frac{m}{n} \cdot 0,2052888894145$

$- \frac{m^3}{n^3} \cdot 0,0065510747882$

$- \frac{m^5}{n^5} \cdot 0,0003450292554$

$- \frac{m^7}{n^7} \cdot 0,0000202791061$ [1]

$- \frac{m^9}{n^9} \cdot 0,0000012366527$

$- \frac{m^{11}}{n^{11}} \cdot 0,0000000764959$

$- \frac{m^{13}}{n^{13}} \cdot 0,0000000047597$

$- \frac{m^{15}}{n^{15}} \cdot 0,0000000002969$

$- \frac{m^{17}}{n^{17}} \cdot 0,0000000000185$

$- \frac{m^{19}}{n^{19}} \cdot 0,0000000000012$ [1]

quarum serierum ratio infra [§ 198 a] fusius exponetur.

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1) In editione principe ultima figura huius fractionis decimalis est unitate minor. Correxit A. K.

2) In editione principe tres ultimae figurae sunt 245. Correxit A. K.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum sinus et cosinus, inde simul omnium angulorum maiorum sinus et cosinus haberi. Verum si tantum angulorum 30^0 minorum habeantur sinus et cosinus, ex iis per solam additionem et subtractionem omnium angulorum maiorum sinus et cosinus inveniri possunt. Cum enim sit

$$\sin. 30^0 = \frac{1}{2},$$

erit posito $y = 30^0$ ex § 130

$$\cos. z = \sin.(30 + z) + \sin.(30 - z)$$

et

$$\sin. z = \cos.(30 - z) - \cos.(30 + z)$$

ideoque ex sinibus et cosinibus angulorum z et $30 - z$ reperiuntur

$$\sin.(30 + z) = \cos. z - \sin.(30 - z)$$

et

$$\cos.(30 + z) = \cos.(30 - z) - \sin. z,$$

unde sinus et cosinus angulorum a 30^0 ad 60^0 hincque omnes maiores definiuntur.

137. In tangentibus et cotangentibus simile subsidium usu venit. Cum enim sit

$$\text{tang.}(a + b) = \frac{\text{tang.}\, a + \text{tang.}\, b}{1 - \text{tang.}\, a\ \text{tang.}\, b},$$

erit

$$\text{tang.}\, 2a = \frac{2\ \text{tang.}\, a}{1 - \text{tang.}\, a\ \text{tang.}\, a} \quad \text{et} \quad \cot. 2a = \frac{\cot.\, a - \text{tang.}\, a}{2},$$

unde ex tangentibus et cotangentibus arcuum 30^0 minorum inveniuntur cotangentes usque ad 60^0.

Sit iam $a = 30 - b$; erit $2a = 60 - 2b$ et $\cot. 2a = \text{tang.}(30 + 2b)$; erit ergo

$$\text{tang.}(30 + 2b) = \frac{\cot.(30 - b) - \text{tang.}(30 - b)}{2},$$

unde etiam tangentes arcuum 30^0 maiorum obtinentur.

Secantes autem et cosecantes ex tangentibus per solam subtractionem inveniuntur; est enim

$$\text{cosec.}\, z = \text{cot.}\, \frac{1}{2} z - \text{cot.}\, z$$

et hinc

$$\text{sec.}\, z = \text{cot.} \left(45^0 - \frac{1}{2} z\right) - \text{tang.}\, z.$$

Ex his ergo luculenter perspicitur, quomodo canones sinuum construi potuerint.

138. Ponatur denuo in formulis § 133 arcus z infinite parvus et sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$ et $z = \frac{v}{i}$, unde $\text{sin.}\, z = \frac{v}{i}$ et $\text{cos.}\, z = 1$; his substitutis fit

$$\text{cos.}\, v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2}$$

atque

$$\text{sin.}\, v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$

In capite autem praecedente vidimus esse

$$\left(1 + \frac{z}{i}\right)^i = e^z$$

denotante e basin logarithmorum hyperbolicorum; scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit

$$\text{cos.}\, v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

et

$$\text{sin.}\, v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur, quomodo quantitates exponentiales imaginariae ad sinus et cosinus arcuum realium reducantur.[1]) Erit vero

1) Has celeberrimas formulas, quas ab inventore *Formulas* EULERIANAS nominare solemus, EULERUS distincte primum exposuit in Commentatione 61 (indicis ENESTROEMIANI): *De summis*

$$e^{+v\sqrt{-1}} = \cos. v + \sqrt{-1} \cdot \sin. v$$

et

$$e^{-v\sqrt{-1}} = \cos. v - \sqrt{-1} \cdot \sin. v.$$

139. Sit iam in iisdem formulis § 133 n numerus infinite parvus seu $n = \frac{1}{i}$ existente i numero infinite magno; erit

$$\cos. nz = \cos. \frac{z}{i} = 1 \quad \text{et} \quad \sin. nz = \sin. \frac{z}{i} = \frac{z}{i};$$

arcus enim evanescentis $\frac{z}{i}$ sinus est ipsi aequalis, cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} + (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2}$$

et

$$\frac{z}{i} = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} - (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Sumendis autem logarithmis hyperbolicis supra (§ 125) ostendimus esse

$$l(1+x) = i(1+x)^{\frac{1}{i}} - i \quad \text{seu} \quad y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$$

posito y loco $1 + x$. Nunc igitur posito loco y partim $\cos. z + \sqrt{-1} \cdot \sin. z$ partim $\cos. z - \sqrt{-1} \cdot \sin. z$ prodibit

serierum reciprocarum ex potestatibus numerorum naturalium ortarum, Miscellanea Berolin. 7, 1743, p. 172; *Leonhardi Euleri Opera omnia,* series I, vol. 14. Iam antea quidem cum amico Chr. Goldbach (1690—1764) formulas huc pertinentes, partim speciales partim generaliores, communicaverat. Sic in epistola d. 9. Dec. 1741 scripta invenitur haec formula

$$\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2} = \text{Cos. Arc. } l\,2$$

et in epistola d. 8. Maii 1742 scripta haec

$$a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = 2 \text{ Cos. Arc. } pla.$$

Vide *Correspondance math. et phys. publiée par P. H. Fuss,* St.-Pétersbourg 1843, t. I, p. 110 et 123; *Leonhardi Euleri Opera omnia,* series III. Confer etiam Commentationem 170 nota 1 p. 35 laudatam, imprimis § 90 et 91. A. K.

$$1 = \frac{1 + \frac{1}{i} l(\cos. z + \sqrt{-1} \cdot \sin. z) + 1 + \frac{1}{i} l(\cos. z - \sqrt{-1} \cdot \sin. z)}{2} = 1$$

ob logarithmos evanescentes, ita ut hinc nil sequatur. Altera vero aequatio pro sinu suppeditat

$$\frac{z}{i} = \frac{\frac{1}{i} l(\cos. z + \sqrt{-1} \cdot \sin. z) - \frac{1}{i} l(\cos. z - \sqrt{-1} \cdot \sin. z)}{2\sqrt{-1}}$$

ideoque

$$z = \frac{1}{2\sqrt{-1}} l \frac{\cos. z + \sqrt{-1} \cdot \sin. z}{\cos. z - \sqrt{-1} \cdot \sin. z},$$

unde patet, quemadmodum logarithmi imaginarii ad arcus circulares revocentur.

140. Cum sit $\frac{\sin. z}{\cos. z} = \text{tang.}\, z$, arcus z per suam tangentem ita exprimetur, ut sit

$$z = \frac{1}{2\sqrt{-1}} l \frac{1 + \sqrt{-1} \cdot \text{tang.}\, z}{1 - \sqrt{-1} \cdot \text{tang.}\, z}.$$

Supra vero (§ 123) vidimus esse

$$l \frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \text{etc.}$$

Posito ergo $x = \sqrt{-1} \cdot \text{tang.}\, z$ fiet

$$z = \frac{\text{tang.}\, z}{1} - \frac{(\text{tang.}\, z)^3}{3} + \frac{(\text{tang.}\, z)^5}{5} - \frac{(\text{tang.}\, z)^7}{7} + \text{etc.}$$

Si ergo ponamus $\text{tang.}\, z = t$, ut sit z arcus, cuius tangens est t, quem ita indicabimus A. tang. t, ideoque erit

$$z = \text{A. tang.}\, t.$$

Cognita ergo tangente t erit arcus respondens [1])

1) Seriem sequentem primus J. GREGORY (1638—1675) d. 15. Febr. 1671 cum J. COLLINS (1625—1683) communicavit. Vide *Commercium epistolicum J. Collins et aliorum de analysi pro-*

$$z = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

Cum igitur, si tangens t aequetur radio 1, fiat arcus $z =$ arcui 45^0 seu $z = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.},$$

quae est series a LEIBNITZIO primum producta ad valorem peripheriae circuli exprimendum.[1])

141. Quo autem ex hiusmodi serie longitudo arcus circuli expedite definiri possit, perspicuum est pro tangente t fractionem satis parvam substitui debere. Sic ope huius seriei facile reperietur longitudo arcus z, cuius tangens t aequetur $\frac{1}{10}$; foret enim iste arcus

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mota etc., publié par J.-B. BIOT et F. LEFORT, Paris 1856, p. 79. Observandum quidem est apud GREGORY hanc seriem ita se habere

$$a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \cdots,$$

ubi r, t, a *lineas* denotant, scilicet radium, tangentem et arcum correspondentem. EULERUS primus loco priorum *linearum* trigonometricarum *rationes* earum ad radium hocque modo *functiones* trigonometricas in analysin introduxit. Vide F. RUDIO, *ARCHIMEDES, HUYGENS, LAMBERT, LEGENDRE, Vier Abhandlungen über die Kreismessung etc.*, Leipzig 1892, p. 43 et 46—53. A. K.

1) G. LEIBNIZ, *De vera proportione circuli ad quadratum circumscriptum in numeris rationalibus expressa.* Acta erud. 1682, p. 41; *LEIBNIZENS Mathematische Schriften*, herausg. von C. I. GERHARDT, 2. Abt., Bd. 1, Halle 1858, p. 118.

LEIBNIZ seriem suam plus quam octo annis ante cum amicis communicaverat. Vide epistolas, quas CHR. HUYGENS (1629—1695) et H. OLDENBURG (1626—1678) d. 6. Nov. 1674 et 12. Apr. 1675 ad G. LEIBNIZ scripserunt, *LEIBNIZENS Mathematische Schriften*, 1. Abt., Bd. 2, Berlin 1850, p. 16, et 1. Abt., Bd. 1, Berlin 1849, p. 60. Vide etiam epistolam a LEIBNIZ d. 27. Aug. 1676 ad OLDENBURG scriptam, *LEIBNIZENS Mathematische Schriften*, 1. Abt., Bd. 1, Berlin 1849, p. 114. Hac in epistola legitur: *Unde, posito Quadrato Circumscripto 1, erit Circulus*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Quae expressio, iam Triennio abhinc et ultra a me communicata amicis, haud dubie omnium possibilium simplicissima est maximeque afficiens mentem.

Epistolae tres supra laudatae inveniuntur etiam in libro, qui inscribitur *Commercium epistolicum* (vide notam praecedentem), p. 93 et 198 (ubi imprimis notabilis est nota adiecta) et 112. Vide porro M. CANTOR, *Vorlesungen über Geschichte der Mathematik*, 3. Bd., 2. Aufl., p. 75 et sq. A. K.

$$z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} - \text{etc.},$$

cuius seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali arcu cognito nihil pro longitudine totius circumferentiae concludere licebit, cum ratio, quam arcus, cuius tangens est $= \frac{1}{10}$, ad totam peripheriam tenet, non sit assignabilis. Hanc ob rem ad peripheriam indagandam eiusmodi arcus quaeri debet, qui sit simul pars aliquota peripheriae et cuius tangens satis exigua commode exprimi queat. Ad hoc ergo institutum sumi solet arcus 30^0, cuius tangens est $= \frac{1}{\sqrt{3}}$, quia minorum arcuum cum peripheria commensurabilium tangentes nimis fiunt irrationales. Quare ob arcum $30^0 = \frac{\pi}{6}$ erit

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2\sqrt{3}} - \text{etc.}$$

et

$$\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} + \text{etc.},$$

cuius seriei ope valor ipsius π ante [§ 126] exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo maior est, quod primum singuli termini sint irrationales, tum vero quisque tantum circiter triplo sit minor quam praecedens. Huic itaque incommodo ita occurri poterit. Sumatur arcus 45^0 seu $\frac{\pi}{4}$; cuius valor etsi per seriem vix convergentem

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$

exprimitur, tamen is retineatur atque in duos arcus a et b dispertiatur, ut sit $a + b = \frac{\pi}{4} = 45^0$. Cum igitur sit

$$\text{tang.}\,(a + b) = 1 = \frac{\text{tang.}\,a + \text{tang.}\,b}{1 - \text{tang.}\,a\,\text{tang.}\,b},$$

erit

$$1 - \text{tang.}\,a\,\text{tang.}\,b = \text{tang.}\,a + \text{tang.}\,b$$

et

$$\text{tang.}\,b = \frac{1 - \text{tang.}\,a}{1 + \text{tang.}\,a}.$$

Sit nunc tang. $a = \frac{1}{2}$; erit tang. $b = \frac{1}{3}$; hinc uterque arcus a et b per seriem rationalem multo magis quam superior convergentem exprimetur eorumque summa dabit valorem arcus $\frac{\pi}{4}$; hinc itaque erit

$$\pi = 4 \cdot \left\{ \begin{array}{l} \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \frac{1}{9 \cdot 2^9} - \text{etc.} \\ \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} - \text{etc.} \end{array} \right\}.$$

Hoc ergo modo multo expeditius longitudo semicircumferentiae π inveniri potuisset, quam quidem factum est ope seriei ante commemoratae.[1])

1) Quomodo numerus π per series infinitas, producta infinita, fractiones continuas infinitas exprimi possit et quales relationes analyticae pro hoc numero valeant, EULERUS permultis investigationibus exposuit. Longius autem est has omnes multis verbis enumerare. Commemorentur tantum exempli gratia capita X, XI, XV, XVIII huius *Introductionis* et Commentationes 41, 59, 61, 63, 72, 74, 125, 130, 275, 561, 664, 705, 706, 745, 809 indicis ENESTROEMIANI. Cetera hic index ipse indicabit. A. K.

CAPUT IX

DE INVESTIGATIONE FACTORUM TRINOMIALIUM

143. Quemadmodum factores simplices cuiusque functionis integrae inveniri oporteat, supra [§ 29] quidem ostendimus hoc fieri per resolutionem aequationum. Si enim proposita sit functio quaecunque integra

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.}$$

huiusque quaerantur factores simplices formae $p - qz$, manifestum est, si $p - qz$ fuerit factor functionis $\alpha + \beta z + \gamma z^2 + \text{etc.}$, tum posito $z = \frac{p}{q}$, quo casu factor $p - qz$ fit $= 0$, etiam ipsam functionem propositam evanescere debere. Hinc $p - qz$ erit factor vel divisor functionis

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.};$$

sequitur fore hanc expressionem

$$\alpha + \frac{\beta p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\varepsilon p^4}{q^4} + \text{etc.} = 0.$$

Unde vicissim, si omnes radices $\frac{p}{q}$ huius aequationis eruantur, singulae dabunt totidem factores simplices functionis integrae propositae

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.},$$

nempe $p - qz$. Patet autem simul numerum factorum huiusmodi simplicium ex maxima potestate ipsius z definiri.

144. Hoc autem modo plerumque difficulter factores imaginarii eruuntur, quamobrem hoc capite methodum peculiarem tradam, cuius ope saepenumero factores simplices imaginarii inveniri queant. Quoniam vero factores simplices imaginarii ita sunt comparati, ut binorum productum fiat reale, hos ipsos factores imaginarios reperiemus, si factores investigemus duplices seu huius formae

$$p - qz + rzz,$$

reales quidem, sed quorum factores simplices sint imaginarii. Quodsi enim functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 +$ etc. constent omnes factores reales duplices huius formae trinomiales $p - qz + rzz$, simul omnes factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ factores simplices habebit imaginarios, si fuerit $4pr > qq$ seu

$$\frac{q}{2\sqrt{pr}} < 1.$$

Cum igitur sinus et cosinus angulorum sint unitate minores, formula $p - qz + rzz$ factores simplices habebit imaginarios, si fuerit $\frac{q}{2\sqrt{pr}} =$ sinui vel cosinui cuiuspiam anguli. Sit ergo

$$\frac{q}{2\sqrt{pr}} = \cos.\varphi \quad \text{seu} \quad q = 2\sqrt{pr}\cdot\cos.\varphi$$

atque trinomium $p - qz + rzz$ continebit factores simplices imaginarios. Ne autem irrationalitas molestiam facessat, assumo hanc formam

$$pp - 2pqz\cos.\varphi + qqzz,$$

cuius factores simplices imaginarii erunt hi

$$qz - p(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi) \quad \text{et} \quad qz - p(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi).$$

Ubi quidem patet, si fuerit $\cos.\varphi = \pm 1$, tum ambos factores ob $\sin.\varphi = 0$ fieri aequales et reales.

146. Proposita ergo functione integra $\alpha+\beta z+\gamma z^2+\delta z^3+\text{etc.}$ eius factores simplices imaginarii eruentur, si determinentur litterae p et q cum angulo φ, ut hoc trinomium $pp-2pqz\cos.\varphi+qqzz$ fiat factor functionis. Tum enim simul inerunt isti factores simplices imaginarii

$$qz-p(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi) \quad \text{et} \quad qz-p(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi).$$

Quamobrem functio proposita evanescet, si ponatur tam

$$z=\frac{p}{q}(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)$$

quam

$$z=\frac{p}{q}(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi).$$

Hinc facta substitutione utraque duplex nascetur aequatio, ex quibus tam fractio $\frac{p}{q}$ quam arcus φ definiri poterunt.

147. Hae autem substitutiones loco z faciendae, etiamsi primo intuitu difficiles videantur, tamen per ea, quae in capite praecedente sunt tradita, satis expedite absolventur. Cum enim fuerit ostensum esse

$$(\cos.\varphi\pm\sqrt{-1}\cdot\sin.\varphi)^n=\cos.n\varphi\pm\sqrt{-1}\cdot\sin.n\varphi,$$

sequentes formulae loco singularum ipsius z potestatum habebuntur substituendae:

pro priori factore	pro altero factore
$z=\frac{p}{q}(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)$	$z=\frac{p}{q}(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi)$
$z^2=\frac{p^2}{q^2}(\cos.2\varphi+\sqrt{-1}\cdot\sin.2\varphi)$	$z^2=\frac{p^2}{q^2}(\cos.2\varphi-\sqrt{-1}\cdot\sin.2\varphi)$
$z^3=\frac{p^3}{q^3}(\cos.3\varphi+\sqrt{-1}\cdot\sin.3\varphi)$	$z^3=\frac{p^3}{q^3}(\cos.3\varphi-\sqrt{-1}\cdot\sin.3\varphi)$
$z^4=\frac{p^4}{q^4}(\cos.4\varphi+\sqrt{-1}\cdot\sin.4\varphi)$	$z^4=\frac{p^4}{q^4}(\cos.4\varphi-\sqrt{-1}\cdot\sin.4\varphi)$
etc.	etc.

Ponatur brevitatis gratia $\frac{p}{q} = r$ factaque substitutione sequentes duae nascentur aequationes:

$$0 = \left\{ \begin{array}{l} \alpha + \beta r \cos.\varphi \qquad + \gamma r^2 \cos.2\varphi \qquad + \delta r^3 \cos.3\varphi \qquad + \text{etc.} \\ \quad + \beta r \sqrt{-1}\cdot\sin.\varphi + \gamma r^2 \sqrt{-1}\cdot\sin.2\varphi + \delta r^3 \sqrt{-1}\cdot\sin.3\varphi + \text{etc.} \end{array} \right\}$$

$$0 = \left\{ \begin{array}{l} \alpha + \beta r \cos.\varphi \qquad + \gamma r^2 \cos.2\varphi \qquad + \delta r^3 \cos.3\varphi \qquad + \text{etc.} \\ \quad - \beta r \sqrt{-1}\cdot\sin.\varphi - \gamma r^2 \sqrt{-1}\cdot\sin.2\varphi - \delta r^3 \sqrt{-1}\cdot\sin.3\varphi - \text{etc.} \end{array} \right\}$$

148. Quodsi hae duae aequationes invicem addantur et subtrahantur et posteriori casu per $2\sqrt{-1}$ dividantur, prodibunt hae duae aequationes reales:

$$0 = \alpha + \beta r \cos.\varphi + \gamma r^2 \cos.2\varphi + \delta r^3 \cos.3\varphi + \text{etc.},$$

$$0 = \qquad \beta r \sin.\varphi + \gamma r^2 \sin.2\varphi + \delta r^3 \sin.3\varphi + \text{etc.},$$

quae statim ex forma functionis propositae

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.}$$

formari possunt ponendo primum pro unaquaque ipsius z potestate

$$z^n = r^n \cos.n\varphi,$$

deinceps

$$z^n = r^n \sin.n\varphi.$$

Sic enim ob $\sin.0\varphi = 0$ et $\cos.0\varphi = 1$ pro z^0 seu 1 in termino constanti priori casu ponitur 1, posteriori autem 0.

Si ergo ex his duabus aequationibus definiantur incognitae r et φ, ob $r = \frac{p}{q}$ habebitur factor functionis trinomialis

$$pp - 2pqz\cos.\varphi + qqzz$$

duos factores simplices imaginarios involvens.

149. Si aequatio prior multiplicetur per $\sin.m\varphi$, posterior per $\cos.m\varphi$ atque producta vel addantur vel subtrahantur, prodibunt istae duae aequationes:

$$0 = \alpha \sin. m\varphi + \beta r \sin. (m+1)\varphi + \gamma r^2 \sin. (m+2)\varphi + \delta r^3 \sin. (m+3)\varphi + \text{etc.},$$

$$0 = \alpha \sin. m\varphi + \beta r \sin. (m-1)\varphi + \gamma r^2 \sin. (m-2)\varphi + \delta r^3 \sin. (m-3)\varphi + \text{etc.}$$

Sin autem aequatio prior multiplicetur per $\cos. m\varphi$ et posterior per $\sin. m\varphi$, per additionem ac subtractionem sequentes emergent aequationes:

$$0 = \alpha \cos. m\varphi + \beta r \cos. (m-1)\varphi + \gamma r^2 \cos. (m-2)\varphi + \delta r^3 \cos. (m-3)\varphi + \text{etc.},$$

$$0 = \alpha \cos. m\varphi + \beta r \cos. (m+1)\varphi + \gamma r^2 \cos. (m+2)\varphi + \delta r^3 \cos. (m+3)\varphi + \text{etc.}$$

Huiusmodi ergo duae aequationes quaecunque coniunctae determinabunt incognitas r et φ; quod cum plerumque pluribus modis fieri possit, simul plures factores trinomiales obtinentur iique adeo omnes, quos functio proposita in se complectitur.

150. Quo usus harum regularum clarius appareat, quarumdam functionum saepius occurrentium factores trinomiales hic indagabimus, ut eos, quoties occasio postulaverit, hinc depromere liceat. Sit itaque proposita haec functio

$$a^n + z^n,$$

cuius factores trinomiales formae

$$pp - 2pqz \cos. \varphi + qqzz$$

determinari oporteat. Posito ergo $r = \frac{p}{q}$ habebuntur hae duae aequationes

$$0 = a^n + r^n \cos. n\varphi \quad \text{et} \quad 0 = r^n \sin. n\varphi,$$

quarum posterior dat

$$\sin. n\varphi = 0;$$

unde erit $n\varphi$ arcus vel huius formae $(2k+1)\pi$ vel $2k\pi$ denotante k numerum integrum. Casus hos ideo distinguo, quod eorum cosinus sint differentes; priori enim casu erit $\cos. (2k+1)\pi = -1$, posteriori casu autem $\cos. 2k\pi = +1$. Patet autem priorem formam

$$n\varphi = (2k+1)\pi$$

sumi debere, quippe quae dat $\cos. n\varphi = -1$, unde fit

$$0 = a^n - r^n$$

hincque porro

$$r = a = \frac{p}{q}.$$

Erit ergo

$$p = a, \quad q = 1$$

et

$$\varphi = \frac{2k+1}{n}\pi,$$

unde functionis $a^n + z^n$ factor erit

$$aa - 2az \cos. \frac{2k+1}{n}\pi + zz.$$

Cum igitur pro k numerum quemque integrum ponere liceat, prodeunt hoc modo plures factores neque tamen infiniti, quoniam, si $2k+1$ ultra n augetur, factores priores recurrunt, quod ex exemplis clarius patebit, cum sit $\cos. (2\pi \pm \varphi) = \cos. \varphi$. Deinde si n est numerus impar, posito $2k+1 = n$ erit factor quadratus $aa + 2az + zz$; neque vero hinc sequitur quadratum $(a+z)^2$ esse factorem functionis $a^n + z^n$, quoniam (in § 148) unica aequatio resultat, qua tantum patet $a + z$ esse divisorem formulae $a^n + z^n$; quae regula semper est tenenda, quoties $\cos. \varphi$ fit vel $+1$ vel -1.

EXEMPLUM

Evolvamus aliquot casus, quo isti factores clarius ob oculos ponantur, atque hos casus in duas classes distribuamus, prout n fuerit numerus vel par vel impar.

Si

$$n = 1,$$

formulae

$$a + z$$

factor est

$$a + z.$$

Si

$$n = 2,$$

formulae

$$a^2 + z^2$$

factor est

$$a^2 + z^2.$$

Si
$$n = 3,$$
formulae
$$a^3 + z^3$$
factores sunt
$$aa - 2az\cos.\tfrac{1}{3}\pi + zz,$$
$$a + z.$$

Si
$$n = 4,$$
formulae
$$a^4 + z^4$$
factores sunt
$$aa - 2az\cos.\tfrac{1}{4}\pi + zz,$$
$$aa - 2az\cos.\tfrac{3}{4}\pi + zz.$$

Si
$$n = 5,$$
formulae
$$a^5 + z^5$$
factores sunt
$$aa - 2az\cos.\tfrac{1}{5}\pi + zz,$$
$$aa - 2az\cos.\tfrac{3}{5}\pi + zz,$$
$$a + z.$$

Si
$$n = 6,$$
formulae
$$a^6 + z^6$$
factores sunt
$$aa - 2az\cos.\tfrac{1}{6}\pi + zz,$$
$$aa - 2az\cos.\tfrac{3}{6}\pi + zz,$$
$$aa - 2az\cos.\tfrac{5}{6}\pi + zz.$$

Ex quibus exemplis patet omnes factores obtineri, si loco $2k+1$ omnes numeri impares non maiores quam exponens n substituantur, iis vero casibus, quibus factor quadratus prodit, tantum eius radicem factoribus annumerari debere.

151. Si proposita sit haec functio
$$a^n - z^n,$$
eius factor trinomialis erit
$$pp - 2pqz\cos.\varphi + qqzz,$$
si posito $r = \frac{p}{q}$ fuerit
$$0 = a^n - r^n\cos.n\varphi \quad \text{et} \quad 0 = r^n\sin.n\varphi.$$
Erit ergo iterum
$$\sin.n\varphi = 0$$

ideoque $n\varphi = (2k+1)\pi$ vel $n\varphi = 2k\pi$. Hoc autem casu valor posterior sumi debet, ut sit $\cos. n\varphi = +1$, qui dat

$$0 = a^n - r^n$$

et

$$r = \frac{p}{q} = a.$$

Habebitur itaque

$$p = a, \quad q = 1$$

et

$$\varphi = \frac{2k}{n}\pi,$$

unde factor trinomialis formulae propositae erit

$$aa - 2az\cos.\frac{2k}{n}\pi + zz;$$

quae forma, si loco $2k$ omnes numeri pares non maiores quam n ponantur, simul dabit omnes factores; ubi de factoribus quadratis idem est tenendum, quod ante monuimus. Ac primo quidem posito $k = 0$ prodit factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, si n fuerit numerus par et ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ esse divisorem formae $a^n - z^n$.

EXEMPLUM

Casus exponentis n ut ante tractati ita se habebunt, prout n fuerit numerus vel impar vel par.

Si

$$n = 1,$$

formulae

$$a - z$$

ipsa erit factor

$$a - z.$$

Si

$$n = 2,$$

formulae

$$a^2 - z^2$$

factores erunt

$$a - z,$$

$$a + z.$$

Si $n = 3$, formulae $a^3 - z^3$ factores erunt $a - z$, $aa - 2az\cos.\frac{2}{3}\pi + zz$.	Si $n = 4$, formulae $a^4 - z^4$ factores erunt $a - z$, $aa - 2az\cos.\frac{2}{4}\pi + zz$, $a + z$.
Si $n = 5$, formulae $a^5 - z^5$ factores erunt $a - z$, $aa - 2az\cos.\frac{2}{5}\pi + zz$, $aa - 2az\cos.\frac{4}{5}\pi + zz$.	Si $n = 6$, formulae $a^6 - z^6$ factores erunt $a - z$, $aa - 2az\cos.\frac{2}{6}\pi + zz$, $aa - 2az\cos.\frac{4}{6}\pi + zz$, $a + z$.

152. His igitur confirmatur id, quod supra [§ 32] iam innuimus, omnem functionem integram, si non in factores simplices reales, tamen in factores duplices reales resolvi posse. Vidimus enim hanc functionem indefinitae dimensionis $a^n \pm z^n$ semper in factores duplices reales praeter simplices reales resolvi posse.

Progrediamur ergo ad functiones magis compositas, uti $\alpha + \beta z^n + \gamma z^{2n}$, cuius quidem, si duos habeat factores formae $\eta + \theta z^n$, resolutio ex praecedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formae $\alpha + \beta z^n + \gamma z^{2n}$ eo casu, quo non habet duos factores reales formae $\eta + \theta z^n$, resolutionem in factores reales, vel simplices vel duplices, doceamus.

153. Consideremus ergo hanc functionem

$$a^{2n} - 2a^n z^n \cos. g + z^{2n},$$

quae in duos factores formae $\eta + \theta z^n$ reales resolvi nequit. Quodsi ergo ponamus huius functionis factorem duplicem realem esse

$$pp - 2pqz \cos. \varphi + qqzz,$$

posito $r = \frac{p}{q}$ duae sequentes aequationes erunt resolvendae

$$0 = a^{2n} - 2a^n r^n \cos. g \cos. n\varphi + r^{2n} \cos. 2n\varphi$$

et

$$0 = \quad - 2a^n r^n \cos. g \sin. n\varphi + r^{2n} \sin. 2n\varphi.$$

Vel loco prioris aequationis sumatur ex § 149 (ponendo $m = 2n$) haec

$$0 = a^{2n} \sin. 2n\varphi - 2a^n r^n \cos. g \sin. n\varphi,$$

quae cum posteriori collata dat

$$r = a;$$

tum vero erit

$$\sin. 2n\varphi = 2 \cos. g \sin. n\varphi.$$

At est

$$\sin. 2n\varphi = 2 \sin. n\varphi \cos. n\varphi,$$

unde fit

$$\cos. n\varphi = \cos. g.$$

At est semper $\cos. (2k\pi \pm g) = \cos. g$, ex quo habetur

$$n\varphi = 2k\pi \pm g$$

et

$$\varphi = \frac{2k\pi \pm g}{n}.$$

Hinc ergo factor generalis duplex formae propositae erit

$$= aa - 2az \cos. \frac{2k\pi \pm g}{n} + zz$$

atque omnes factores prodibunt, si pro $2k$ omnes numeri pares non maiores quam n successive substituantur, uti ex applicatione ad casus videre licebit.

EXEMPLUM

Consideremus ergo casus, quibus n est 1, 2, 3, 4 etc., ut ratio factorum appareat. Erit ergo

formulae

$$aa - 2az\cos. g + zz$$

unicus factor

$$aa - 2az\cos. g + zz;$$

formulae

$$a^4 - 2a^2z^2\cos. g + z^4$$

factores duo

$$aa - 2az\cos.\frac{g}{2} + zz,$$

$$aa - 2az\cos.\frac{2\pi \pm g}{2} + zz \quad \text{seu} \quad aa + 2az\cos.\frac{g}{2} + zz;$$

formulae

$$a^6 - 2a^3z^3\cos. g + z^6$$

factores tres

$$aa - 2az\cos.\frac{g}{3} + zz,$$

$$aa - 2az\cos.\frac{2\pi - g}{3} + zz,$$

$$aa - 2az\cos.\frac{2\pi + g}{3} + zz;$$

formulae

$$a^8 - 2a^4z^4\cos. g + z^8$$

factores quatuor

$$aa - 2az\cos.\frac{g}{4} + zz,$$

$$aa - 2az\cos.\frac{2\pi - g}{4} + zz,$$

$$aa - 2az\cos.\frac{2\pi + g}{4} + zz,$$

$$aa - 2az\cos.\frac{4\pi \pm g}{4} + zz \quad \text{seu} \quad aa + 2az\cos.\frac{g}{4} + zz;$$

formulae

$$a^{10} - 2a^5z^5 \cos. g + z^{10}$$

factores quinque

$$aa - 2az \cos. \frac{g}{5} + zz,$$

$$aa - 2az \cos. \frac{2\pi - g}{5} + zz,$$

$$aa - 2az \cos. \frac{2\pi + g}{5} + zz,$$

$$aa - 2az \cos. \frac{4\pi - g}{5} + zz,$$

$$aa - 2az \cos. \frac{4\pi + g}{5} + zz.$$

Confirmatur ergo etiam his exemplis omnem functionem integram in factores reales sive simplices sive duplices resolvi posse.

154. Hinc ulterius progredi licebit ad functionem hanc

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n},$$

quae certo habebit unum factorem realem formae $\eta + \theta z^n$, cuius igitur factores reales vel simplices vel duplices exhiberi possunt; alter vero multiplicator formae $\iota + \varkappa z^n + \lambda z^{2n}$, utcunque fuerit comparatus, per paragraphum praecedentem pari modo in factores resolvi poterit.

Deinde haec functio

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n},$$

cum perpetuo habeat duos factores reales formae huius $\eta + \theta z^n + \iota z^{2n}$, similiter in factores vel simplices vel duplices reales resolvitur.

Quin etiam progredi licet ad formam

$$\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \zeta z^{5n};$$

quae cum certo habeat unum factorem formae $\eta + \theta z^n$, alter factor erit formae praecedentis, unde etiam haec functio resolutionem in factores reales vel simplices vel duplices admittet.

Quare si ullum dubium mansisset circa huiusmodi resolutionem omnium functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam potest haec in factores resolutio ad series infinitas; scilicet quia vidimus supra [§ 123] esse

$$1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}=e^x,$$

at vero esse

$$e^x=\left(1+\frac{x}{i}\right)^i$$

denotante i numerum infinitum, perspicuum est seriem

$$1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\text{etc.}$$

habere factores infinitos simplices inter se aequales nempe $1+\frac{x}{i}$. At si ab eadem serie primus terminus dematur, erit

$$\frac{x}{1}+\frac{x^2}{1\cdot 2}+\frac{x^3}{1\cdot 2\cdot 3}+\text{etc.}=e^x-1=\left(1+\frac{x}{i}\right)^i-1,$$

cuius formae cum § 151 comparatae, quo fit

$$a=1+\frac{x}{i},\quad n=i\quad \text{et}\quad z=1,$$

factor quicunque erit

$$=\left(1+\frac{x}{i}\right)^2-2\left(1+\frac{x}{i}\right)\cos.\frac{2k}{i}\pi+1,$$

unde substituendo pro $2k$ omnes numeros pares simul omnes factores prodibunt.

Posito autem $2k=0$ prodit factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ sumi debet; erit ergo x factor expressionis

$e^x - 1$, quod quidem sponte patet. Ad reliquos factores inveniendos notari oportet esse ob arcum $\frac{2k}{i}\pi$ infinite parvum

$$\cos.\frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$$

(§ 134) terminis sequentibus ob i numerum infinitum in nihilum abeuntibus. Hinc erit factor quilibet

$$\frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$$

atque adeo forma $e^x - 1$ erit divisibilis per

$$1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}.$$

Quare expressio

$$e^x - 1 = x\left(1 + \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3} + \frac{x^3}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right)$$

praeter factorem x habebit hos infinitos

$$\left(1 + \frac{x}{i} + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{x}{i} + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{x}{i} + \frac{xx}{36\pi\pi}\right)\left(1 + \frac{x}{i} + \frac{xx}{64\pi\pi}\right) \text{ etc.}$$

156. Cum autem hi factores contineant partem infinite parvam $\frac{x}{i}$, quae, cum in singulis insit atque per multiplicationem omnium, quorum numerus est $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non potest, ad hoc ergo incommodum vitandum consideremus hanc expressionem

$$e^x - e^{-x} = \left(1 + \frac{x}{i}\right)^i - \left(1 - \frac{x}{i}\right)^i = 2\left(\frac{x}{1} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}\right);$$

est enim

$$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} + \text{etc.}$$

Quae cum § 151 comparata dat

$$n = i, \quad a = 1 + \frac{x}{i} \quad \text{et} \quad z = 1 - \frac{x}{i},$$

unde huius expressionis factor erit

$$= aa - 2az\cos.\frac{2k}{n}\pi + zz$$

$$= 2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right)\cos.\frac{2k}{i}\pi = \frac{4xx}{ii} + \frac{4kk}{ii}\pi\pi - \frac{4kk\pi\pi xx}{i^4}$$

ob

$$\cos.\frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi.$$

Functio ergo $e^x - e^{-x}$ divisibilis erit per

$$1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii},$$

ubi autem terminus $\frac{xx}{ii}$ tuto omittitur, quia, etsi per i multiplicetur, tamen manet infinite parvus. Praeterea vero ut ante, si $k = 0$, erit primus factor $= x$. Quocirca his factoribus in ordinem redactis erit

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{\pi\pi}\right)\left(1 + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{xx}{9\pi\pi}\right)\left(1 + \frac{xx}{16\pi\pi}\right)\left(1 + \frac{xx}{25\pi\pi}\right) \text{ etc.}$$

$$= x\left(1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.}\right).$$

Singulis scilicet factoribus per multiplicationem constantis eiusmodi formam dedi, ut per actualem multiplicationem primus terminus x resultet.

157. Eodem modo cum sit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.} = \frac{\left(1 + \frac{x}{i}\right)^i + \left(1 - \frac{x}{i}\right)^i}{2},$$

huius expressionis cum superiori [§ 150] $a^n + z^n$ comparatio dabit

$$a = 1 + \frac{x}{i}, \quad z = 1 - \frac{x}{i} \quad \text{et} \quad n = i;$$

erit ergo factor quicunque

$$= aa - 2az\cos.\frac{2k+1}{n}\pi + zz = 2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right)\cos.\frac{2k+1}{i}\pi.$$

Est autem

$$\cos.\frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2}{2ii}\pi\pi,$$

unde forma factoris erit

$$\frac{4xx}{ii} + \frac{(2k+1)^2}{ii}\pi\pi$$

evanescente termino, cuius denominator est i^4. Quoniam ergo omnis factor expressionis

$$1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

huiusmodi formam habere debet $1 + \alpha xx$, quo factor inventus ad hanc formam reducatur, dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$; hinc factor formae propositae erit

$$= 1 + \frac{4xx}{(2k+1)^2\pi\pi}$$

ex eoque omnes factores infiniti invenientur, si loco $2k+1$ successive omnes numeri impares substituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

$$= \left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \text{ etc.}$$

158. Si x fiat quantitas imaginaria, formulae hae exponentiales in sinum et cosinum cuiuspiam arcus realis abeunt. Sit enim $x = z\sqrt{-1}$; erit

$$\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} = \sin. z$$

$$= z - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{z^7}{1\cdot 2\cdot 3\cdots 7} + \text{etc.},$$

quae adeo expressio hos habet factores numero infinitos

$$z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right) \text{ etc.},$$

seu erit

$$\sin. z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right) \text{ etc.}$$

Quoties ergo arcus z ita est comparatus, ut quispiam factor evanescat, quod fit, si $z=0$, $z=\pm\pi$, $z=\pm 2\pi$ et generaliter si $z=\pm k\pi$ denotante k numerum quemcunque integrum, simul sinus eius arcus debet esse $=0$, quod quidem ita patet, ut hinc istos factores a posteriori eruere licuisset.

Simili modo cum sit

$$\frac{e^{z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2}=\cos. z,$$

erit quoque

$$\cos. z=\left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right) \text{ etc.}$$

seu his factoribus in binos resolvendis erit quoque

$$\cos. z=\left(1-\frac{2z}{\pi}\right)\left(1+\frac{2z}{\pi}\right)\left(1-\frac{2z}{3\pi}\right)\left(1+\frac{2z}{3\pi}\right)\left(1-\frac{2z}{5\pi}\right)\left(1+\frac{2z}{5\pi}\right) \text{ etc.},$$

ex qua pari modo patet, si fuerit $z=\pm\frac{2k+1}{2}\pi$, fore $\cos. z=0$, id quod etiam ex natura circuli liquet.

159. Ex § 153 etiam inveniri possunt factores huius expressionis

$$e^x-2\cos. g+e^{-x}=2\left(1-\cos. g+\frac{xx}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}\right).$$

Transit enim haec expressio in hanc

$$\left(1+\frac{x}{i}\right)^i-2\cos. g+\left(1-\frac{x}{i}\right)^i,$$

quae cum illa forma comparata dat

$$2n=i,\quad a=1+\frac{x}{i}\quad \text{et}\quad z=1-\frac{x}{i},$$

unde factor quicunque huius formulae erit

$$=aa-2az\cos.\frac{2k\pi\pm g}{n}+zz=2+\frac{2xx}{ii}-2\left(1-\frac{xx}{ii}\right)\cos.\frac{2(2k\pi\pm g)}{i};$$

at est

$$\cos.\frac{2(2k\pi\pm g)}{i}=1-\frac{2(2k\pi\pm g)^2}{ii},$$

unde factor erit $=\frac{4xx}{ii}+\frac{4(2k\pi\pm g)^2}{ii}$ seu huius formae

$$1+\frac{xx}{(2k\pi\pm g)^2}.$$

Si ergo expressio per $2(1-\cos.g)$ dividatur, ut in serie infinita terminus constans sit $=1$, erit sumendis omnibus factoribus

$$\frac{e^x-2\cos.g+e^{-x}}{2(1-\cos.g)}$$
$$=\left(1+\frac{xx}{gg}\right)\left(1+\frac{xx}{(2\pi-g)^2}\right)\left(1+\frac{xx}{(2\pi+g)^2}\right)\left(1+\frac{xx}{(4\pi-g)^2}\right)\left(1+\frac{xx}{(4\pi+g)^2}\right)$$
$$\left(1+\frac{xx}{(6\pi-g)^2}\right)\left(1+\frac{xx}{(6\pi+g)^2}\right)\text{ etc.}$$

Atque si loco x ponatur $z\sqrt{-1}$, erit

$$\frac{\cos.z-\cos.g}{1-\cos.g}$$
$$=\left(1-\frac{z}{g}\right)\left(1+\frac{z}{g}\right)\left(1-\frac{z}{2\pi-g}\right)\left(1+\frac{z}{2\pi-g}\right)\left(1-\frac{z}{2\pi+g}\right)\left(1+\frac{z}{2\pi+g}\right)$$
$$\left(1-\frac{z}{4\pi-g}\right)\left(1+\frac{z}{4\pi-g}\right)\text{ etc.}$$
$$=1-\frac{zz}{1\cdot 2(1-\cos.g)}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-\cos.g)}-\frac{z^6}{1\cdot 2\cdots 6(1-\cos.g)}+\text{etc.}$$

Huius adeo seriei in infinitum continuatae factores omnes cognoscuntur.

160. Commode etiam huiusmodi functionis

$$e^{b+x}\pm e^{c-x}$$

factores inveniri omnesque assignari possunt. Transmutatur enim in hanc formam

$$\left(1+\frac{b+x}{i}\right)^i\pm\left(1+\frac{c-x}{i}\right)^i,$$

quae comparata cum forma $a^i\pm z^i$ factorem habebit

$$aa-2az\cos.\frac{m\pi}{i}+zz$$

denotante m numerum imparem, si valeat signum superius, contra vero numerum parem [§ 150 et 151]. Cum autem ob i numerum infinite magnum sit

$$\cos.\frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii},$$

erit factor ille generalis

$$= (a-z)^2 + \frac{mm\pi\pi}{ii}az.$$

At hoc casu erit

$$a = 1 + \frac{b+x}{i} \quad \text{et} \quad z = 1 + \frac{c-x}{i},$$

unde fit

$$(a-z)^2 = \frac{(b-c+2x)^2}{ii} \quad \text{et} \quad az = 1 + \frac{b+c}{i} + \frac{bc+(c-b)x-xx}{ii};$$

ideoque factor erit per ii multiplicatus

$$= (b-c)^2 + 4(b-c)x + 4xx + mm\pi\pi$$

neglectis terminis per i vel ii divisis, quoniam iam omnis generis termini adsunt, prae quibus hi evanescerent. Termino ergo constante ad unitatem per divisionem reducto erit factor

$$= 1 + \frac{4(b-c)x + 4xx}{mm\pi\pi + (b-c)^2}.$$

161. Nunc quoniam in omnibus factoribus terminus constans est $=1$, ipsa functio $e^{b+x} \pm e^{c-x}$ per eiusmodi constantem dividi debet, ut terminus constans fiat $=1$ seu ut eius valor posito $x=0$ fiat $=1$; talis divisor erit $e^b \pm e^c$ et hanc ob rem expressio haec

$$\frac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$$

per factores numero infinitos exponi poterit. Erit ergo, si valeat signum superius atque m denotet numerum imparem,

$$\frac{e^{b+x} + e^{c-x}}{e^b + e^c}$$

$$= \left(1 + \frac{4(b-c)x + 4xx}{\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{9\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{25\pi\pi + (b-c)^2}\right) \text{etc.};$$

sin autem signum inferius valeat atque ideo m denotet numerum parem casuque $m = 0$ radix factoris quadrati ponatur, erit

$$\frac{e^{b+x} - e^{c-x}}{e^b - e^c}$$

$$= \left(1 + \frac{2x}{b-c}\right)\left(1 + \frac{4(b-c)x + 4xx}{4\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{16\pi\pi + (b-c)^2}\right)\left(1 + \frac{4(b-c)x + 4xx}{36\pi\pi + (b-c)^2}\right) \text{ etc.}$$

162. Ponatur $b = 0$, quod sine detrimento universalitatis fieri potest, eritque

$$\frac{e^x + e^c e^{-x}}{1 + e^c}$$

$$= \left(1 - \frac{4cx - 4xx}{\pi\pi + cc}\right)\left(1 - \frac{4cx - 4xx}{9\pi\pi + cc}\right)\left(1 - \frac{4cx - 4xx}{25\pi\pi + cc}\right) \text{ etc.},$$

$$\frac{e^x - e^c e^{-x}}{1 - e^c}$$

$$= \left(1 - \frac{2x}{c}\right)\left(1 - \frac{4cx - 4xx}{4\pi\pi + cc}\right)\left(1 - \frac{4cx - 4xx}{16\pi\pi + cc}\right)\left(1 - \frac{4cx - 4xx}{36\pi\pi + cc}\right) \text{ etc.}$$

Iam ponatur c negativum atque habebuntur hae duae aequationes

$$\frac{e^x + e^{-c} e^{-x}}{1 + e^{-c}}$$

$$= \left(1 + \frac{4cx + 4xx}{\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{9\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{25\pi\pi + cc}\right) \text{ etc.},$$

$$\frac{e^x - e^{-c} e^{-x}}{1 - e^{-c}}$$

$$= \left(1 + \frac{2x}{c}\right)\left(1 + \frac{4cx + 4xx}{4\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{16\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{36\pi\pi + cc}\right) \text{ etc.}$$

Multiplicetur forma prima per tertiam ac prodibit

$$\frac{e^{2x} + e^{-2x} + e^c + e^{-c}}{2 + e^c + e^{-c}};$$

ponatur vero y loco $2x$ eritque

$$\frac{e^y+e^{-y}+e^c+e^{-c}}{2+e^c+e^{-c}}$$

$$=\left(1-\frac{2cy-yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{9\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{25\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\text{ etc.}$$

Multiplicetur prima forma per quartam; erit productum

$$=\frac{e^{2x}-e^{-2x}+e^c-e^{-c}}{e^c-e^{-c}};$$

ponatur y pro $2x$ eritque

$$\frac{e^y-e^{-y}+e^c-e^{-c}}{e^c-e^{-c}}$$

$$=\left(1+\frac{y}{c}\right)\left(1-\frac{2cy-yy}{\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{9\pi\pi+cc}\right)$$
$$\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{25\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right)\text{ etc.}$$

Si secunda forma per tertiam multiplicetur, prodibit eadem aequatio, nisi quod c capiendum sit negativum; erit nempe

$$\frac{e^c-e^{-c}-e^y+e^{-y}}{e^c-e^{-c}}$$

$$=\left(1-\frac{y}{c}\right)\left(1+\frac{2cy+yy}{\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{9\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{25\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right)\text{ etc.}$$

Multiplicetur denique forma secunda per quartam eritque

$$\frac{e^y+e^{-y}-e^c-e^{-c}}{2-e^c-e^{-c}}$$

$$=\left(1-\frac{yy}{cc}\right)\left(1-\frac{2cy-yy}{4\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{4\pi\pi+cc}\right)\left(1-\frac{2cy-yy}{16\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{16\pi\pi+cc}\right)$$
$$\left(1-\frac{2cy-yy}{36\pi\pi+cc}\right)\left(1+\frac{2cy+yy}{36\pi\pi+cc}\right)\text{ etc.}$$

163. Hae quatuor combinationes nunc commode ad circulum transferri possunt ponendo

$$c = g\sqrt{-1} \quad \text{et} \quad y = v\sqrt{-1};$$

erit enim

$$e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} = 2\cos. v, \quad e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} = 2\sqrt{-1}\cdot \sin. v$$

et

$$e^{g\sqrt{-1}} + e^{-g\sqrt{-1}} = 2\cos. g, \quad e^{g\sqrt{-1}} - e^{-g\sqrt{-1}} = 2\sqrt{-1}\cdot \sin. g.$$

Hinc prima combinatio dabit

$$\frac{\cos. v + \cos. g}{1 + \cos. g}$$

$$= 1 - \frac{vv}{1\cdot 2(1+\cos. g)} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4(1+\cos. g)} - \frac{v^6}{1\cdot 2\cdots 6(1+\cos. g)} + \text{etc.}$$

$$= \left(1 + \frac{2gv - vv}{\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{\pi\pi - gg}\right)\left(1 + \frac{2gv - vv}{9\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{9\pi\pi - gg}\right)$$

$$\left(1 + \frac{2gv - vv}{25\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{25\pi\pi - gg}\right) \text{etc.}$$

$$= \left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 - \frac{v}{\pi - g}\right)\left(1 + \frac{v}{\pi + g}\right)$$

$$\left(1 + \frac{v}{3\pi - g}\right)\left(1 - \frac{v}{3\pi + g}\right)\left(1 - \frac{v}{3\pi - g}\right)\left(1 + \frac{v}{3\pi + g}\right) \text{etc.}$$

$$= \left(1 - \frac{vv}{(\pi - g)^2}\right)\left(1 - \frac{vv}{(\pi + g)^2}\right)\left(1 - \frac{vv}{(3\pi - g)^2}\right)\left(1 - \frac{vv}{(3\pi + g)^2}\right)\left(1 - \frac{vv}{(5\pi - g)^2}\right) \text{etc.}$$

Quarta vero combinatio dat

$$\frac{\cos. v - \cos. g}{1 - \cos. g}$$

$$= 1 - \frac{vv}{1\cdot 2(1-\cos. g)} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4(1-\cos. g)} - \frac{v^6}{1\cdot 2\cdots 6(1-\cos. g)} + \text{etc.}$$

$$= \left(1 - \frac{vv}{gg}\right)\left(1 + \frac{2gv - vv}{4\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{4\pi\pi - gg}\right)\left(1 + \frac{2gv - vv}{16\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{16\pi\pi - gg}\right) \text{etc.}$$

$$= \left(1 - \frac{v}{g}\right)\left(1 + \frac{v}{g}\right)\left(1 + \frac{v}{2\pi - g}\right)\left(1 - \frac{v}{2\pi + g}\right)\left(1 - \frac{v}{2\pi - g}\right)\left(1 + \frac{v}{2\pi + g}\right)$$

$$\left(1 + \frac{v}{4\pi - g}\right)\left(1 - \frac{v}{4\pi + g}\right) \text{etc.}$$

$$= \left(1 - \frac{vv}{gg}\right)\left(1 - \frac{vv}{(2\pi - g)^2}\right)\left(1 - \frac{vv}{(2\pi + g)^2}\right)\left(1 - \frac{vv}{(4\pi - g)^2}\right)\left(1 - \frac{vv}{(4\pi + g)^2}\right) \text{etc.}$$

Secunda combinatio dat

$$\frac{\sin. g + \sin. v}{\sin. g}$$

$$= 1 + \frac{v}{\sin. g} - \frac{v^3}{1 \cdot 2 \cdot 3 \sin. g} + \frac{v^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \sin. g} - \text{etc.}$$

$$= \left(1 + \frac{v}{g}\right)\left(1 + \frac{2gv - vv}{\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{4\pi\pi - gg}\right)\left(1 + \frac{2gv - vv}{9\pi\pi - gg}\right)\left(1 - \frac{2gv + vv}{16\pi\pi - gg}\right) \text{etc.}$$

$$= \left(1 + \frac{v}{g}\right)\left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 - \frac{v}{2\pi - g}\right)\left(1 + \frac{v}{2\pi + g}\right)$$

$$\left(1 + \frac{v}{3\pi - g}\right)\left(1 - \frac{v}{3\pi + g}\right)\left(1 - \frac{v}{4\pi - g}\right) \text{etc.}$$

Ac sumto v negativo prodit tertia combinatio.

164. Ipsae vero etiam expressiones in § 162 primum inventae ad arcus circulares traduci possunt hoc modo. Cum sit

$$\frac{e^x + e^c e^{-x}}{1 + e^c} = \frac{(1 + e^{-c})(e^x + e^c e^{-x})}{2 + e^c + e^{-c}} = \frac{e^x + e^{-x} + e^{c-x} + e^{-c+x}}{2 + e^c + e^{-c}},$$

si ponamus

$$c = g\sqrt{-1} \quad \text{et} \quad x = z\sqrt{-1},$$

haec expressio abit in hanc

$$\frac{\cos. z + \cos. (g - z)}{1 + \cos. g} = \cos. z + \frac{\sin. g \sin. z}{1 + \cos. g}.$$

Erit ergo ob $\frac{\sin. g}{1 + \cos. g} = \text{tang.} \frac{1}{2} g$

$$\cos. z + \text{tang.} \frac{1}{2} g \sin. z$$

$$= 1 + \frac{z}{1} \text{tang.} \frac{1}{2} g - \frac{zz}{1 \cdot 2} - \frac{z^3}{1 \cdot 2 \cdot 3} \text{tang.} \frac{1}{2} g + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{z^5}{1 \cdot 2 \cdots 5} \text{tang.} \frac{1}{2} g - \text{etc.}$$

$$= \left(1 + \frac{4gz - 4zz}{\pi\pi - gg}\right)\left(1 + \frac{4gz - 4zz}{9\pi\pi - gg}\right)\left(1 + \frac{4gz - 4zz}{25\pi\pi - gg}\right) \text{etc.}$$

$$= \left(1 + \frac{2z}{\pi - g}\right)\left(1 - \frac{2z}{\pi + g}\right)\left(1 + \frac{2z}{3\pi - g}\right)\left(1 - \frac{2z}{3\pi + g}\right)\left(1 + \frac{2z}{5\pi - g}\right)\left(1 - \frac{2z}{5\pi + g}\right) \text{etc.}$$

Simili modo altera expressio, si numerator et denominator per $1 - e^{-c}$ multiplicetur, abit in

$$\frac{e^x + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^c - e^{-c}},$$

quae facto $c = g\sqrt{-1}$ et $x = z\sqrt{-1}$ dat

$$\frac{\cos. z - \cos.(g-z)}{1-\cos. g} = \cos. z - \frac{\sin. g \sin. z}{1-\cos. g} = \cos. z - \frac{\sin. z}{\operatorname{tang}. \frac{1}{2}g}.$$

Erit ergo

$$\cos. z - \cot. \frac{1}{2}g \sin. z$$

$$= 1 - \frac{z}{1}\cot. \frac{1}{2}g - \frac{zz}{1\cdot 2} + \frac{z^3}{1\cdot 2\cdot 3}\cot. \frac{1}{2}g + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cot. \frac{1}{2}g + \text{etc.}$$

$$= \left(1-\frac{2z}{g}\right)\left(1+\frac{4gz-4zz}{4\pi\pi-gg}\right)\left(1+\frac{4gz-4zz}{16\pi\pi-gg}\right)\left(1+\frac{4gz-4zz}{36\pi\pi-gg}\right) \text{ etc.}$$

$$= \left(1-\frac{2z}{g}\right)\left(1+\frac{2z}{2\pi-g}\right)\left(1-\frac{2z}{2\pi+g}\right)\left(1+\frac{2z}{4\pi-g}\right)\left(1-\frac{2z}{4\pi+g}\right) \text{ etc.}$$

Quodsi ergo ponatur $v = 2z$ seu $z = \frac{1}{2}v$, habebitur

$$\frac{\cos. \frac{1}{2}(g-v)}{\cos. \frac{1}{2}g} = \cos. \frac{1}{2}v + \operatorname{tang}. \frac{1}{2}g \sin. \frac{1}{2}v$$

$$= \left(1+\frac{v}{\pi-g}\right)\left(1-\frac{v}{\pi+g}\right)\left(1+\frac{v}{3\pi-g}\right)\left(1-\frac{v}{3\pi+g}\right) \text{ etc.,}$$

$$\frac{\cos. \frac{1}{2}(g+v)}{\cos. \frac{1}{2}g} = \cos. \frac{1}{2}v - \operatorname{tang}. \frac{1}{2}g \sin. \frac{1}{2}v$$

$$= \left(1-\frac{v}{\pi-g}\right)\left(1+\frac{v}{\pi+g}\right)\left(1-\frac{v}{3\pi-g}\right)\left(1+\frac{v}{3\pi+g}\right) \text{ etc.,}$$

$$\frac{\sin. \frac{1}{2}(g-v)}{\sin. \frac{1}{2}g} = \cos. \frac{1}{2}v - \cot. \frac{1}{2}g \sin. \frac{1}{2}v$$

$$= \left(1-\frac{v}{g}\right)\left(1+\frac{v}{2\pi-g}\right)\left(1-\frac{v}{2\pi+g}\right)\left(1+\frac{v}{4\pi-g}\right)\left(1-\frac{v}{4\pi+g}\right) \text{ etc.,}$$

$$\frac{\sin. \frac{1}{2}(g+v)}{\sin. \frac{1}{2}g} = \cos. \frac{1}{2}v + \cot. \frac{1}{2}g \sin. \frac{1}{2}v$$

$$= \left(1+\frac{v}{g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)\left(1-\frac{v}{4\pi-g}\right)\left(1+\frac{v}{4\pi+g}\right) \text{ etc.}$$

Quorum factorum lex progressionis satis est simplex et uniformis; atque ex his expressionibus per multiplicationem oriuntur eae ipsae, quae paragrapho praecedente sunt inventae.

CAPUT X

DE USU FACTORUM INVENTORUM IN DEFINIENDIS SUMMIS SERIERUM INFINITARUM

165. Si fuerit

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + \text{etc.} = (1 + \alpha z)(1 + \beta z)(1 + \gamma z)(1 + \delta z) \text{ etc.},$$

hi factores, sive sint numero finiti sive infiniti, si in se actu multiplicentur, illam expressionem $1 + Az + Bz^2 + Cz^3 + Dz^4 +$ etc. producere debent. Aequabitur ergo coefficiens A summae omnium quantitatum

$$\alpha + \beta + \gamma + \delta + \varepsilon + \text{ etc.}$$

Coefficiens vero B aequalis erit summae productorum ex binis eritque

$$B = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \text{ etc.}$$

Tum vero coefficiens C aequabitur summae productorum ex ternis, nempe erit

$$C = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \text{ etc.}$$

Atque ita porro erit $D =$ summae productorum ex quaternis, $E =$ summae productorum ex quinis etc., id quod ex Algebra communi constat.

166. Quia summa quantitatum $\alpha + \beta + \gamma + \delta +$ etc. datur una cum summa productorum ex binis, hinc summa quadratorum $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 +$ etc. inveniri poterit, quippe quae aequalis est quadrato summae demptis duplicibus

productis ex binis. Simili modo summa cuborum, biquadratorum et altiorum potestatum definiri potest; si enim ponamus

$$\begin{aligned}
P &= \alpha + \beta + \gamma + \delta + \varepsilon + \text{ etc.},\\
Q &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \text{ etc.},\\
R &= \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \text{ etc.},\\
S &= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \text{ etc.},\\
T &= \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \text{ etc.},\\
V &= \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \varepsilon^6 + \text{ etc.}\\
&\text{etc.},
\end{aligned}$$

valores P, Q, R, S, T, V etc. sequenti modo ex cognitis A, B, C, D etc. determinabuntur:

$$\begin{aligned}
P &= A,\\
Q &= AP - 2B,\\
R &= AQ - BP + 3C,\\
S &= AR - BQ + CP - 4D,\\
T &= AS - BR + CQ - DP + 5E,\\
V &= AT - BS + CR - DQ + EP - 6F\\
&\text{etc.},
\end{aligned}$$

quarum formularum veritas examine instituto facile agnoscitur; interim tamen in Calculo differentiali summo cum rigore demonstrabitur.[1])

1) Has formulas, quae ab EULERO formulae NEUTONIANAE nominari solent, usque ad quartam potestatem primus A. GIRARD (?—1632) exposuit in libro, qui inscribitur *Invention nouvelle en l'algèbre*, Amsterdam 1629; réimpression par D. BIERENS DE HAAN, Leiden 1884, fol. F 2. Vide etiam I. NEWTON, *Arithmetica universalis*, Cantabrigiae 1707, p. 251; 3. ed. (ed. G. I. 's GRAVESANDE) Lugd. Batav. 1732, p. 192. Demonstrationes dedit EULERUS in Commentatione 153 (indicis ENESTROEMIANI): *Demonstratio gemina theorematis NEUTONIANI, quo traditur relatio inter coefficientes cuiusvis aequationis algebraicae et summas potestatum radicum eiusdem,* Opuscula varii argumenti 2, 1750, p. 108; *LEONHARDI EULERI Opera omnia*, series I, vol. 6, p. 20 (confer imprimis notas p. 20—22 adiectas). Vide praeterea Commentationes 406 et 560 (indicis ENESTROEMIANI): *Observationes circa radices aequationum,* Novi comment. acad. sc. Petrop. 15 (1770), 1771, p. 51, et *Miscellanea analytica,* Opuscula analytica 1, 1783, p. 329, imprimis p. 337; *LEONHARDI EULERI Opera omnia,* series I, vol. 6, p. 263 (confer imprimis notas p. 265 et 267 adiectas) et vol. 4. A. K.

167. Cum igitur supra (§ 156) invenerimus esse

$$\frac{e^x - e^{-x}}{2} = x\left(1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdots 7} + \text{etc.}\right)$$

$$= x\left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\left(1+\frac{xx}{25\pi\pi}\right) \text{ etc.,}$$

erit

$$1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdot 3\cdots 7} + \text{etc.}$$

$$= \left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\left(1+\frac{xx}{25\pi\pi}\right) \text{ etc.}$$

Ponatur $xx = \pi\pi z$ eritque

$$1 + \frac{\pi\pi}{1\cdot 2\cdot 3}z + \frac{\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}z^2 + \frac{\pi^6}{1\cdot 2\cdot 3\cdots 7}z^3 + \text{etc.}$$

$$= (1+z)\left(1+\frac{1}{4}z\right)\left(1+\frac{1}{9}z\right)\left(1+\frac{1}{16}z\right)\left(1+\frac{1}{25}z\right) \text{ etc.}$$

Facta ergo applicatione superioris regulae ad hunc casum erit

$$A = \frac{\pi\pi}{6}, \quad B = \frac{\pi^4}{120}, \quad C = \frac{\pi^6}{5040}, \quad D = \frac{\pi^8}{362880} \quad \text{etc.}$$

Quodsi ergo ponatur

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \text{etc.,}$$

$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \text{etc.,}$$

$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \text{etc.,}$$

$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \text{etc.,}$$

$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \text{etc.}$$

etc.

atque harum litterarum valores ex A, B, C, D etc. determinentur, prodibit:

$$P = \frac{\pi\pi}{6}\ ^{1)},$$

$$Q = \frac{\pi^4}{90},$$

$$R = \frac{\pi^6}{945},$$

$$S = \frac{\pi^8}{9450},$$

$$T = \frac{\pi^{10}}{93555}$$

etc.

168. Patet ergo omnium serierum infinitarum in hac forma generali

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}$$

contentarum [summas], quoties n fuerit numerus par, ope semiperipheriae circuli π exhiberi posse; habebit enim semper summa seriei ad π^n rationem

1) Hanc celeberrimam formulam

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

EULERUS a. 1736 cum D. BERNOULLI (1700—1782) communicaverat. Vide epistolam a D. BERNOULLI d. 12. Sept. 1736 ad EULERUM scriptam, *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. II, p. 433. Confer G. ENESTRÖM, *Note historique sur la somme des valeurs inverses des nombres carrés, Der Briefwechsel zwischen LEONHARD EULER und JOHANN I. BERNOULLI, Der Briefwechsel zwischen LEONHARD EULER und DANIEL BERNOULLI,* Biblioth. Mathem. 4_2, 1890, p. 22, 5_3, 1904, p. 248, 7_3, 1906—1907, p. 126. Vide porro EULERI Commentationes 41, 63, 130 (indicis ENESTROEMIANI): *De summis serierum reciprocarum*, Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 123, *Démonstration de la somme de cette suite:* $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.}$, Journal littéraire d'Allemagne etc. 2: 1, 1743, p. 115, *De seriebus quibusdam considerationes*, Comment acad. sc. Petrop. 12 (1740), 1750, p. 53. Ad Commentationem 63 pertinet P. STÄCKEL, *Eine vergessene Abhandlung LEONHARD EULERS über die Summe der reziproken Quadrate der natürlichen Zahlen*, Biblioth. Mathem. 8_3, 1907—1908, p. 37. A. K.

rationalem. Quo autem valor harum summarum clarius perspiciatur, plures huiusmodi serierum summas commodiori modo expressas hic adiiciam.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{2^0}{1\cdot 2\cdot 3}\cdot\frac{1}{1}\pi^2,$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{2^2}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1}{3}\pi^4,$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{2^4}{1\cdot 2\cdot 3\cdots 7}\cdot\frac{1}{3}\pi^6,$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{2^6}{1\cdot 2\cdot 3\cdots 9}\cdot\frac{3}{5}\pi^8,$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^8}{1\cdot 2\cdot 3\cdots 11}\cdot\frac{5}{3}\pi^{10},$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{10}}{1\cdot 2\cdot 3\cdots 13}\cdot\frac{691}{105}\pi^{12},$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{12}}{1\cdot 2\cdot 3\cdots 15}\cdot\frac{35}{1}\pi^{14},$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{14}}{1\cdot 2\cdot 3\cdots 17}\cdot\frac{3617}{15}\pi^{16},$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{16}}{1\cdot 2\cdot 3\cdots 19}\cdot\frac{43867}{21}\pi^{18},$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{18}}{1\cdot 2\cdot 3\cdots 21}\cdot\frac{1222277}{55}\pi^{20},$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{20}}{1\cdot 2\cdot 3\cdots 23}\cdot\frac{854513}{3}\pi^{22},$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{22}}{1\cdot 2\cdot 3\cdots 25}\cdot\frac{1181820455}{273}\pi^{24},$$

$$1+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\frac{1}{4^{26}}+\frac{1}{5^{26}}+\text{etc.}=\frac{2^{24}}{1\cdot 2\cdot 3\cdots 27}\cdot\frac{76977927}{1}\pi^{26}.$$

Hucusque istos potestatum ipsius π exponentes artificio alibi[1]) exponendo

1) Vide L. EULERI *Institutiones calculi differentialis*, Petropoli 1755, partis posterioris cap. V, § 121 et sq.; *LEONHARDI EULERI Opera omnia*, series I, vol. 10, p. 319 et sq. His locis summae serierum reciprocarum litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ etc. significantur, factores $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$ etc. autem

continuare licuit, quod ideo hic adiunxi, quod seriei fractionum primo intuitu perquam irregularis

$$1, \frac{1}{3}, \frac{1}{3}, \frac{3}{5}, \frac{5}{3}, \frac{691}{105}, \frac{35}{1} \text{ etc.}$$

in plurimis occasionibus eximius est usus.

169. Tractemus eodem modo aequationem § 157 inventam, ubi erat

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{etc.}$$

$$= \left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4xx}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \text{ etc.}$$

Posito ergo $xx = \frac{\pi\pi z}{4}$ erit

$$1 + \frac{\pi\pi}{1 \cdot 2 \cdot 4} z + \frac{\pi^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4^2} zz + \frac{\pi^6}{1 \cdot 2 \cdots 6 \cdot 4^3} z^3 + \text{etc.}$$

$$= (1 + z)\left(1 + \frac{1}{9} z\right)\left(1 + \frac{1}{25} z\right)\left(1 + \frac{1}{49} z\right) \text{ etc.}$$

Unde facta applicatione erit

$$A = \frac{\pi\pi}{1 \cdot 2 \cdot 4}, \quad B = \frac{\pi^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4^2}, \quad C = \frac{\pi^6}{1 \cdot 2 \cdot 3 \cdots 6 \cdot 4^3} \quad \text{etc.}$$

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sunt dupla quantitatum α, β, γ, δ etc.; numeri BERNOULLIANI $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. denique definiuntur aequationibus

$$\mathfrak{A} = \frac{\alpha}{3} = \frac{1}{6}, \quad \mathfrak{B} = \frac{\beta}{5} = \frac{1}{30}, \quad \mathfrak{C} = \frac{\gamma}{7} = \frac{1}{42}, \quad \mathfrak{D} = \frac{\delta}{9} = \frac{1}{30} \quad \text{etc.}$$

Confer etiam praeter dissertationes nota praecedenti laudatas EULERI Commentationes 61 et 736 (indicis ENESTROEMIANI): *De summis serierum reciprocarum ex potestatibus numerorum naturalium ortarum dissertatio altera*, Miscellanea Berolin. **7**, 1743, p. 172, et *De summatione serierum in hac forma contentarum*

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{16} + \frac{a^5}{25} + \frac{a^6}{36} + \text{etc.},$$

Mém. de l'acad. d. sc. de St.-Pétersbourg **3** (1809/10), 1811, p. 26; *LEONHARDI EULERI Opera omnia*, series I, vol. 14 et 16. Vide porro *LEONHARDI EULERI Opera postuma* **1**, Petropoli 1862, p. 519; *LEONHARDI EULERI Opera omnia*, series III. A. K.

Quodsi ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.},$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \text{etc.},$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \text{etc.},$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \text{etc.}$$

etc.,

reperientur sequentes pro P, Q, R, S etc. valores:

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3},$$

$$Q = \frac{2}{1 \cdot 2 \cdot 3} \cdot \frac{\pi^4}{2^5},$$

$$R = \frac{16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{\pi^6}{2^7},$$

$$S = \frac{272}{1 \cdot 2 \cdot 3 \cdots 7} \cdot \frac{\pi^8}{2^9},$$

$$T = \frac{7936}{1 \cdot 2 \cdot 3 \cdots 9} \cdot \frac{\pi^{10}}{2^{11}},$$

$$V = \frac{353792}{1 \cdot 2 \cdot 3 \cdots 11} \cdot \frac{\pi^{12}}{2^{13}},$$

$$W = \frac{22368256}{1 \cdot 2 \cdot 3 \cdots 13} \cdot \frac{\pi^{14}}{2^{15}}$$

etc.

170. Eaedem summae potestatum numerorum imparium inveniri possunt ex summis praecedentibus, in quibus omnes numeri occurrunt. Si enim fuerit

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.},$$

erit ubique per $\frac{1}{2^n}$ multiplicando

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \text{etc.};$$

quae series numeros tantum pares continens si a priori subtrahatur, relinquet numeros impares eritque ideo

$$M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

Quodsi autem series $\frac{M}{2^n}$ bis sumta subtrahatur ab M, signa prodibunt alternantia eritque

$$M - \frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.}$$

Per tradita ergo praecepta summari poterunt hae series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{etc.},$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{etc.},$$

si quidem n sit numerus par, atque summa erit $= A\pi^n$ existente A numero rationali.

171. Praeterea vero expressiones § 164 exhibitae simili modo series notatu dignas suppeditabunt. Cum enim sit

$$\cos. \frac{1}{2} v + \text{tang.} \frac{1}{2} g \sin. \frac{1}{2} v$$

$$= \left(1 + \frac{v}{\pi - g}\right)\left(1 - \frac{v}{\pi + g}\right)\left(1 + \frac{v}{3\pi - g}\right)\left(1 - \frac{v}{3\pi + g}\right) \text{ etc.},$$

si ponamus $v = \frac{x}{n}\pi$ et $g = \frac{m}{n}\pi$, erit

$$\left(1 + \frac{x}{n - m}\right)\left(1 - \frac{x}{n + m}\right)\left(1 + \frac{x}{3n - m}\right)\left(1 - \frac{x}{3n + m}\right)\left(1 + \frac{x}{5n - m}\right)\left(1 - \frac{x}{5n + m}\right) \text{ etc.}$$

$$= \cos. \frac{x\pi}{2n} + \text{tang.} \frac{m\pi}{2n} \sin. \frac{x\pi}{2n}$$

$$= 1 + \frac{\pi x}{2n} \text{tang.} \frac{m\pi}{2n} - \frac{\pi\pi xx}{2 \cdot 4nn} - \frac{\pi^3 x^3}{2 \cdot 4 \cdot 6n^3} \text{tang.} \frac{m\pi}{2n} + \frac{\pi^4 x^4}{2 \cdot 4 \cdot 6 \cdot 8n^4} + \text{etc.}$$

Haec expressio infinita cum § 165 collata dabit hos valores

$$A = \frac{\pi}{2n} \text{ tang. } \frac{m\pi}{2n},$$

$$B = \frac{-\pi\pi}{2 \cdot 4nn},$$

$$C = \frac{-\pi^3}{2 \cdot 4 \cdot 6n^3} \text{ tang. } \frac{m\pi}{2n},$$

$$D = \frac{\pi^4}{2 \cdot 4 \cdot 6 \cdot 8n^4},$$

$$E = \frac{\pi^5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10n^5} \text{ tang. } \frac{m\pi}{2n}$$

etc.

Tum vero erit

$$\alpha = \frac{1}{n-m}, \quad \beta = -\frac{1}{n+m}, \quad \gamma = \frac{1}{3n-m}, \quad \delta = -\frac{1}{3n+m},$$

$$\varepsilon = \frac{1}{5n-m}, \quad \zeta = -\frac{1}{5n+m} \quad \text{etc.}$$

172. Hinc ergo ad normam § 166 sequentes series exorientur:

$$P = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \frac{1}{5n-m} - \text{etc.},$$

$$Q = \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(3n-m)^2} + \frac{1}{(3n+m)^2} + \frac{1}{(5n-m)^2} + \text{etc.},$$

$$R = \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} + \frac{1}{(3n-m)^3} - \frac{1}{(3n+m)^3} + \frac{1}{(5n-m)^3} - \text{etc.},$$

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} + \frac{1}{(5n-m)^4} + \text{etc.},$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} + \frac{1}{(5n-m)^5} - \text{etc.},$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} + \frac{1}{(5n-m)^6} + \text{etc.}$$

etc.

Posito autem tang. $\frac{m\pi}{2n}=k$ erit, uti ostendimus,

$$P=A=\frac{k\pi}{2n} \qquad =\frac{k\pi}{2n},$$

$$Q=\frac{(kk+1)\pi\pi}{4nn} \qquad =\frac{(2kk+2)\pi^2}{2\cdot 4nn},$$

$$R=\frac{(k^3+k)\pi^3}{8n^3} \qquad =\frac{(6k^3+6k)\pi^3}{2\cdot 4\cdot 6n^3},$$

$$S=\frac{(3k^4+4kk+1)\pi^4}{48n^4}=\frac{(24k^4+32k^2+8)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$

$$T=\frac{(3k^5+5k^3+2k)\pi^5}{96n^5}=\frac{(120k^5+200k^3+80k)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5}.$$

etc.

173. Pari modo ultima forma § 164

$$\cos.\frac{1}{2}v+\cot.\frac{1}{2}g\sin.\frac{1}{2}v$$

$$=\left(1+\frac{v}{g}\right)\left(1-\frac{v}{2\pi-g}\right)\left(1+\frac{v}{2\pi+g}\right)\left(1-\frac{v}{4\pi-g}\right)\left(1+\frac{v}{4\pi+g}\right)\text{ etc.},$$

si ponamus $v=\frac{x}{n}\pi$, $g=\frac{m}{n}\pi$ et tang. $\frac{m\pi}{2n}=k$, ut sit cot. $\frac{1}{2}g=\frac{1}{k}$, dabit

$$\cos.\frac{x\pi}{2n}+\frac{1}{k}\sin.\frac{x\pi}{2n}$$

$$=1+\frac{\pi x}{2nk}-\frac{\pi\pi xx}{2\cdot 4nn}-\frac{\pi^3x^3}{2\cdot 4\cdot 6n^3k}+\frac{\pi^4x^4}{2\cdot 4\cdot 6\cdot 8n^4}+\frac{\pi^5x^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k}-\text{ etc.}$$

$$=\left(1+\frac{x}{m}\right)\left(1-\frac{x}{2n-m}\right)\left(1+\frac{x}{2n+m}\right)\left(1-\frac{x}{4n-m}\right)\left(1+\frac{x}{4n+m}\right)\text{ etc.}$$

Comparatione ergo cum forma generali (§ 165) instituta erit

$$A=\frac{\pi}{2nk},\quad B=\frac{-\pi\pi}{2\cdot 4n^2},\quad C=\frac{-\pi^3}{2\cdot 4\cdot 6n^3k},\quad D=\frac{\pi^4}{2\cdot 4\cdot 6\cdot 8n^4},$$

$$E=\frac{\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k}\quad\text{etc.};$$

ex factoribus vero habebitur

$$\alpha=\frac{1}{m},\quad \beta=\frac{-1}{2n-m},\quad \gamma=\frac{1}{2n+m},\quad \delta=\frac{-1}{4n-m},\quad \varepsilon=\frac{1}{4n+m}\quad\text{etc.}$$

174. Hinc ergo ad normam § 166 sequentes series formabuntur earumque summae assignabuntur

$$P = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{etc.},$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.},$$

$$R = \frac{1}{m^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} - \frac{1}{(4n-m)^3} + \frac{1}{(4n+m)^3} - \text{etc.},$$

$$S = \frac{1}{m^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \frac{1}{(4n-m)^4} + \frac{1}{(4n+m)^4} + \text{etc.},$$

$$T = \frac{1}{m^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} - \frac{1}{(4n-m)^5} + \frac{1}{(4n+m)^5} - \text{etc.}$$

Hae autem summae P, Q, R, S etc. ita se habebunt

$$P = A = \frac{\pi}{2nk} = \frac{1\pi}{2nk},$$

$$Q = \frac{(kk+1)\pi\pi}{4nnkk} = \frac{(2+2kk)\pi^2}{2\cdot 4n^2k^2},$$

$$R = \frac{(kk+1)\pi^3}{8n^3k^3} = \frac{(6+6kk)\pi^3}{2\cdot 4\cdot 6n^3k^3},$$

$$S = \frac{(k^4+4kk+3)\pi^4}{48n^4k^4} = \frac{(24+32kk+8k^4)\pi^4}{2\cdot 4\cdot 6\cdot 8n^4k^4},$$

$$T = \frac{(2k^4+5kk+3)\pi^5}{96n^5k^5} = \frac{(120+200kk+80k^4)\pi^5}{2\cdot 4\cdot 6\cdot 8\cdot 10n^5k^5},$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960n^6k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12n^6k^6}$$

etc.

175. Series istae generales merentur, ut casus quosdam particulares inde derivemus, qui prodibunt, si rationem m ad n in numeris determinemus. Sit igitur primum

$$m = 1 \quad \text{et} \quad n = 2;$$

fiet

$$k = \text{tang.}\frac{\pi}{4} = \text{tang.}\,45^0 = 1$$

atque ambae serierum classes inter se congruent. Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.},$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.},$$

$$\frac{5\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.},$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}$$

etc.

Harum serierum primam iam supra (§ 140) elicuimus, reliquarum illae, quae pares habent dignitates, modo ante (§ 169) sunt erutae; ceterae, in quibus exponentes sunt numeri impares, hic primum occurrunt. Constat ergo omnium quoque istarum serierum

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \text{etc.}$$

summas per valorem ipsius π assignari posse.

176. Sit nunc

$$m = 1, \quad n = 3;$$

erit

$$k = \text{tang.}\,\frac{\pi}{6} = \text{tang.}\,30^0 = \frac{1}{\sqrt{3}}$$

atque series § 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \text{etc.},$$

$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \text{etc.},$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \text{etc.}$$

etc.

sive

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \text{etc.},$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \text{etc.},$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \text{etc.}$$

etc.

In his seriebus desunt omnes numeri per ternarium divisibiles; hinc pares dimensiones ex iam inventis deducentur hoc modo. Cum sit [§ 167, 168]

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.},$$

erit

$$\frac{\pi\pi}{6 \cdot 9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \text{etc.} = \frac{\pi\pi}{54};$$

quae posterior series continens omnes numeros per ternarium divisibiles si subtrahatur a priori, remanebunt omnes numeri non divisibiles per 3 sicque erit

$$\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.},$$

uti invenimus.

177. Eadem hypothesis

$$m = 1, \quad n = 3 \quad \text{et} \quad k = \frac{1}{\sqrt{3}}$$

ad § 174 accommodata has praebebit summationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.},$$

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.},$$

$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \text{etc.}$$

etc.,

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares dimensiones ex iam cognitis deduci possunt; cum enim sit [§ 169]

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$

erit

$$\frac{\pi\pi}{8\cdot 9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.} = \frac{\pi\pi}{72};$$

quae series omnes numeros impares per 3 divisibiles continens si subtrahatur a superiore, relinquet seriem quadratorum numerorum imparium per 3 non divisibilium eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

178. Si series in §§ 172 et 174 inventae vel addantur vel subtrahantur, obtinebuntur aliae series notatu dignae. Erit scilicet

$$\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \text{etc.} = \frac{(kk+1)\pi}{2nk};$$

at est

$$k = \text{tang.}\frac{m\pi}{2n} = \frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{m\pi}{2n}} \quad \text{et} \quad 1 + kk = \frac{1}{\left(\cos.\frac{m\pi}{2n}\right)^2},$$

unde

$$\frac{2k}{1+kk} = 2\sin.\frac{m\pi}{2n}\cos.\frac{m\pi}{2n} = \sin.\frac{m\pi}{n},$$

quo valore substituto habebimus

$$\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \text{etc.}$$

Simili modo per subtractionem erit

$$\frac{\pi}{2nk} - \frac{k\pi}{2n} = \frac{(1-kk)\pi}{2nk}$$

$$= \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \text{etc.};$$

at est

$$\frac{2k}{1-kk} = \text{tang.}\, 2\frac{m\pi}{2n} = \text{tang.}\frac{m\pi}{n} = \frac{\sin.\frac{m\pi}{n}}{\cos.\frac{m\pi}{n}};$$

hinc erit

$$\frac{\pi\cos.\frac{m\pi}{n}}{n\sin.\frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Series quadratorum et altiorum potestatum hinc ortae facilius per differentiationem hinc deducentur infra.

179. Quoniam casus, quibus $m=1$ et $n=2$ vel 3, iam evolvimus, ponamus

$$m = 1 \quad \text{et} \quad n = 4;$$

erit

$$\sin.\frac{m\pi}{n} = \sin.\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{et} \quad \cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hinc itaque habebitur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

et

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

Sit

$$m = 1 \quad \text{et} \quad n = 8;$$

erit

$$\frac{m\pi}{n} = \frac{\pi}{8} \quad \text{et} \quad \sin.\frac{\pi}{8} = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)} \quad \text{et} \quad \cos.\frac{\pi}{8} = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)}$$

et

$$\frac{\cos.\frac{\pi}{8}}{\sin.\frac{\pi}{8}} = 1 + \sqrt{2}.$$

Hinc itaque erit

$$\frac{\pi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \text{etc.},$$

$$\frac{\pi}{8(\sqrt{2}-1)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \text{etc.}$$

Sit nunc

$$m = 3 \quad \text{et} \quad n = 8;$$

erit

$$\frac{m\pi}{n} = \frac{3\pi}{8} \quad \text{et} \quad \sin.\frac{3\pi}{8} = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)} \quad \text{et} \quad \cos.\frac{3\pi}{8} = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)},$$

unde

$$\frac{\cos.\frac{3\pi}{8}}{\sin.\frac{3\pi}{8}} = \frac{1}{\sqrt{2}+1},$$

ac prodibunt hae series

$$\frac{\pi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \text{etc.},$$

$$\frac{\pi}{8(\sqrt{2}+1)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \text{etc.}$$

180. Ex his seriebus per combinationem nascuntur

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{etc.},$$

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \text{etc.},$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}+\sqrt{2}-1)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} - \text{etc.},$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}-\sqrt{2}+1)}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \text{etc.},$$

$$\frac{\pi(\sqrt{2}+1+\sqrt{(4-2\sqrt{2})})}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{etc.},$$

$$\frac{\pi(\sqrt{2}+1-\sqrt{(4-2\sqrt{2})})}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \text{etc.}$$

Simili modo ponendo $n = 16$ et m vel 1 vel 3 vel 5 vel 7 ulterius progredi licet hocque modo summae reperientur serierum 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$ etc., in quibus signorum + et — vicissitudines alias leges sequantur.

181. Si in seriebus § 178 inventis bini termini in unam summam colligantur, erit

$$\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16nn-mm}+\text{etc.}$$

ideoque

$$\frac{1}{nn-mm}-\frac{1}{4nn-mm}+\frac{1}{9nn-mm}-\text{etc.}=\frac{\pi}{2mn\sin.\frac{m\pi}{n}}-\frac{1}{2mm}.$$

Altera vero series dabit

$$\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}}=\frac{1}{m}-\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}-\frac{2m}{9nn-mm}-\text{etc.}$$

hincque

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn\,\text{tang.}\frac{m\pi}{n}}.$$

Ex his autem coniunctis nascitur haec

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi\,\text{tang.}\frac{m\pi}{2n}}{4mn}.$$

Si in hac serie sit $n=1$ et m numerus par quicunque $=2k$, ob $\text{tang.}\,k\pi=0$ erit semper, nisi sit $k=0$,

$$\frac{1}{1-4kk}+\frac{1}{9-4kk}+\frac{1}{25-4kk}+\frac{1}{49-4kk}+\text{etc.}=0;$$

sin autem in illa serie fiat $n=2$ et m fuerit numerus quicunque impar $=2k+1$, ob $\frac{1}{\text{tang.}\frac{m\pi}{n}}=0$ erit

$$\frac{1}{4-(2k+1)^2}+\frac{1}{16-(2k+1)^2}+\frac{1}{36-(2k+1)^2}+\text{etc.}=\frac{1}{2(2k+1)^2}.$$

182. Multiplicentur series inventae per nn sitque $\frac{m}{n}=p$; habebuntur istae formae

$$\frac{1}{1-pp}-\frac{1}{4-pp}+\frac{1}{9-pp}-\frac{1}{16-pp}+\text{etc.}=\frac{\pi}{2p\sin.p\pi}-\frac{1}{2pp},$$

$$\frac{1}{1-pp}+\frac{1}{4-pp}+\frac{1}{9-pp}+\frac{1}{16-pp}+\text{etc.}=\frac{1}{2pp}-\frac{\pi}{2p\,\text{tang.}p\pi}.$$

Sit $pp=a$ atque nascentur hae series

$$\frac{1}{1-a}-\frac{1}{4-a}+\frac{1}{9-a}-\frac{1}{16-a}+\text{etc.}=\frac{\pi\sqrt{a}}{2a\sin.\pi\sqrt{a}}-\frac{1}{2a},$$

$$\frac{1}{1-a}+\frac{1}{4-a}+\frac{1}{9-a}+\frac{1}{16-a}+\text{etc.}=\frac{1}{2a}-\frac{\pi\sqrt{a}}{2a\,\text{tang.}\pi\sqrt{a}}.$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, summa harum serierum per circulum exhiberi poterit.

183. Per reductionem autem exponentialium imaginariorum ad sinus et cosinus arcuum circularium supra [§ 138] traditam poterimus quoque summas harum serierum assignare, si a sit numerus negativus. Cum enim sit

$$e^{x\sqrt{-1}}=\cos.x+\sqrt{-1}\cdot\sin.x \quad \text{et} \quad e^{-x\sqrt{-1}}=\cos.x-\sqrt{-1}\cdot\sin.x,$$

erit vicissim posito $y\sqrt{-1}$ loco x

$$\cos.y\sqrt{-1}=\frac{e^{-y}+e^{y}}{2} \quad \text{et} \quad \sin.y\sqrt{-1}=\frac{e^{-y}-e^{y}}{2\sqrt{-1}}.$$

Quodsi ergo $a=-b$ et $y=\pi\sqrt{b}$, erit

$$\cos.\pi\sqrt{-b}=\frac{e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}}}{2} \quad \text{et} \quad \sin.\pi\sqrt{-b}=\frac{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}}{2\sqrt{-1}}$$

ideoque

$$\text{tang.}\pi\sqrt{-b}=\frac{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}}{(e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}})\sqrt{-1}}.$$

Hinc erit

$$\frac{\pi\sqrt{-b}}{\sin.\pi\sqrt{-b}}=\frac{-2\pi\sqrt{b}}{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}} \quad \text{et} \quad \frac{\pi\sqrt{-b}}{\text{tang.}\pi\sqrt{-b}}=\frac{-(e^{-\pi\sqrt{b}}+e^{\pi\sqrt{b}})\pi\sqrt{b}}{e^{-\pi\sqrt{b}}-e^{\pi\sqrt{b}}}.$$

His ergo notatis erit

$$\frac{1}{1+b}-\frac{1}{4+b}+\frac{1}{9+b}-\frac{1}{16+b}+\text{etc.}=\frac{1}{2b}-\frac{\pi\sqrt{b}}{(e^{\pi\sqrt{b}}-e^{-\pi\sqrt{b}})b},$$

$$\frac{1}{1+b}+\frac{1}{4+b}+\frac{1}{9+b}+\frac{1}{16+b}+\text{etc.}=\frac{(e^{\pi\sqrt{b}}+e^{-\pi\sqrt{b}})\pi\sqrt{b}}{2b(e^{\pi\sqrt{b}}-e^{-\pi\sqrt{b}})}-\frac{1}{2b}.$$

Eaedem hae series deduci possunt ex § 162 adhibendo eandem methodum, qua in hoc capite sum usus. Quoniam vero hoc pacto reductio sinuum et cosinuum arcuum imaginariorum ad quantitates exponentiales reales non mediocriter illustratur, hanc explicationem alteri praeferendam duxi.

CAPUT XI

DE ALIIS ARCUUM ATQUE SINUUM EXPRESSIONIBUS INFINITIS

184. Quoniam supra (§ 158) denotante z arcum circuli quemcunque vidimus esse

$$\sin. z = z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right) \text{ etc.}$$

et

$$\cos. z = \left(1-\frac{4zz}{\pi\pi}\right)\left(1-\frac{4zz}{9\pi\pi}\right)\left(1-\frac{4zz}{25\pi\pi}\right)\left(1-\frac{4zz}{49\pi\pi}\right) \text{ etc.,}$$

ponamus esse arcum $z=\frac{m\pi}{n}$; erit

$$\sin. \frac{m\pi}{n} = \frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right) \text{ etc.}$$

et

$$\cos. \frac{m\pi}{n} = \left(1-\frac{4mm}{nn}\right)\left(1-\frac{4mm}{9nn}\right)\left(1-\frac{4mm}{25nn}\right)\left(1-\frac{4mm}{49nn}\right) \text{ etc.}$$

Vel ponatur $2n$ loco n, ut prodeant hae expressiones

$$\sin. \frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot \text{etc.,}$$

$$\cos. \frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot \text{etc.,}$$

quae in factores simplices resolutae dant

$$\sin.\frac{m\pi}{2n}=\frac{m\pi}{2n}\cdot\frac{2n-m}{2n}\cdot\frac{2n+m}{2n}\cdot\frac{4n-m}{4n}\cdot\frac{4n+m}{4n}\cdot\frac{6n-m}{6n}\cdot\text{etc.},$$

$$\cos.\frac{m\pi}{2n}=\frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\frac{5n-m}{5n}\cdot\frac{5n+m}{5n}\cdot\text{etc.}$$

Ponatur $n-m$ loco m; quia est

$$\sin.\frac{(n-m)\pi}{2n}=\cos.\frac{m\pi}{2n}\quad\text{et}\quad\cos.\frac{(n-m)\pi}{2n}=\sin.\frac{m\pi}{2n},$$

provenient hae expressiones

$$\cos.\frac{m\pi}{2n}=\frac{(n-m)\pi}{2n}\cdot\frac{n+m}{2n}\cdot\frac{3n-m}{2n}\cdot\frac{3n+m}{4n}\cdot\frac{5n-m}{4n}\cdot\frac{5n+m}{6n}\cdot\text{etc.},$$

$$\sin.\frac{m\pi}{2n}=\frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{5n}\cdot\frac{6n-m}{5n}\cdot\text{etc.}$$

185. Cum igitur pro sinu et cosinu anguli $\frac{m\pi}{2n}$ binae habeantur expressiones, si eae inter se comparentur dividendo, erit

$$1=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}\cdot\text{etc.}$$

ideoque

$$\frac{\pi}{2}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12\cdot12\cdot\text{etc.}}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot\ 9\cdot11\cdot11\cdot13\cdot\text{etc.}},$$

quae est expressio pro peripheria circuli, quam WALLISIUS invenit in *Arithmetica infinitorum.*[1]) Similes autem huic innumeras expressiones exhibere licet ope primae expressionis pro sinu; ex ea enim deducitur fore

$$\frac{\pi}{2}=\frac{n}{m}\sin.\frac{m\pi}{2n}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\frac{6n}{6n-m}\cdot\text{etc.},$$

1) J. WALLIS, *Arithmetica infinitorum sive nova methodus inquirendi in curvilineorum quadraturam aliaque difficiliora Matheseos problemata*, Oxoniae 1655; *Opera mathematica*, t. I, Oxoniae 1695, p. 355, imprimis p. 469. A. K.

quae posito $\frac{m}{n}=1$ praebet illam ipsam WALLISII formulam. Sit ergo $\frac{m}{n}=\frac{1}{2}$; ob sin. $\frac{1}{4}\pi=\frac{1}{\sqrt{2}}$ erit

$$\frac{\pi}{2}=\frac{\sqrt{2}}{1}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{16}{15}\cdot\frac{16}{17}\cdot\text{etc.}$$

Sit $\frac{m}{n}=\frac{1}{3}$; ob sin. $\frac{1}{6}\pi=\frac{1}{2}$ erit

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{18}{17}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\text{etc.}$$

Quodsi expressio WALLISIANA dividatur per illam, ubi $\frac{m}{n}=\frac{1}{2}$, erit

$$\sqrt{2}=\frac{2\cdot2\cdot6\cdot6\cdot10\cdot10\cdot14\cdot14\cdot18\cdot18\cdot\text{etc.}}{1\cdot3\cdot5\cdot7\cdot\ 9\cdot11\cdot13\cdot15\cdot17\cdot19\cdot\text{etc.}}.$$

186. Quoniam tangens cuiusque anguli aequatur sinui per cosinum diviso, tangens quoque per huiusmodi factores infinitos exprimi poterit. Quodsi autem prima sinus expressio dividatur per alteram cosinus expressionem, erit

$$\text{tang.}\frac{m\pi}{2n}=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{n-m}{m}\cdot\frac{n+m}{2n-m}\cdot\frac{3n-m}{2n+m}\cdot\frac{3n+m}{4n-m}\cdot\frac{5n-m}{4n+m}\cdot\text{etc.}$$

Simili modo autem secantes et cosecantes exprimentur

$$\text{sec.}\frac{m\pi}{2n}=\frac{n}{n-m}\cdot\frac{n}{n+m}\cdot\frac{3n}{3n-m}\cdot\frac{3n}{3n+m}\cdot\frac{5n}{5n-m}\cdot\frac{5n}{5n+m}\cdot\text{etc.},$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{n}{2n-m}\cdot\frac{3n}{2n+m}\cdot\frac{3n}{4n-m}\cdot\frac{5n}{4n+m}\cdot\frac{5n}{6n-m}\cdot\text{etc.}$$

Sin autem alterae sinuum et cosinuum formulae combinentur, erit

$$\text{tang.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{m}{n-m}\cdot\frac{1(2n-m)}{2(n+m)}\cdot\frac{3(2n+m)}{2(3n-m)}\cdot\frac{3(4n-m)}{4(3n+m)}\cdot\text{etc.},$$

$$\text{cot.}\frac{m\pi}{2n}=\frac{\pi}{2}\cdot\frac{n-m}{m}\cdot\frac{1(n+m)}{2(2n-m)}\cdot\frac{3(3n-m)}{2(2n+m)}\cdot\frac{3(3n+m)}{4(4n-m)}\cdot\text{etc.},$$

$$\text{sec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{n-m}\cdot\frac{2n}{n+m}\cdot\frac{2n}{3n-m}\cdot\frac{4n}{3n+m}\cdot\frac{4n}{5n-m}\cdot\text{etc.},$$

$$\text{cosec.}\frac{m\pi}{2n}=\frac{2}{\pi}\cdot\frac{n}{m}\cdot\frac{2n}{2n-m}\cdot\frac{2n}{2n+m}\cdot\frac{4n}{4n-m}\cdot\frac{4n}{4n+m}\cdot\text{etc.}$$

187. Si loco m scribatur k similique modo anguli $\frac{k\pi}{2n}$ sinus et cosinus definiantur ac per has expressiones illae priores dividantur, prodibunt istae formulae

$$\frac{\sin.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}} = \frac{m}{k}\cdot\frac{2n-m}{2n-k}\cdot\frac{2n+m}{2n+k}\cdot\frac{4n-m}{4n-k}\cdot\frac{4n+m}{4n+k}\cdot\text{etc.},$$

$$\frac{\sin.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}} = \frac{m}{n-k}\cdot\frac{2n-m}{n+k}\cdot\frac{2n+m}{3n-k}\cdot\frac{4n-m}{3n+k}\cdot\frac{4n+m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\cos.\frac{k\pi}{2n}} = \frac{n-m}{n-k}\cdot\frac{n+m}{n+k}\cdot\frac{3n-m}{3n-k}\cdot\frac{3n+m}{3n+k}\cdot\frac{5n-m}{5n-k}\cdot\text{etc.},$$

$$\frac{\cos.\frac{m\pi}{2n}}{\sin.\frac{k\pi}{2n}} = \frac{n-m}{k}\cdot\frac{n+m}{2n-k}\cdot\frac{3n-m}{2n+k}\cdot\frac{3n+m}{4n-k}\cdot\frac{5n-m}{4n+k}\cdot\text{etc.}$$

Sumpto ergo pro $\frac{k\pi}{2n}$ eiusmodi angulo, cuius sinus et cosinus dentur, per hos licebit alius cuiuscunque anguli $\frac{m\pi}{2n}$ sinum et cosinum determinare.

188. Vicissim igitur huiusmodi expressionum, quae ex factoribus infinitis constant, valores veri vel per circuli peripheriam vel per sinus et cosinus angulorum datorum assignari possunt, quod ipsum non parvi est momenti, cum etiamnunc aliae methodi non constent, quarum ope huiusmodi productorum infinitorum valores exhiberi queant. Ceterum vero huiusmodi expressiones parum utilitatis afferunt ad valores cum ipsius π tum sinuum cosinuumve angulorum $\frac{m\pi}{2n}$ per approximationem eruendos. Quanquam enim isti factores

$$\frac{\pi}{2} = 2\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\text{ etc.}$$

in fractionibus decimalibus non difficulter in se multiplicantur, tamen nimis multi termini in computum duci deberent, si valorem ipsius π ad decem tantum figuras iustum invenire vellemus.

189. Praecipuus autem usus huiusmodi expressionum etsi infinitarum in inventione logarithmorum versatur, in quo negotio factorum utilitas tanta est, ut sine illis logarithmorum supputatio esset difficillima. Ac primo quidem, cum sit

$$\pi = 4\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right) \text{ etc.},$$

erit sumendis logarithmis

$$l\pi = l4 + l\left(1-\frac{1}{9}\right) + l\left(1-\frac{1}{25}\right) + l\left(1-\frac{1}{49}\right) + \text{etc.}$$

vel

$$l\pi = l2 - l\left(1-\frac{1}{4}\right) - l\left(1-\frac{1}{16}\right) - l\left(1-\frac{1}{36}\right) - \text{etc.},$$

sive logarithmi communes sive hyperbolici sumantur. Quoniam vero ex logarithmis hyperbolicis vulgares facile reperiuntur, insigne compendium adhiberi poterit ad logarithmum hyperbolicum ipsius π inveniendum.

190. Cum igitur logarithmis hyperbolicis sumendis sit

$$l(1-x) = -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{etc.},$$

si hoc modo singuli termini evolvantur, erit

$$\begin{aligned} l\pi = l4 &- \frac{1}{9} - \frac{1}{2\cdot 9^2} - \frac{1}{3\cdot 9^3} - \frac{1}{4\cdot 9^4} - \text{etc.} \\ &- \frac{1}{25} - \frac{1}{2\cdot 25^2} - \frac{1}{3\cdot 25^3} - \frac{1}{4\cdot 25^4} - \text{etc.} \\ &- \frac{1}{49} - \frac{1}{2\cdot 49^2} - \frac{1}{3\cdot 49^3} - \frac{1}{4\cdot 49^4} - \text{etc.} \\ &\qquad \text{etc.} \end{aligned}$$

In his seriebus numero infinitis verticaliter descendendo eiusmodi prodeunt series, quarum summas supra [§ 169, 170] iam invenimus; quare, si brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.},$$

$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.},$$

$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.},$$

$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.}$$

etc.,

erit

$$l\pi = l4 - (A-1) - \tfrac{1}{2}(B-1) - \tfrac{1}{3}(C-1) - \tfrac{1}{4}(D-1) - \text{etc.}$$

Est vero summis supra inventis proxime exprimendis

$$A = 1{,}23370\,05501\,36169\,82735\,431,$$
$$B = 1{,}01467\,80316\,04192\,05454\,625,$$
$$C = 1{,}00144\,70766\,40942\,12190\,647,$$
$$D = 1{,}00015\,51790\,25296\,11930\,298,$$
$$E = 1{,}00001\,70413\,63044\,82548\,818\,\text{[1]},$$
$$F = 1{,}00000\,18858\,48583\,11957\,590,$$
$$G = 1{,}00000\,02092\,40519\,21150\,010,$$
$$H = 1{,}00000\,00232\,37157\,37915\,670,$$
$$I = 1{,}00000\,00025\,81437\,55665\,977,$$
$$K = 1{,}00000\,00002\,86807\,69745\,558,$$
$$L = 1{,}00000\,00000\,31866\,77514\,044,$$
$$M = 1{,}00000\,00000\,03540\,72294\,392,$$
$$N = 1{,}00000\,00000\,00393\,41246\,691,$$
$$O = 1{,}00000\,00000\,00043\,71244\,859,$$
$$P = 1{,}00000\,00000\,00004\,85693\,682,$$
$$Q = 1{,}00000\,00000\,00000\,53965\,957,$$
$$R = 1{,}00000\,00000\,00000\,05996\,217,$$

1) In editione principe quinque ultimae figurae sunt 50816. Correxit A. K.

$$S = 1{,}00000\,00000\,00000\,00666\,246,$$
$$T = 1{,}00000\,00000\,00000\,00074\,027,$$
$$V = 1{,}00000\,00000\,00000\,00008\,225,$$
$$W = 1{,}00000\,00000\,00000\,00000\,914\ ^{1)},$$
$$X = 1{,}00000\,00000\,00000\,00000\,102\ ^{2)}.$$

Hinc sine taedioso calculo reperitur logarithmus hyperbolicus ipsius π

$$= 1{,}14472\,98858\,49400\,17414\,345\ ^{3)};$$

qui si multiplicetur per 0,43429 etc., prodit logarithmus vulgaris ipsius π

$$= 0{,}49714\,98726\,94133\,85435\,128\ ^{4)}.$$

191. Quia porro tam sinum quam cosinum anguli $\frac{m\pi}{2n}$ expressum habemus per factores numero infinitos, utriusque logarithmum commode exprimere poterimus. Erit autem ex formulis primo [§ 184] inventis

$$l\sin.\frac{m\pi}{2n} = l\pi + l\frac{m}{2n} + l\left(1-\frac{mm}{4nn}\right) + l\left(1-\frac{mm}{16nn}\right) + l\left(1-\frac{mm}{36nn}\right) + \text{etc.},$$
$$l\cos.\frac{m\pi}{2n} = l\left(1-\frac{mm}{nn}\right) + l\left(1-\frac{mm}{9nn}\right) + l\left(1-\frac{mm}{25nn}\right) + l\left(1-\frac{mm}{49nn}\right) + \text{etc.}$$

Hinc primum logarithmi hyperbolici ut ante per series maxime convergentes facile exprimuntur. Ne autem praeter necessitatem series infinitas multiplicemus, terminos priores actu in logarithmis involutos relinquamus eritque

$$\begin{aligned} l\sin.\frac{m\pi}{2n} = {} & l\pi + lm + l(2n-m) + l(2n+m) - l8 - 3ln \\ & - \frac{mm}{16nn} - \frac{m^4}{2\cdot 16^2n^4} - \frac{m^6}{3\cdot 16^3n^6} - \frac{m^8}{4\cdot 16^4n^8} - \text{etc.} \\ & - \frac{mm}{36nn} - \frac{m^4}{2\cdot 36^2n^4} - \frac{m^6}{3\cdot 36^3n^6} - \frac{m^8}{4\cdot 36^4n^8} - \text{etc.} \\ & - \frac{mm}{64nn} - \frac{m^4}{2\cdot 64^2n^4} - \frac{m^6}{3\cdot 64^3n^6} - \frac{m^8}{4\cdot 64^4n^8} - \text{etc.} \\ & \text{etc.,} \end{aligned}$$

1) In editione principe ultima figura est 3. 2) In editione principe ultima figura est 1. 3) In editione principe ultima figura est 2. 4) In editione principe ultima figura est 6. Correxit A. K.

$$l\cos.\frac{m\pi}{2n}=l(n-m)+l(n+m)-2ln$$
$$-\frac{mm}{9nn}-\frac{m^4}{2\cdot 9^2n^4}-\frac{m^6}{3\cdot 9^3n^6}-\frac{m^8}{4\cdot 9^4n^8}-\text{etc.}$$
$$-\frac{mm}{25nn}-\frac{m^4}{2\cdot 25^2n^4}-\frac{m^6}{3\cdot 25^3n^6}-\frac{m^8}{4\cdot 25^4n^8}-\text{etc.}$$
$$-\frac{mm}{49nn}-\frac{m^4}{2\cdot 49^2n^4}-\frac{m^6}{3\cdot 49^3n^6}-\frac{m^8}{4\cdot 49^4n^8}-\text{etc.}$$
$$\text{etc.}$$

192. Occurrunt ergo in his seriebus singulae potestates pares ipsius $\frac{m}{n}$, quae sunt multiplicatae per series, quarum summas iam supra assignavimus. Erit nempe

$$l\sin.\frac{m\pi}{2n}=lm+l(2n-m)+l(2n+m)-3ln+l\pi-l8$$
$$-\frac{mm}{nn}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{12^2}+\text{etc.}\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\frac{1}{10^4}+\frac{1}{12^4}+\text{etc.}\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\frac{1}{10^6}+\frac{1}{12^6}+\text{etc.}\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\frac{1}{10^8}+\frac{1}{12^8}+\text{etc.}\right)$$
$$\text{etc.},$$

$$l\cos.\frac{m\pi}{2n}=l(n-m)+l(n+m)-2ln$$
$$-\frac{mm}{nn}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{etc.}\right)$$
$$-\frac{m^4}{2n^4}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}\right)$$
$$-\frac{m^6}{3n^6}\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}\right)$$
$$-\frac{m^8}{4n^8}\left(\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}\right)$$
$$\text{etc.}$$

Serierum posteriorum modo ante (§ 190) summae sunt exhibitae; priores series quidem ex his derivari possent, at, quo facilius ad usum transferri queant, earum summas pariter hic adiiciam.

193. Quodsi ergo brevitatis gratia ponamus

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \text{etc.},$$

$$\beta = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \text{etc.},$$

$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \text{etc.},$$

$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \text{etc.}$$

etc.,

erunt summae in numeris proxime expressae hae:

$$\alpha = 0{,}41123\,35167\,12056\,60911\,810,$$

$$\beta = 0{,}06764\,52021\,06946\,13696\,975,$$

$$\gamma = 0{,}01589\,59853\,43507\,01780\,804,$$

$$\delta = 0{,}00392\,21771\,72648\,22007\,570,$$

$$\varepsilon = 0{,}00097\,75337\,64773\,25984\,896 \text{ [1]},$$

$$\zeta = 0{,}00024\,42007\,04724\,92872\,273 \text{ [2]},$$

$$\eta = 0{,}00006\,10388\,94539\,49332\,915,$$

$$\theta = 0{,}00001\,52590\,22251\,27271\,502 \text{ [3]},$$

$$\iota = 0{,}00000\,38147\,11827\,44318\,008,$$

$$\varkappa = 0{,}00000\,09536\,75226\,17534\,053,$$

$$\lambda = 0{,}00000\,02384\,18635\,95259\,255 \text{ [4]},$$

1) In editione principe ultima figura est 8. Correxit A. K.
2) In editione principe ultima figura est 4. Correxit A. K.
3) In editione principe quinque ultimae figurae sunt 69977. Correxit A. K.
4) In editione principe tres ultimae figurae sunt 154. Correxit A. K.

$$\mu = 0{,}00000\,00596\,04648\,32831\,556\ ^{1)},$$
$$\nu = 0{,}00000\,00149\,01161\,41589\,813,$$
$$\xi = 0{,}00000\,00037\,25290\,31233\,986,$$
$$o = 0{,}00000\,00009\,31322\,57548\,284,$$
$$\pi = 0{,}00000\,00002\,32830\,64370\,808\ ^{1)},$$
$$\varrho = 0{,}00000\,00000\,58207\,66091\,686\ ^{1)},$$
$$\sigma = 0{,}00000\,00000\,14551\,91522\,858,$$
$$\tau = 0{,}00000\,00000\,03637\,97880\,710,$$
$$v = 0{,}00000\,00000\,00909\,49470\,177,$$
$$\varphi = 0{,}00000\,00000\,00227\,37367\,544,$$
$$\chi = 0{,}00000\,00000\,00056\,84341\,886,$$
$$\psi = 0{,}00000\,00000\,00014\,21085\,472\ ^{1)},$$
$$\omega = 0{,}00000\,00000\,00003\,55271\,368\ ^{1)}.$$

Reliquae summae in ratione quadrupla decrescunt.

194. His ergo in subsidium vocatis erit

$$l\sin.\frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\pi - l8$$
$$-\frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(\beta - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right) - \text{etc.},$$

$$l\cos.\frac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$
$$-\frac{mm}{nn}(A-1) - \frac{m^4}{2n^4}(B-1) - \frac{m^6}{3n^6}(C-1) - \text{etc.};$$

quoniam igitur logarithmi $l\pi$ et $l8$ dantur, erit

1) In editione principe ultima figura est unitate minor. Correxit A. K.

Logarithmus hyperbolicus sinus anguli $\frac{m}{n} 90^0$

$$= lm + l(2n - m) + l(2n + m) - 3ln$$

$$- \quad 0{,}93471\,16558\,30435\,75411 \text{ }^{1)}$$

$$- \frac{m^2}{n^2} \cdot 0{,}16123\,35167\,12056\,60912 \text{ }^{1)}$$

$$- \frac{m^4}{n^4} \cdot 0{,}00257\,26010\,53473\,06848$$

$$- \frac{m^6}{n^6} \cdot 0{,}00009\,03284\,47835\,67260$$

$$- \frac{m^8}{n^8} \cdot 0{,}00000\,39817\,93162\,05502 \text{ }^{1)}$$

$$- \frac{m^{10}}{n^{10}} \cdot 0{,}00000\,01942\,52954\,65197 \text{ }^{1)}$$

$$- \frac{m^{12}}{n^{12}} \cdot 0{,}00000\,00100\,13287\,48812$$

$$- \frac{m^{14}}{n^{14}} \cdot 0{,}00000\,00005\,34041\,35619 \text{ }^{1)}$$

$$- \frac{m^{16}}{n^{16}} \cdot 0{,}00000\,00000\,29148\,59659 \text{ }^{1)}$$

$$- \frac{m^{18}}{n^{18}} \cdot 0{,}00000\,00000\,01617\,97980 \text{ }^{2)}$$

$$- \frac{m^{20}}{n^{20}} \cdot 0{,}00000\,00000\,00090\,97691 \text{ }^{1)}$$

$$- \frac{m^{22}}{n^{22}} \cdot 0{,}00000\,00000\,00005\,16828 \text{ }^{1)}$$

$$- \frac{m^{24}}{n^{24}} \cdot 0{,}00000\,00000\,00000\,29608 \text{ }^{1)}$$

$$- \frac{m^{26}}{n^{26}} \cdot 0{,}00000\,00000\,00000\,01708$$

$$- \frac{m^{28}}{n^{28}} \cdot 0{,}00000\,00000\,00000\,00099$$

$$- \frac{m^{30}}{n^{30}} \cdot 0{,}00000\,00000\,00000\,00006 \text{ }^{1)}.$$

1) In editione principe ultima figura est unitate minor. Correxit A. K.

2) In editione principe duae ultimae figurae sunt 79. Correxit A. K.

At

$$\textit{Logarithmus hyperbolicus cosinus anguli } \frac{m}{n}\,90^{\circ}$$
$$\begin{aligned}
&= l(n-m) + l(n+m) - 2ln \\
&- \frac{m^2}{n^2}\cdot 0{,}23370\,05501\,36169\,82735 \\
&- \frac{m^4}{n^4}\cdot 0{,}00733\,90158\,02096\,02727 \\
&- \frac{m^6}{n^6}\cdot 0{,}00048\,23588\,80314\,04064\,^{1)} \\
&- \frac{m^8}{n^8}\cdot 0{,}00003\,87947\,56324\,02983\,^{1)} \\
&- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,34082\,72608\,96510 \\
&- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,03143\,08097\,18660\,^{2)} \\
&- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00298\,91502\,74450 \\
&- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00029\,04644\,67239 \\
&- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00002\,86826\,39518 \\
&- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,28680\,76975\,^{1)} \\
&- \frac{m^{22}}{n^{22}}\cdot 0{,}00000\,00000\,02896\,97956 \\
&- \frac{m^{24}}{n^{24}}\cdot 0{,}00000\,00000\,00295\,06025\,^{1)} \\
&- \frac{m^{26}}{n^{26}}\cdot 0{,}00000\,00000\,00030\,26250\,^{3)} \\
&- \frac{m^{28}}{n^{28}}\cdot 0{,}00000\,00000\,00003\,12232 \\
&- \frac{m^{30}}{n^{30}}\cdot 0{,}00000\,00000\,00000\,32380\,^{4)} \\
&- \frac{m^{32}}{n^{32}}\cdot 0{,}00000\,00000\,00000\,03373 \\
&- \frac{m^{34}}{n^{34}}\cdot 0{,}00000\,00000\,00000\,00353\,^{1)} \\
&- \frac{m^{36}}{n^{36}}\cdot 0{,}00000\,00000\,00000\,00037 \\
&- \frac{m^{38}}{n^{38}}\cdot 0{,}00000\,00000\,00000\,00004.
\end{aligned}$$

1) In editione principe ultima figura est unitate minor. Correxit A. K.
2) In editione principe duae ultimae figurae sunt 59. Correxit A. K.
3) In editione principe duae ultimae figurae sunt 49. Correxit A. K.
4) In editione principe duae ultimae figurae sunt 79. Correxit A. K.

195. Si isti sinuum et cosinuum logarithmi hyperbolici multiplicentur per 0,43429 44819 etc., prodibunt eorundem logarithmi vulgares ad radium $= 1$ relati. Quoniam vero in tabulis logarithmus sinus totius statui solet $= 10$, quo logarithmi tabulares sinuum et cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

$$\textit{Logarithmus tabularis sinus anguli } \frac{m}{n}90^{\circ}$$

$$= lm + l(2n - m) + l(2n + m) - 3ln$$

$$\begin{aligned}
&+ \quad 9{,}59405\,98857\,02190 \\
&- \frac{m^2}{n^2}\cdot 0{,}07002\,28266\,05902\,^{1)} \\
&- \frac{m^4}{n^4}\cdot 0{,}00111\,72664\,41662\,^{1)} \\
&- \frac{m^6}{n^6}\cdot 0{,}00003\,92291\,46454\,^{1)} \\
&- \frac{m^8}{n^8}\cdot 0{,}00000\,17292\,70798 \\
&- \frac{m^{10}}{n^{10}}\cdot 0{,}00000\,00843\,62986 \\
&- \frac{m^{12}}{n^{12}}\cdot 0{,}00000\,00043\,48715 \\
&- \frac{m^{14}}{n^{14}}\cdot 0{,}00000\,00002\,31931 \\
&- \frac{m^{16}}{n^{16}}\cdot 0{,}00000\,00000\,12659 \\
&- \frac{m^{18}}{n^{18}}\cdot 0{,}00000\,00000\,00703\,^{1)} \\
&- \frac{m^{20}}{n^{20}}\cdot 0{,}00000\,00000\,00040\,^{2)}.
\end{aligned}$$

1) In editione principe ultima figura est unitate minor. Correxit A. K.

2) In editione principe duae ultimae figurae sunt 39. Correxit A. K.

Logarithmus tabularis cosinus anguli $\frac{m}{n}90^{\circ}$

$$= l(n-m) + l(n+m) - 2ln$$

$$+ \quad 10{,}00000\,00000\,00000$$

$$- \frac{m^2}{n^2} \cdot 0{,}10149\,48593\,41893\,^{1)}$$

$$- \frac{m^4}{n^4} \cdot 0{,}00318\,72940\,65451$$

$$- \frac{m^6}{n^6} \cdot 0{,}00020\,94858\,00017$$

$$- \frac{m^8}{n^8} \cdot 0{,}00001\,68483\,48598\,^{1)}$$

$$- \frac{m^{10}}{n^{10}} \cdot 0{,}00000\,14801\,93987^{1)}$$

$$- \frac{m^{12}}{n^{12}} \cdot 0{,}00000\,01365\,02272$$

$$- \frac{m^{14}}{n^{14}} \cdot 0{,}00000\,00129\,81715$$

$$- \frac{m^{16}}{n^{16}} \cdot 0{,}00000\,00012\,61471$$

$$- \frac{m^{18}}{n^{18}} \cdot 0{,}00000\,00001\,24567$$

$$- \frac{m^{20}}{n^{20}} \cdot 0{,}00000\,00000\,12456$$

$$- \frac{m^{22}}{n^{22}} \cdot 0{,}00000\,00000\,01258$$

$$- \frac{m^{24}}{n^{24}} \cdot 0{,}00000\,00000\,00128$$

$$- \frac{m^{26}}{n^{26}} \cdot 0{,}00000\,00000\,00013.$$

196. Harum ergo formularum ope inveniri possunt logarithmi sinuum et cosinuum quorumvis angulorum tam hyperbolici quam vulgares etiam ignoratis ipsis sinibus et cosinibus. Ex logarithmis autem sinuum et cosinuum

1) In editione principe ultima figura est unitate minor. A. K.

per solam subtractionem inveniuntur logarithmi tangentium, cotangentium et secantium cosecantiumque, quamobrem pro his peculiaribus formulis non erit opus. Ceterum notandum est numerorum m, n, $n-m$, $n+m$ etc. logarithmos hyperbolicos accipi oportere, cum logarithmi hyperbolici sinuum cosinuumque quaeruntur, vulgares autem, cum tales ope posteriorum formularum sunt indagandi. Praeterea $m:n$ denotat rationem, quam angulus propositus habet ad angulum rectum; sicque cum sinus angulorum semirecto maiorum aequentur cosinibus angulorum semirecto minorum ac vicissim, fractio $\frac{m}{n}$ nunquam maior accipienda erit quam $\frac{1}{2}$ hancque ob rem termini illi multo magis convergent, ut semissis instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem aperiamus modum tangentes et secantes quorumvis angulorum inveniendi, quam caput praecedens suppeditat. Quanquam enim tangentes et secantes per sinus et cosinus determinantur, tamen hoc fit per divisionem, quae operatio in tantis numeris nimis est operosa. Ac tangentes quidem et cotangentes iam supra (§ 135) exhibuimus, verum illo loco rationem formularum reddere non licuit, quam huic capiti reservavimus.

198. Ex § 181 ergo primum expressionem pro tangente anguli $\frac{m\pi}{2n}$ elicimus. Cum enim sit

$$\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}=\frac{\pi}{4mn}\operatorname{tang.}\frac{m\pi}{2n},$$

erit

$$\operatorname{tang.}\frac{m\pi}{2n}=\frac{4mn}{\pi}\left(\frac{1}{nn-mm}+\frac{1}{9nn-mm}+\frac{1}{25nn-mm}+\text{etc.}\right).$$

Cum deinde sit

$$\frac{1}{nn-mm}+\frac{1}{4nn-mm}+\frac{1}{9nn-mm}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot.\frac{m\pi}{n},$$

si pro n scribamus $2n$, erit

$$\cot.\frac{m\pi}{2n}=\frac{2n}{m\pi}-\frac{4mn}{\pi}\left(\frac{1}{4nn-mm}+\frac{1}{16nn-mm}+\frac{1}{36nn-mm}+\text{etc.}\right).$$

Convertantur hae fractiones praeter primas, quippe quae facile in computum ducuntur, in series infinitas; erit

$$\begin{aligned}\operatorname{tang.}\frac{m\pi}{2n} = \frac{mn}{nn-mm}\cdot\frac{4}{\pi} &+ \frac{4}{\pi}\left(\frac{m}{3^2 n}+\frac{m^3}{3^4 n^3}+\frac{m^5}{3^6 n^5}+\text{etc.}\right)\\ &+ \frac{4}{\pi}\left(\frac{m}{5^2 n}+\frac{m^3}{5^4 n^3}+\frac{m^5}{5^6 n^5}+\text{etc.}\right)\\ &+ \frac{4}{\pi}\left(\frac{m}{7^2 n}+\frac{m^3}{7^4 n^3}+\frac{m^5}{7^6 n^5}+\text{etc.}\right)\\ &\qquad\text{etc.,}\end{aligned}$$

$$\begin{aligned}\cot.\frac{m\pi}{2n} = \frac{n}{m}\cdot\frac{2}{\pi} - \frac{mn}{4nn-mm}\cdot\frac{4}{\pi} &- \frac{4}{\pi}\left(\frac{m}{4^2 n}+\frac{m^3}{4^4 n^3}+\frac{m^5}{4^6 n^5}+\text{etc.}\right)\\ &- \frac{4}{\pi}\left(\frac{m}{6^2 n}+\frac{m^3}{6^4 n^3}+\frac{m^5}{6^6 n^5}+\text{etc.}\right)\\ &- \frac{4}{\pi}\left(\frac{m}{8^2 n}+\frac{m^3}{8^4 n^3}+\frac{m^5}{8^6 n^5}+\text{etc.}\right)\\ &\qquad\text{etc.}\end{aligned}$$

198 [a][1]). At ex valore ipsius π cognito reperitur

$$\frac{1}{\pi} = 0{,}31830\,98861\,83790\,67153\,77675\,26745\,028724\ ^{2)},$$

deinde hic eaedem series occurrunt, quas supra [§ 190 et 193] litteris A, B, C, D etc. et α, β, γ, δ etc. indicavimus. His ergo notatis erit

$$\operatorname{tang.}\frac{m\pi}{2n} = \frac{mn}{nn-mm}\cdot\frac{4}{\pi}$$

$$+\frac{m}{n}\cdot\frac{4}{\pi}(A-1)+\frac{m^3}{n^3}\cdot\frac{4}{\pi}(B-1)+\frac{m^5}{n^5}\cdot\frac{4}{\pi}(C-1)+\frac{m^7}{n^7}\cdot\frac{4}{\pi}(D-1)+\text{etc.}$$

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1) In editione principe numerus 198 per errorem iteratur. A. K.

2) In editione principe figura vicesima quinta est 9. Correxit A. K.

Deinde erit pro cotangente

$$\cot.\frac{m\pi}{2n}=\frac{n}{m}\cdot\frac{2}{\pi}-\frac{4mn}{4nn-mm}\cdot\frac{1}{\pi}$$
$$-\frac{m}{n}\cdot\frac{4}{\pi}\left(\alpha-\frac{1}{2^2}\right)-\frac{m^3}{n^3}\cdot\frac{4}{\pi}\left(\beta-\frac{1}{2^4}\right)-\frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma-\frac{1}{2^6}\right)-\text{etc.},$$

atque ex his formulis natae sunt expressiones, quas supra (§ 135) pro tangente et cotangente dedimus; simul vero (§ 137) ostendimus, quomodo ex tangentibus et cotangentibus inventis per solam additionem et subtractionem secantes et cosecantes reperiantur. Harum ergo regularum ope universus canon sinuum, tangentium et secantium, eorumque logarithmorum multo facilius supputari posset, quam quidem hoc a primis conditoribus est factum.

CAPUT XII

DE REALI FUNCTIONUM FRACTARUM EVOLUTIONE

199. Iam supra, in capite secundo, methodus est tradita functionem quamcunque fractam in tot partes resolvendi, quot eius denominator habeat factores simplices; hi enim praebent denominatores fractionum illarum partialium. Ex quo manifestum est, si denominator quos habeat factores simplices imaginarios, fractiones quoque inde ortas fore imaginarias; his ergo casibus parum iuvabit fractionem realem in imaginarias resolvisse. Cum igitur ostendissem [cap. IX] omnem functionem integram, qualis est denominator cuiusvis fractionis, quantumvis factoribus simplicibus imaginariis scateat, tamen in factores duplices, seu secundae dimensionis, reales semper resolvi posse, hoc modo in resolutione fractionum quantitates imaginariae evitari poterunt, si pro denominatoribus fractionum partialium non factores denominatoris principalis simplices, sed duplices reales assumamus.

200. Sit igitur proposita haec functio fracta $\frac{M}{N}$, ex qua tot fractiones simplices secundum methodum supra [cap. II] expositam eliciantur, quot denominator N habuerit factores simplices reales. Sit autem loco imaginariorum haec expressio

$$pp - 2pqz\cos.\varphi + qqzz$$

factor ipsius N, et quoniam in hoc negotio numeratorem et denominatorem in forma evoluta contemplari oportet, sit haec fractio proposita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}}{(pp - 2pqz\cos.\varphi + qqzz)(\alpha + \beta z + \gamma zz + \delta z^3 + \text{etc.})}$$

ac ponatur fractio partialis ex denominatoris factore $pp - 2pqz \cos.\varphi + qqzz$ oriunda haec

$$\frac{\mathfrak{A} + \mathfrak{a}z}{pp - 2pqz \cos.\varphi + qqzz};$$

quoniam enim variabilis z in denominatore duas habet dimensiones, in numeratore unam habere poterit, non vero plures; alias enim integra functio contineretur, quam seorsim elici oportet.

201. Sit brevitatis gratia numerator

$$A + Bz + Cz^2 + \text{etc.} = M$$

et alter denominatoris factor

$$\alpha + \beta z + \gamma z^2 + \text{etc.} = Z;$$

ponatur altera pars ex denominatoris factore Z oriunda $= \frac{Y}{Z}$ eritque

$$Y = \frac{M - \mathfrak{A}Z - \mathfrak{a}zZ}{pp - 2pqz \cos.\varphi + qqzz},$$

quae expressio functio integra ipsius z esse debet, ideoque necesse est, ut

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

divisibile sit per $pp - 2pqz \cos.\varphi + qqzz$. Evanescet ergo $M - \mathfrak{A}Z - \mathfrak{a}zZ$, si ponatur

$$pp - 2pqz \cos.\varphi + qqzz = 0,$$

hoc est, si ponatur [§ 146] tam

$$z = \frac{p}{q}(\cos.\varphi + \sqrt{-1} \cdot \sin.\varphi)$$

quam

$$z = \frac{p}{q}(\cos.\varphi - \sqrt{-1} \cdot \sin.\varphi);$$

sit $\frac{p}{q} = f$ eritque [§ 133]

$$z^n = f^n(\cos. n\varphi \pm \sqrt{-1} \cdot \sin. n\varphi).$$

Duplex ergo hic valor pro z substitutus duplicem dabit aequationem, unde ambas incognitas constantes $\mathfrak{A}$ et $\mathfrak{a}$ definire licet.

202. Facta ergo hac substitutione aequatio

$$M = \mathfrak{A}Z + \mathfrak{a}zZ$$

evoluta hanc duplicem dabit aequationem

$$\begin{aligned}
&A + Bf\cos.\varphi + Cff\cos.2\varphi + Df^3\cos.3\varphi + \text{etc.}\\
&\quad \pm (Bf\sin.\varphi + Cff\sin.2\varphi + Df^3\sin.3\varphi + \text{etc.})\sqrt{-1}\\
&= \mathfrak{A}(\alpha + \beta f\cos.\varphi + \gamma ff\cos.2\varphi + \delta f^3\cos.3\varphi + \text{etc.})\\
&\quad \pm \mathfrak{A}(\beta f\sin.\varphi + \gamma ff\sin.2\varphi + \delta f^3\sin.3\varphi + \text{etc.})\sqrt{-1}\\
&\quad + \mathfrak{a}(\alpha f\cos.\varphi + \beta ff\cos.2\varphi + \gamma f^3\cos.3\varphi + \text{etc.})\\
&\quad \pm \mathfrak{a}(\alpha f\sin.\varphi + \beta ff\sin.2\varphi + \gamma f^3\sin.3\varphi + \text{etc.})\sqrt{-1}.
\end{aligned}$$

Sit ad calculum abbreviandum

$$\begin{aligned}
A + Bf\cos.\varphi + Cff\cos.2\varphi + Df^3\cos.3\varphi + \text{etc.} &= \mathfrak{P},\\
Bf\sin.\varphi + Cff\sin.2\varphi + Df^3\sin.3\varphi + \text{etc.} &= \mathfrak{p},\\
\alpha + \beta f\cos.\varphi + \gamma ff\cos.2\varphi + \delta f^3\cos.3\varphi + \text{etc.} &= \mathfrak{Q},\\
\beta f\sin.\varphi + \gamma ff\sin.2\varphi + \delta f^3\sin.3\varphi + \text{etc.} &= \mathfrak{q},\\
\alpha f\cos.\varphi + \beta ff\cos.2\varphi + \gamma f^3\cos.3\varphi + \text{etc.} &= \mathfrak{R},\\
\alpha f\sin.\varphi + \beta ff\sin.2\varphi + \gamma f^3\sin.3\varphi + \text{etc.} &= \mathfrak{r}
\end{aligned}$$

eritque his positis

$$\mathfrak{P} \pm \mathfrak{p}\sqrt{-1} = \mathfrak{A}\mathfrak{Q} \pm \mathfrak{A}\mathfrak{q}\sqrt{-1} + \mathfrak{a}\mathfrak{R} \pm \mathfrak{a}\mathfrak{r}\sqrt{-1}.$$

203. Ob signorum ambiguitatem hae duae oriuntur aequationes

$$\begin{aligned}
\mathfrak{P} &= \mathfrak{A}\mathfrak{Q} + \mathfrak{a}\mathfrak{R},\\
\mathfrak{p} &= \mathfrak{A}\mathfrak{q} \;\, + \mathfrak{a}\mathfrak{r},
\end{aligned}$$

ex quibus incognitae $\mathfrak{A}$ et $\mathfrak{a}$ ita definiuntur, ut sit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \quad \text{et} \quad \mathfrak{a} = \frac{\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}}{\mathfrak{q}\mathfrak{R} - \mathfrak{Q}\mathfrak{r}}.$$

Proposita ergo fractione

$$\frac{M}{(pp - 2pqz \cos.\varphi + qqzz)Z}$$

per sequentem regulam fractio partialis ex ea oriunda

$$\frac{\mathfrak{A} + \mathfrak{a}z}{pp - 2pqz \cos.\varphi + qqzz}$$

definietur. Posito $f = \frac{p}{q}$ et evolutis singulis terminis fiat, ut sequitur:

Posito $z^n = f^n \cos. n\varphi$ sit $M = \mathfrak{P}$,

$z^n = f^n \sin. n\varphi$ sit $M = \mathfrak{p}$,

$z^n = f^n \cos. n\varphi$ sit $Z = \mathfrak{Q}$,

$z^n = f^n \sin. n\varphi$ sit $Z = \mathfrak{q}$,

$z^n = f^n \cos. n\varphi$ sit $zZ = \mathfrak{R}$,

$z^n = f^n \sin. n\varphi$ sit $zZ = \mathfrak{r}$.

Inventis hoc modo valoribus $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ erit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}} \quad \text{et} \quad \mathfrak{a} = \frac{\mathfrak{p}\mathfrak{Q} - \mathfrak{P}\mathfrak{q}}{\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R}}.$$

EXEMPLUM 1

Sit proposita haec functio fracta

$$\frac{zz}{(1 - z + zz)(1 + z^4)},$$

ex qua partem a denominatoris factore $1 - z + zz$ oriundam definire oporteat, quae sit

$$\frac{\mathfrak{A} + \mathfrak{a}z}{1 - z + zz}.$$

Ac primo quidem hic factor cum forma generali $pp - 2pqz\cos.\varphi + qqzz$ comparatus dat

$$p = 1, \quad q = 1 \quad \text{et} \quad \cos.\varphi = \frac{1}{2},$$

unde fit

$$\varphi = 60^0 = \frac{\pi}{3}.$$

Quia itaque est

$$M = zz, \quad Z = 1 + z^4 \quad \text{et} \quad f = 1,$$

erit

$$\mathfrak{P} = \cos.\frac{2\pi}{3} = -\frac{1}{2}, \quad \mathfrak{p} = \frac{\sqrt{3}}{2},$$

$$\mathfrak{Q} = 1 + \cos.\frac{4\pi}{3} = \frac{1}{2}, \quad \mathfrak{q} = -\frac{\sqrt{3}}{2},$$

$$\mathfrak{R} = \cos.\frac{\pi}{3} + \cos.\frac{5\pi}{3} = 1, \quad \mathfrak{r} = 0.$$

Ex his invenitur

$$\mathfrak{A} = -1 \quad \text{et} \quad \mathfrak{a} = 0$$

ideoque fractio quaesita est

$$\frac{-1}{1 - z + zz}$$

huiusque complementum erit

$$\frac{1 + z + zz}{1 + z^4};$$

cuius denominator $1 + z^4$ cum habeat factores $1 + z\sqrt{2} + zz$ et $1 - z\sqrt{2} + zz$, resolutio denuo suscipi potest; fit autem $\varphi = \frac{\pi}{4}$ et priori casu $f = -1$, posteriori $f = +1$.

EXEMPLUM 2

Sit igitur proposita haec fractio resolvenda

$$\frac{1 + z + zz}{(1 + z\sqrt{2} + zz)(1 - z\sqrt{2} + zz)}$$

et erit

$$M = 1 + z + zz;$$

et pro priore factore habebitur

$$f = -1, \quad \varphi = \frac{\pi}{4} \quad \text{et} \quad Z = 1 - z\sqrt{2} + zz,$$

unde erit

$$\mathfrak{P} = 1 - \cos.\frac{\pi}{4} + \cos.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}},$$

$$\mathfrak{p} = \quad -\sin.\frac{\pi}{4} + \sin.\frac{2\pi}{4} = \frac{\sqrt{2}-1}{\sqrt{2}},$$

$$\mathfrak{Q} = 1 + \sqrt{2}\cdot\cos.\frac{\pi}{4} + \cos.\frac{2\pi}{4} = 2,$$

$$\mathfrak{q} = \quad +\sqrt{2}\cdot\sin.\frac{\pi}{4} + \sin.\frac{2\pi}{4} = 2,$$

$$\mathfrak{R} = -\cos.\frac{\pi}{4} - \sqrt{2}\cdot\cos.\frac{2\pi}{4} - \cos.\frac{3\pi}{4} = 0,$$

$$\mathfrak{r} = -\sin.\frac{\pi}{4} - \sqrt{2}\cdot\sin.\frac{2\pi}{4} - \sin.\frac{3\pi}{4} = -2\sqrt{2}.$$

Ex his reperitur

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -4\sqrt{2}$$

et

$$\mathfrak{A} = \frac{\sqrt{2}-1}{2\sqrt{2}} \quad \text{et} \quad \mathfrak{a} = 0,$$

unde ex denominatoris factore $1 + z\sqrt{2} + zz$ haec orietur fractio partialis

$$\frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz}.$$

Alter autem factor dabit simili modo hanc

$$\frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

Hinc functio primum proposita

$$\frac{zz}{(1-z+zz)(1+z^4)}$$

resolvitur in has

$$\frac{-1}{1-z+zz} + \frac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz} + \frac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}.$$

EXEMPLUM 3

Sit proposita haec fractio resolvenda

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}.$$

Pro factore denominatoris $1-\frac{8}{5}z+zz$ oriatur ista fractio

$$\frac{\mathfrak{A}+\mathfrak{a}z}{1-\frac{8}{5}z+zz}$$

eritque

$$p=1, \quad q-1, \quad \cos.\varphi=\frac{4}{5},$$

unde

$$f=1, \quad M=1+2z+zz, \quad Z=1+2z+3zz.$$

Quia vero hic ratio anguli φ ad rectum non constat, sinus et cosinus eius multiplorum seorsim debent investigari. Cum sit

$$\cos.\varphi=\frac{4}{5}, \qquad \text{erit} \qquad \sin.\varphi=\frac{3}{5},$$

$$\cos.2\varphi=\frac{7}{25}, \qquad \sin.2\varphi=\frac{24}{25},$$

$$\cos.3\varphi=-\frac{44}{125}, \qquad \sin.3\varphi=\frac{117}{125};$$

hinc fit

$$\mathfrak{P}=1+2\cdot\frac{4}{5}+\frac{7}{25}=\frac{72}{25},$$

$$\mathfrak{p}=2\cdot\frac{3}{5}+\frac{24}{25}=\frac{54}{25},$$

$$\mathfrak{Q}=1+2\cdot\frac{4}{5}+3\cdot\frac{7}{25}=\frac{86}{25},$$

$$\mathfrak{q}=2\cdot\frac{3}{5}+3\cdot\frac{24}{25}=\frac{102}{25},$$

$$\mathfrak{R}=\frac{4}{5}+2\cdot\frac{7}{25}-3\cdot\frac{44}{125}=\frac{38}{125},$$

$$\mathfrak{r}=\frac{3}{5}+2\cdot\frac{24}{25}+3\cdot\frac{117}{125}=\frac{666}{125}$$

ideoque

$$\mathfrak{Q}\mathfrak{r}-\mathfrak{q}\mathfrak{R}=\frac{53400}{25\cdot125}=\frac{2136}{125}.$$

Ergo

$$\mathfrak{A} = \frac{1836}{2136} = \frac{153}{178}, \quad \mathfrak{a} = -\frac{540}{2136} = -\frac{45}{178}.$$

Quare fractio ex factore $1 - \frac{8}{5}z + zz$ oriunda erit

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz}.$$

Quaeramus simili modo fractionem alteri factori respondentem; erit

$$p = 1, \quad q = -\sqrt{3} \quad \text{et} \quad \cos.\varphi = \frac{1}{\sqrt{3}},$$

ergo

$$f = -\frac{1}{\sqrt{3}}, \quad M = 1 + 2z + zz \quad \text{et} \quad Z = 1 - \frac{8}{5}z + zz.$$

Fiet autem ob

$$\cos.\varphi = \frac{1}{\sqrt{3}} \qquad \sin.\varphi = \frac{\sqrt{2}}{\sqrt{3}},$$

$$\cos.2\varphi = -\frac{1}{3}, \qquad \sin.2\varphi = \frac{2\sqrt{2}}{3},$$

$$\cos.3\varphi = -\frac{5}{3\sqrt{3}}, \qquad \sin.3\varphi = \frac{\sqrt{2}}{3\sqrt{3}},$$

consequenter

$$\mathfrak{P} = 1 - \frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{2}{9},$$

$$\mathfrak{p} = -\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9},$$

$$\mathfrak{Q} = 1 + \frac{8}{5\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\cdot -\frac{1}{3} = \frac{64}{45},$$

$$\mathfrak{q} = +\frac{8}{5\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3}\cdot\frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45},$$

$$\mathfrak{R} = -\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot -\frac{1}{3} - \frac{1}{3\sqrt{3}}\cdot -\frac{5}{3\sqrt{3}} = \frac{4}{135},$$

$$\mathfrak{r} = -\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5\cdot 3}\cdot\frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}}\cdot\frac{\sqrt{2}}{3\sqrt{3}} = -\frac{98\sqrt{2}}{135}$$

ideoque

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = -\frac{712\sqrt{2}}{675};$$

fiet ergo

$$\mathfrak{A} = \frac{100}{712} = \frac{25}{178}, \quad \mathfrak{a} = \frac{540}{712} = \frac{135}{178}.$$

Fractio ergo proposita

$$\frac{1+2z+zz}{(1-\frac{8}{5}z+zz)(1+2z+3zz)}$$

resolvitur in

$$\frac{9(17-5z):178}{1-\frac{8}{5}z+zz} + \frac{5(5+27z):178}{1+2z+3zz}.$$

204. Possunt autem valores litterarum $\mathfrak{R}$ et $\mathfrak{r}$ ex litteris $\mathfrak{Q}$ et $\mathfrak{q}$ definiri. Cum enim sit

$$\mathfrak{Q} = \alpha + \beta f \cos.\varphi + \gamma f^2 \cos.2\varphi + \delta f^3 \cos.3\varphi + \text{etc.},$$

$$\mathfrak{q} = \quad \beta f \sin.\varphi + \gamma f^2 \sin.2\varphi + \delta f^3 \sin.3\varphi + \text{etc.},$$

erit

$$\mathfrak{Q}\cos.\varphi - \mathfrak{q}\sin.\varphi = \alpha\cos.\varphi + \beta f\cos.2\varphi + \gamma f^2\cos.3\varphi + \text{etc.}$$

ideoque

$$\mathfrak{R} = f(\mathfrak{Q}\cos.\varphi - \mathfrak{q}\sin.\varphi).$$

Deinde erit

$$\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi = \alpha\sin.\varphi + \beta f\sin.2\varphi + \gamma f^2\sin.3\varphi + \text{etc.},$$

ergo

$$\mathfrak{r} = f(\mathfrak{Q}\sin.\varphi + \mathfrak{q}\cos.\varphi).$$

Ex his porro fit

$$\mathfrak{Q}\mathfrak{r} - \mathfrak{q}\mathfrak{R} = (\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})f\sin.\varphi,$$

$$\mathfrak{P}\mathfrak{r} - \mathfrak{p}\mathfrak{R} = (\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})f\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})f\cos.\varphi$$

eritque consequenter

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}} + \frac{\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}}{\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q}}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a} = \frac{-\mathfrak{P}\mathfrak{q}+\mathfrak{p}\mathfrak{Q}}{(\mathfrak{Q}\mathfrak{Q}+\mathfrak{q}\mathfrak{q})f\sin.\varphi}.$$

Quare ex denominatoris factore $pp - 2pqz\cos.\varphi + qqzz$ nascitur ista fractio partialis

$$\frac{(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})f\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})(f\cos.\varphi - z)}{(pp - 2pqz\cos.\varphi + qqzz)(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})f\sin.\varphi}$$

seu ob $f = \frac{p}{q}$ haec

$$\frac{(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})p\sin.\varphi + (\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})(p\cos.\varphi - qz)}{(pp - 2pqz\cos.\varphi + qqzz)(\mathfrak{Q}\mathfrak{Q} + \mathfrak{q}\mathfrak{q})p\sin.\varphi}.$$

205. Oritur ergo haec fractio partialis ex functionis propositae

$$\frac{M}{(pp - 2pqz\cos.\varphi + qqzz)Z}$$

factore denominatoris $pp - 2pqz\cos.\varphi + qqzz$ atque litterae $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{Q}$ et $\mathfrak{q}$ sequenti modo ex functionibus M et Z inveniuntur: Posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M = \mathfrak{P} \quad \text{et} \quad Z = \mathfrak{Q}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M = \mathfrak{p} \quad \text{et} \quad Z = \mathfrak{q},$$

ubi notandum est functiones M et Z, antequam haec substitutio fiat, omnino evolvi debere, ut huiusmodi habeant formas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc.}$$

et

$$Z = \alpha + \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \text{etc.};$$

eritque ideo

$$\mathfrak{P} = A + B\frac{p}{q}\cos.\varphi + C\frac{p^2}{q^2}\cos.2\varphi + D\frac{p^3}{q^3}\cos.3\varphi + \text{etc.},$$

$$\mathfrak{p} = \quad B\frac{p}{q}\sin.\varphi + C\frac{p^2}{q^2}\sin.2\varphi + D\frac{p^3}{q^3}\sin.3\varphi + \text{etc.},$$

$$\mathfrak{Q} = \alpha + \beta\frac{p}{q}\cos.\varphi + \gamma\frac{p^2}{q^2}\cos.2\varphi + \delta\frac{p^3}{q^3}\cos.3\varphi + \text{etc.},$$

$$\mathfrak{q} = \quad \beta\frac{p}{q}\sin.\varphi + \gamma\frac{p^2}{q^2}\sin.2\varphi + \delta\frac{p^3}{q^3}\sin.3\varphi + \text{etc.}$$

206. Ex praecedentibus autem intelligitur hanc resolutionem locum habere non posse, si functio Z eundem factorem $pp - 2pqz \cos.\varphi + qqzz$ adhuc in se complectatur; hoc enim casu in aequatione

$$M = \mathfrak{A}Z + \mathfrak{a}zZ$$

facta substitutione

$$z^n = f^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi)$$

ipsa quantitas Z evanesceret nihilque propterea colligi posset. Quamobrem si functionis fractae $\frac{M}{N}$ denominator habeat factorem $(pp - 2pqz \cos.\varphi + qqzz)^2$ vel altiorem potestatem, peculiari opus erit resolutione. Sit igitur

$$N = (pp - 2pqz \cos.\varphi + qqzz)^2 Z$$

atque ex denominatoris factore $(pp - 2pqz \cos.\varphi + qqzz)^2$ orientur huiusmodi duae fractiones partiales

$$\frac{\mathfrak{A} + \mathfrak{a}z}{(pp - 2pqz \cos.\varphi + qqzz)^2} + \frac{\mathfrak{B} + \mathfrak{b}z}{pp - 2pqz \cos.\varphi + qqzz},$$

ubi litteras constantes $\mathfrak{A}$, $\mathfrak{a}$, $\mathfrak{B}$, $\mathfrak{b}$ determinari oportet.

207. His positis debebit ista expressio

$$\frac{M - (\mathfrak{A} + \mathfrak{a}z)Z - (\mathfrak{B} + \mathfrak{b}z)Z(pp - 2pqz \cos.\varphi + qqzz)}{(pp - 2pqz \cos.\varphi + qqzz)^2}$$

esse functio integra et hanc ob rem numerator divisibilis erit per denominatorem [§ 43]. Primum ergo haec expressio

$$M - \mathfrak{A}Z - \mathfrak{a}zZ$$

divisibilis esse debet per $pp - 2pqz \cos.\varphi + qqzz$; qui cum sit casus praecedens, eodem quoque modo litterae $\mathfrak{A}$ et $\mathfrak{a}$ determinabuntur.

Quare posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M = \mathfrak{P} \quad \text{et} \quad Z = \mathfrak{N}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M = \mathfrak{p} \quad \text{et} \quad Z = \mathfrak{n}.$$

Hisque factis secundum regulam supra datam erit

$$\mathfrak{A} = \frac{\mathfrak{P}\mathfrak{N} + \mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{P}\mathfrak{n} - \mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a} = \frac{-\mathfrak{P}\mathfrak{n} + \mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

208. Inventis ergo hoc modo $\mathfrak{A}$ et $\mathfrak{a}$ fiet

$$\frac{M - (\mathfrak{A} + \mathfrak{a}z)Z}{pp - 2pqz\cos.\varphi + qqzz}$$

functio integra, quae sit $= P$, atque superest, ut

$$P - \mathfrak{B}Z - \mathfrak{b}zZ$$

divisibile evadat per $pp - 2pqz\cos.\varphi + qqzz$; quae expressio cum similis sit praecedenti, si posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{vocetur} \quad P = \mathfrak{R}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{vocetur} \quad P = \mathfrak{r},$$

erit

$$\mathfrak{B} = \frac{\mathfrak{R}\mathfrak{N} + \mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n} - \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{b} = \frac{-\mathfrak{R}\mathfrak{n} + \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p\sin.\varphi}.$$

209. Hinc iam generaliter concludere licet, quomodo resolutio institui debeat, si denominator functionis propositae $\frac{M}{N}$ factorem habeat

$$(pp - 2pqz\cos.\varphi + qqzz)^k.$$

Sit enim

$$N = (pp - 2pqz\cos.\varphi + qqzz)^k Z,$$

ita ut haec resolvenda sit functio fracta

$$\frac{M}{(pp - 2pqz\cos.\varphi + qqzz)^k Z}.$$

Praebeat ergo factor denominatoris $(pp - 2pqz\cos.\varphi + qqzz)^k$ has partes

$$\frac{\mathfrak{A} + \mathfrak{a}z}{(pp - 2pqz\cos.\varphi + qqzz)^k} + \frac{\mathfrak{B} + \mathfrak{b}z}{(pp - 2pqz\cos.\varphi + qqzz)^{k-1}}$$
$$+ \frac{\mathfrak{C} + \mathfrak{c}z}{(pp - 2pqz\cos.\varphi + qqzz)^{k-2}} + \frac{\mathfrak{D} + \mathfrak{d}z}{(pp - 2pqz\cos.\varphi + qqzz)^{k-3}} + \text{etc.}$$

Iam posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad M = \mathfrak{M} \quad \text{et} \quad Z = \mathfrak{N}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad M = \mathfrak{m} \quad \text{et} \quad Z = \mathfrak{n};$$

erit

$$\mathfrak{A} = \frac{\mathfrak{M}\mathfrak{N} + \mathfrak{m}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{M}\mathfrak{n} - \mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{a} = \frac{-\mathfrak{M}\mathfrak{n} + \mathfrak{m}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Deinde vocetur

$$\frac{M - (\mathfrak{A} + \mathfrak{a}z)Z}{pp - 2pqz\cos.\varphi + qqzz} = P$$

atque posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad P = \mathfrak{P}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad P = \mathfrak{p};$$

erit

$$\mathfrak{B} = \frac{\mathfrak{P}\mathfrak{N} + \mathfrak{p}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{P}\mathfrak{n} - \mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{b} = \frac{-\mathfrak{P}\mathfrak{n} + \mathfrak{p}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2}\cdot\frac{q}{p\sin.\varphi}.$$

Tum vocetur

$$\frac{P - (\mathfrak{B} + \mathfrak{b}z)Z}{pp - 2pqz\cos.\varphi + qqzz} = Q$$

atque posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad Q = \mathfrak{Q}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad Q = \mathfrak{q};$$

erit

$$\mathfrak{C} = \frac{\mathfrak{Q}\mathfrak{N} + \mathfrak{q}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{Q}\mathfrak{n} - \mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{c} = \frac{-\mathfrak{Q}\mathfrak{n} + \mathfrak{q}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p \sin.\varphi}.$$

Porro vocetur

$$\frac{Q - (\mathfrak{C} + \mathfrak{c}z)Z}{pp - 2pqz\cos.\varphi + qqzz} = R$$

atque posito

$$z^n = \frac{p^n}{q^n}\cos.n\varphi \quad \text{sit} \quad R = \mathfrak{R}$$

et posito

$$z^n = \frac{p^n}{q^n}\sin.n\varphi \quad \text{sit} \quad R = \mathfrak{r};$$

erit

$$\mathfrak{D} = \frac{\mathfrak{R}\mathfrak{N} + \mathfrak{r}\mathfrak{n}}{\mathfrak{N}^2 + \mathfrak{n}^2} + \frac{\mathfrak{R}\mathfrak{n} - \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{\cos.\varphi}{\sin.\varphi},$$

$$\mathfrak{d} = \frac{-\mathfrak{R}\mathfrak{n} + \mathfrak{r}\mathfrak{N}}{\mathfrak{N}^2 + \mathfrak{n}^2} \cdot \frac{q}{p \sin.\varphi}.$$

Hocque modo progrediendum est, donec ultimae fractionis, cuius denominator est $pp - 2pqz\cos.\varphi + qqzz$, numerator fuerit determinatus.

EXEMPLUM

Sit ista proposita functio fracta

$$\frac{z - z^3}{(1 + zz)^4(1 + z^4)},$$

ex cuius denominatoris factore $(1 + zz)^4$ oriantur hae fractiones partiales

$$\frac{\mathfrak{A} + \mathfrak{a}z}{(1 + zz)^4} + \frac{\mathfrak{B} + \mathfrak{b}z}{(1 + zz)^3} + \frac{\mathfrak{C} + \mathfrak{c}z}{(1 + zz)^2} + \frac{\mathfrak{D} + \mathfrak{d}z}{1 + zz}.$$

Comparatione ergo instituta erit

$$p = 1, \quad q = 1, \quad \cos.\varphi = 0 \quad \text{ideoque} \quad \varphi = \frac{\pi}{2}$$

porroque

$$M = z - z^3 \quad \text{et} \quad Z = 1 + z^4.$$

Hinc erit

$$\mathfrak{M} = 0, \quad \mathfrak{m} = 2, \quad \mathfrak{N} = 2, \quad \mathfrak{n} = 0 \quad \text{et} \quad \sin. \varphi = 1.$$

Hinc itaque invenitur

$$\mathfrak{A} = -\frac{4}{4} \cdot 0 = 0 \quad \text{et} \quad \mathfrak{a} = 1,$$

ergo

$$\mathfrak{A} + \mathfrak{a}z = z$$

hincque

$$P = \frac{z - z^3 - z - z^5}{1 + zz} = -z^3$$

et

$$\mathfrak{P} = 0, \quad \mathfrak{p} = 1,$$

unde reperitur

$$\mathfrak{B} = 0 \quad \text{et} \quad \mathfrak{b} = \frac{1}{2}.$$

Ergo

$$\mathfrak{B} + \mathfrak{b}z = \frac{1}{2}z$$

et

$$Q = \frac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1 + zz} = -\frac{1}{2}z - \frac{1}{2}z^3,$$

unde

$$\mathfrak{Q} = 0 \quad \text{et} \quad \mathfrak{q} = 0,$$

ergo

$$\mathfrak{C} = 0 \quad \text{et} \quad \mathfrak{c} = 0.$$

Hincque

$$R = \frac{-\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z,$$

ergo

$$\mathfrak{R} = 0 \quad \text{et} \quad \mathfrak{r} = -\frac{1}{2},$$

unde fit

$$\mathfrak{D} = 0 \quad \text{et} \quad \mathfrak{d} = -\frac{1}{4}.$$

Quamobrem fractiones quaesitae sunt hae

$$\frac{z}{(1+zz)^4} + \frac{z}{2(1+zz)^3} - \frac{z}{4(1+zz)}.$$

Reliquae vero fractionis numerator est

$$S = \frac{R - (\mathfrak{D} + \mathfrak{d}z)Z}{1 + zz} = -\frac{1}{4}z + \frac{1}{4}z^3,$$

quae ergo erit

$$= \frac{-z + z^3}{4(1 + z^4)}.$$

210. Hac ergo methodo simul innotescit fractio complementi, quae cum inventis coniuncta producat fractionem propositam ipsam. Scilicet si fractionis

$$\frac{M}{(pp-2pqz\cos.\varphi+qqzz)^k Z}$$

inventae fuerint omnes fractiones partiales ex factore $(pp-2pqz\cos.\varphi+qqzz)^k$ oriundae, pro quibus formati sunt valores functionum P, Q, R, S, T, si harum litterarum series ulterius continuetur, erit ea, quae ultimam, qua opus est ad numeratores inveniendos, sequitur, numerator reliquae fractionis denominatorem Z habentis; nempe, si $k=1$, erit reliqua fractio $\frac{P}{Z}$; si $k=2$, erit reliqua fractio $\frac{Q}{Z}$; si $k=3$, erit ea $\frac{R}{Z}$, et ita porro. Inventa autem hac reliqua fractione denominatorem Z habente ea per has regulas ulterius resolvi poterit.

CAPUT XIII

DE SERIEBUS RECURRENTIBUS

211. Ad hoc serierum genus, quas MOIVREUS[1]) *recurrentes* vocare solet, hic refero omnes series, quae ex evolutione functionis cuiusque fractae per divisionem actualem instituta nascuntur. Supra [cap. IV] enim iam ostendimus has series ita esse comparatas, ut quivis terminus ex aliquot praecedentibus secundum legem quandam constantem determinetur, quae lex a denominatore functionis fractae pendet. Cum autem nunc functionem quamcunque fractam in alias simpliciores resolvere docuerim, hinc series quoque recurrens in alias simpliciores resolvetur. In hoc igitur capite propositum est serierum recurrentium cuiusvis gradus resolutionem in simpliciores exponere.

212. Sit proposita ista functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \text{etc.}}{1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \text{etc.}},$$

quae per divisionem evolvatur in hanc seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.};$$

cuius coefficientes quemadmodum progrediantur, supra est ostensum. Quodsi iam functio illa fracta resolvatur in fractiones suas simplices et unaquaeque in seriem recurrentem evolvatur, manifestum est summam omnium harum

1) Vide notam p. 79. F. R.

serierum ex fractionibus partialibus ortarum aequalem esse debere seriei recurrenti

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.}$$

Fractiones ergo partiales, quas supra [cap. II] invenire docuimus, dabunt series partiales, quarum indoles ob simplicitatem facile perspicitur; omnes autem series partiales iunctim sumptae producent seriem recurrentem propositam, unde et huius natura penitius cognoscetur.

213. Sint series recurrentes ex singulis fractionibus partialibus ortae hae

$$\begin{aligned}
&a + bz + czz + dz^3 + ez^4 + \text{etc.},\\
&a' + b'z + c'zz + d'z^3 + e'z^4 + \text{etc.},\\
&a'' + b''z + c''zz + d''z^3 + e''z^4 + \text{etc.},\\
&a''' + b'''z + c'''zz + d'''z^3 + e'''z^4 + \text{etc.}\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Quoniam hae series iunctim sumptae aequales esse debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \text{etc.},$$

necesse est, ut sit

$$\begin{aligned}
A &= a + a' + a'' + a''' + \text{etc.},\\
B &= b + b' + b'' + b''' + \text{etc.},\\
C &= c + c' + c'' + c''' + \text{etc.},\\
D &= d + d' + d'' + d''' + \text{etc.}\\
&\qquad \text{etc.}
\end{aligned}$$

Hinc, si singularum serierum ex fractionibus partialibus ortarum definiri queant coefficientes potestatis z^n, horum summa dabit coefficientem potestatis z^n in serie recurrente $A + Bz + Cz^2 + Dz^3 + \text{etc.}$

214. Dubium hic suboriri posset, an, si duae huiusmodi series fuerint inter se aequales

$$A + Bz + Cz^2 + Dz^3 + \text{etc.} = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \text{etc.},$$

necessario inde sequatur coefficientes similium potestatum ipsius z inter se esse aequales, seu an sit $A = \mathfrak{A}$, $B = \mathfrak{B}$, $C = \mathfrak{C}$, $D = \mathfrak{D}$ etc. Hoc autem dubium facile tolletur, si perpendamus hanc aequalitatem subsistere debere, quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$ atque manifestum est fore $A = \mathfrak{A}$. His ergo terminis aequalibus utrinque sublatis ac reliqua aequatione per z divisa habebitur

$$B + Cz + Dz^2 + \text{etc.} = \mathfrak{B} + \mathfrak{C}z + \mathfrak{D}z^2 + \text{etc.},$$

unde sequitur fore $B = \mathfrak{B}$; simili autem modo ostendetur esse $C = \mathfrak{C}$, $D = \mathfrak{D}$ et ita porro in infinitum.[1]

215. Contemplemur ergo series, quae ex fractionibus partialibus, in quas fractio quaepiam proposita resolvitur, oriuntur. Ac primo quidem patet fractionem

$$\frac{\mathfrak{A}}{1 - pz}$$

dare seriem

$$\mathfrak{A} + \mathfrak{A}pz + \mathfrak{A}p^2z^2 + \mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$\mathfrak{A}p^nz^n;$$

haec enim expressio vocari solet *terminus generalis*, quoniam ex ea loco n numeros omnes successive substituendo omnes seriei termini nascuntur. Deinde ex fractione

$$\frac{\mathfrak{A}}{(1 - pz)^2}$$

oritur series

$$\mathfrak{A} + 2\mathfrak{A}pz + 3\mathfrak{A}p^2z^2 + 4\mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$(n + 1)\mathfrak{A}p^nz^n.$$

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1) Confer EULERI Commentationem 130 (indicis ENESTROEMIANI): *De seriebus quibusdam considerationes*, Comment. acad. sc. Petrop. **12** (1740), 1750, p. 53, imprimis p. 61; *LEONHARDI EULERI Opera omnia,* series I, vol. 14. F. R.

Tum ex fractione

$$\frac{\mathfrak{A}}{(1-pz)^3}$$

oritur series

$$\mathfrak{A} + 3\mathfrak{A}pz + 6\mathfrak{A}p^2z^2 + 10\mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^nz^n.$$

Generatim autem fractio

$$\frac{\mathfrak{A}}{(1-pz)^k}$$

praebet seriem hanc

$$\mathfrak{A} + k\mathfrak{A}pz + \frac{k(k+1)}{1\cdot 2}\mathfrak{A}p^2z^2 + \frac{k(k+1)(k+2)}{1\cdot 2\cdot 3}\mathfrak{A}p^3z^3 + \text{etc.},$$

cuius terminus generalis est

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}\mathfrak{A}p^nz^n.$$

Ex ipsa autem seriei progressione colligitur hic idem terminus

$$=\frac{k(k+1)(k+2)\cdots(k+n-1)}{1\cdot 2\cdot 3\cdots n}\mathfrak{A}p^nz^n.$$

Haec vero expressio illi est aequalis, id quod multiplicatione per crucem instituta patebit; fiet enim

$$1\cdot 2\cdot 3\cdots n(n+1)\cdots(n+k-1) = 1\cdot 2\cdot 3\cdots(k-1)k\cdots(k+n-1),$$

quae est aequatio identica.

216. Quoties ergo in resolutione functionum fractarum ad huiusmodi fractiones partiales $\frac{\mathfrak{A}}{(1-pz)^k}$ pervenitur, toties seriei recurrentis ex illa functione fracta ortae

$$A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

terminus generalis assignari poterit, quippe qui erit summa terminorum generalium serierum, quae ex fractionibus partialibus nascuntur.

EXEMPLUM 1

Invenire terminum generalem seriei recurrentis, quae ex hac fractione

$$\frac{1-z}{1-z-2zz}$$

nascitur.

Series hinc nata est

$$1+0z+2zz+2z^3+6z^4+10z^5+22z^6+42z^7+86z^8+\text{etc.}$$

Ad coefficientem potestatis generalis z^n inveniendum fractio $\frac{1-z}{1-z-2zz}$ resolvatur in

$$\frac{\frac{2}{3}}{1+z}+\frac{\frac{1}{3}}{1-2z},$$

unde oritur terminus generalis quaesitus

$$\left(\frac{2}{3}(-1)^n+\frac{1}{3}\cdot 2^n\right)z^n=\frac{2^n\pm 2}{3}z^n,$$

ubi signum $+$ valet, si n sit numerus par, signum $-$, si n sit impar.

EXEMPLUM 2

Invenire terminum generalem seriei recurrentis, quae oritur ex fractione

$$\frac{1-z}{1-5z+6zz},$$

seu seriei huius

$$1+4z+14z^2+46z^3+146z^4+454z^5+\text{etc.}$$

Ob denominatorem $=(1-2z)(1-3z)$ resolvitur fractio in has

$$\frac{-1}{1-2z}+\frac{2}{1-3z},$$

ex quibus fit terminus generalis

$$2\cdot 3^n z^n-2^n z^n=(2\cdot 3^n-2^n)z^n.$$

EXEMPLUM 3

Invenire terminum generalem seriei huius

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 + \text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{1+2z}{1-z-zz}.$$

Ob denominatoris factores

$$1 - \frac{1+\sqrt{5}}{2}z \quad \text{et} \quad 1 - \frac{1-\sqrt{5}}{2}z$$

per resolutionem prodeunt

$$\frac{\frac{1+\sqrt{5}}{2}}{1-\frac{1+\sqrt{5}}{2}z} + \frac{\frac{1-\sqrt{5}}{2}}{1-\frac{1-\sqrt{5}}{2}z},$$

unde erit terminus generalis

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} z^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} z^n.$$

EXEMPLUM 4

Invenire terminum generalem seriei huius

$$a + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 + \text{etc.},$$

quae oritur ex evolutione fractionis

$$\frac{a+bz}{1-\alpha z-\beta zz}.$$

Per resolutionem oriuntur hae duae fractiones

$$\frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}+\alpha\right)+2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}z} + \frac{\left(a\left(\sqrt{(\alpha\alpha+4\beta)}-\alpha\right)-2b\right):2\sqrt{(\alpha\alpha+4\beta)}}{1-\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}z};$$

hinc terminus generalis erit

$$\frac{a(\sqrt{(\alpha\alpha+4\beta)}+\alpha)+2b}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha+\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n$$

$$+\frac{a(\sqrt{(\alpha\alpha+4\beta)}-\alpha)-2b}{2\sqrt{(\alpha\alpha+4\beta)}}\left(\frac{\alpha-\sqrt{(\alpha\alpha+4\beta)}}{2}\right)^n z^n.$$

Ex quo omnium serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur, termini generales expedite definiri poterunt.

EXEMPLUM 5

Invenire terminum generalem huius seriei

$$1+z+2z^2+2z^3+3z^4+3z^5+4z^6+4z^7+\text{etc.},$$

quae oritur ex fractione

$$\frac{1}{1-z-zz+z^3}=\frac{1}{(1-z)^2(1+z)}.$$

Quanquam lex progressionis primo intuitu ita est manifesta, ut explicatione non indigeat, tamen fractiones per resolutionem ortae

$$\frac{\frac{1}{2}}{(1-z)^2}+\frac{\frac{1}{4}}{1-z}+\frac{\frac{1}{4}}{1+z}$$

dant hunc terminum generalem

$$\frac{1}{2}(n+1)z^n+\frac{1}{4}z^n+\frac{1}{4}(-1)^n z^n=\frac{2n+3\pm 1}{4}z^n,$$

ubi signum superius valet, si n fuerit numerus par, inferius, si n fuerit impar.

217. Hoc pacto omnium serierum recurrentium termini generales exhiberi possunt, quoniam omnes fractiones in huiusmodi fractiones partiales simplices resolvere licet. Quodsi autem expressiones imaginarias vitare velimus, saepenumero ad huiusmodi fractiones partiales pervenietur

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz},\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2},\quad \ldots\quad \frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^k};$$

ex quarum evolutione cuiusmodi series nascantur, videndum est. Ac primo quidem ob

$$\cos.n\varphi = 2\cos.\varphi\cos.(n-1)\varphi - \cos.(n-2)\varphi$$

fractio

$$\frac{\mathfrak{A}}{1-2pz\cos.\varphi+ppzz}$$

evoluta dabit [§ 61]

$$\begin{array}{l}\mathfrak{A} + 2\mathfrak{A}pz\cos.\varphi + 2\mathfrak{A}ppzz\cos.2\varphi + 2\mathfrak{A}p^3z^3\cos.3\varphi + 2\mathfrak{A}p^4z^4\cos.4\varphi + \text{etc.}\\ \qquad\qquad + \mathfrak{A}ppzz \qquad\qquad + 2\mathfrak{A}p^3z^3\cos.\varphi + 2\mathfrak{A}p^4z^4\cos.2\varphi + \text{etc.}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathfrak{A}p^4z^4 \qquad\qquad + \text{etc.}\\ \qquad\qquad\qquad\qquad\qquad \text{etc.},\end{array}$$

cuius seriei terminus generalis non tam facile apparet.

218. Quo igitur ad scopum perveniamus, consideremus has duas series

$$Ppz\sin.\varphi + Pp^2z^2\sin.2\varphi + Pp^3z^3\sin.3\varphi + Pp^4z^4\sin.4\varphi + \text{etc.},$$

$$Q + Qpz\cos.\varphi + Qp^2z^2\cos.2\varphi + Qp^3z^3\cos.3\varphi + Qp^4z^4\cos.4\varphi + \text{etc.},$$

quae duae series utique nascuntur ex evolutione fractionis, cuius denominator est

$$1-2pz\cos.\varphi+ppzz.$$

Ac prior quidem oritur ex hac fractione

$$\frac{Ppz\sin.\varphi}{1-2pz\cos.\varphi+ppzz},$$

posterior vero ex hac

$$\frac{Q-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}.$$

Addantur hae duae fractiones atque summa

$$\frac{Q+Ppz\sin.\varphi-Qpz\cos.\varphi}{1-2pz\cos.\varphi+ppzz}$$

dabit seriem, cuius terminus generalis erit

$$(P \sin. n\varphi + Q \cos. n\varphi) p^n z^n.$$

Fiat autem haec fractio propositae

$$\frac{\mathfrak{A} + \mathfrak{B}pz}{1 - 2pz \cos. \varphi + ppzz}$$

aequalis; erit

$$Q = \mathfrak{A} \quad \text{et} \quad P = \mathfrak{A} \cot. \varphi + \mathfrak{B} \operatorname{cosec}. \varphi.$$

Seriei ergo ex hac fractione

$$\frac{\mathfrak{A} + \mathfrak{B}pz}{1 - 2pz \cos. \varphi + ppzz}$$

ortae terminus generalis erit

$$\frac{\mathfrak{A} \cos. \varphi \sin. n\varphi + \mathfrak{B} \sin. n\varphi + \mathfrak{A} \sin. \varphi \cos. n\varphi}{\sin. \varphi} p^n z^n = \frac{\mathfrak{A} \sin. (n+1)\varphi + \mathfrak{B} \sin. n\varphi}{\sin. \varphi} p^n z^n.$$

219. Ad terminum generalem inveniendum, si denominator fractionis fuerit potestas ut

$$(1 - 2pz \cos. \varphi + ppzz)^k,$$

conveniet hanc fractionem resolvi in duas etsi imaginarias

$$\frac{a}{(1 - (\cos. \varphi + \sqrt{-1} \cdot \sin. \varphi)pz)^k} + \frac{b}{(1 - (\cos. \varphi - \sqrt{-1} \cdot \sin. \varphi)pz)^k},$$

quarum simul sumptarum terminus generalis seriei ex ipsis ortae erit

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1 \cdot 2 \cdot 3 \cdots (k-1)} (\cos. n\varphi + \sqrt{-1} \cdot \sin. n\varphi) a p^n z^n$$
$$+ \frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1 \cdot 2 \cdot 3 \cdots (k-1)} (\cos. n\varphi - \sqrt{-1} \cdot \sin. n\varphi) b p^n z^n.$$

Sit

$$a + b = f, \quad a - b = \frac{g}{\sqrt{-1}},$$

ut sit

$$a = \frac{f\sqrt{-1} + g}{2\sqrt{-1}} \quad \text{et} \quad b = \frac{f\sqrt{-1} - g}{2\sqrt{-1}},$$

eritque haec expressio

$$\frac{(n+1)(n+2)(n+3)\cdots(n+k-1)}{1\cdot 2\cdot 3\cdots(k-1)}(f\cos. n\varphi + g\sin. n\varphi)p^n z^n$$

terminus generalis seriei, quae oritur ex his fractionibus

$$\frac{\frac{1}{2}f+\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi)pz)^k}+\frac{\frac{1}{2}f-\frac{1}{2\sqrt{-1}}g}{(1-(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi)pz)^k},$$

seu quae oritur ex hac fractione una

$$\frac{\left\{\begin{array}{l} f-kfpz\cos.\varphi+\frac{k(k-1)}{1\cdot 2}fp^2z^2\cos.2\varphi-\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}fp^3z^3\cos.3\varphi+\text{etc.}\\ +kgpz\sin.\varphi-\frac{k(k-1)}{1\cdot 2}gp^2z^2\sin.2\varphi+\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}gp^3z^3\sin.3\varphi-\text{etc.}\end{array}\right\}}{(1-2pz\cos.\varphi+ppzz)^k}$$

220. Posito ergo $k=2$ erit seriei ex hac fractione

$$\frac{f-2pz(f\cos.\varphi-g\sin.\varphi)+ppzz(f\cos.2\varphi-g\sin.2\varphi)}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis

$$(n+1)(f\cos. n\varphi+g\sin. n\varphi)p^n z^n.$$

At seriei ex hac fractione [§ 218]

$$\frac{a}{1-2pz\cos.\varphi+ppzz}$$

seu hac

$$\frac{a-2apz\cos.\varphi+appzz}{(1-2pz\cos.\varphi+ppzz)^2}$$

ortae terminus generalis est

$$\frac{a\sin.(n+1)\varphi}{\sin.\varphi}p^n z^n.$$

Addantur hae fractiones invicem ac ponatur

$$a + f = \mathfrak{A},$$

$$2a \cos.\varphi + 2f \cos.\varphi - 2g \sin.\varphi = -\mathfrak{B}$$

et

$$a + f \cos. 2\varphi - g \sin. 2\varphi = 0;$$

hinc erit

$$g = \frac{\mathfrak{B} + 2\mathfrak{A} \cos.\varphi}{2 \sin.\varphi},$$

$$a = \frac{\mathfrak{A} + \mathfrak{B} \cos.\varphi}{1 - \cos. 2\varphi} = \frac{\mathfrak{A} + \mathfrak{B} \cos.\varphi}{2 (\sin.\varphi)^2}$$

et

$$f = \frac{-\mathfrak{A} \cos. 2\varphi - \mathfrak{B} \cos.\varphi}{2 (\sin.\varphi)^2}$$

et

$$g = \frac{\mathfrak{B} \sin.\varphi + \mathfrak{A} \sin. 2\varphi}{2 (\sin.\varphi)^2}.$$

Hanc ob rem seriei ex hac fractione

$$\frac{\mathfrak{A} + \mathfrak{B}pz}{(1 - 2pz \cos.\varphi + ppzz)^2}$$

ortae terminus generalis est

$$\frac{\mathfrak{A} + \mathfrak{B} \cos.\varphi}{2 (\sin.\varphi)^3} \sin. (n+1)\varphi \cdot p^n z^n$$

$$+ (n+1) \frac{(\mathfrak{B} \sin \varphi \sin. n\varphi + \mathfrak{A} \sin. 2\varphi \sin. n\varphi - \mathfrak{B} \cos.\varphi \cos. n\varphi - \mathfrak{A} \cos. 2\varphi \cos. n\varphi}{2 (\sin.\varphi)^2} p^n z^n$$

$$= -\frac{(n+1)(\mathfrak{A} \cos. (n+2)\varphi + \mathfrak{B} \cos. (n+1)\varphi)}{2 (\sin.\varphi)^2} p^n z^n + \frac{(\mathfrak{A} + \mathfrak{B} \cos.\varphi) \sin. (n+1)\varphi}{2 (\sin.\varphi)^3} p^n z^n$$

$$= \frac{\frac{1}{2}(n+3) \sin. (n+1)\varphi - \frac{1}{2}(n+1) \sin. (n+3)\varphi}{2 (\sin.\varphi)^3} \mathfrak{A} p^n z^n$$

$$+ \frac{\frac{1}{2}(n+2) \sin. n\varphi - \frac{1}{2} n \sin. (n+2)\varphi}{2 (\sin.\varphi)^3} \mathfrak{B} p^n z^n.$$

Est ergo iste terminus generalis quaesitus

$$= \frac{(n+3) \sin. (n+1)\varphi - (n+1) \sin. (n+3)\varphi}{4 (\sin.\varphi)^3} \mathfrak{A} p^n z^n + \frac{(n+2) \sin. n\varphi - n \sin. (n+2)\varphi}{4 (\sin.\varphi)^3} \mathfrak{B} p^n z^n$$

seriei, quae oritur ex fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^2}.$$

221. Sit $k=3$ eritque seriei ex hac fractione ortae

$$\frac{f-3pz(f\cos.\varphi-g\sin.\varphi)+3ppzz(f\cos.2\varphi-g\sin.2\varphi)-p^3z^3(f\cos.3\varphi-g\sin.3\varphi)}{(1-2pz\cos.\varphi+ppzz)^3}$$

terminus generalis

$$=\frac{(n+1)(n+2)}{1\cdot 2}(f\cos.n\varphi+g\sin.n\varphi)p^nz^n.$$

Deinde seriei ex fractione

$$\frac{a+bpz}{(1-2pz\cos.\varphi+ppzz)^2}$$

seu ex hac

$$\frac{a-(2a\cos.\varphi-b)pz+(a-2b\cos.\varphi)ppzz+bp^3z^3}{(1-2pz\cos.\varphi+ppzz)^3}$$

ortae terminus generalis est

$$\frac{(n+3)\sin.(n+1)\varphi-(n+1)\sin.(n+3)\varphi}{4(\sin.\varphi)^3}ap^nz^n+\frac{(n+2)\sin.n\varphi-n\sin.(n+2)\varphi}{4(\sin.\varphi)^3}bp^nz^n$$

Addantur hae fractiones ac ponatur numerator $=\mathfrak{A}$; erit

$$a+f=\mathfrak{A},$$

$$3f\cos.\varphi-3g\sin.\varphi+2a\cos.\varphi-b=0,$$

$$3f\cos.2\varphi-3g\sin.2\varphi+a-2b\cos.\varphi=0$$

et

$$b=f\cos.3\varphi-g\sin.3\varphi;$$

hinc erit

$$a=\frac{f\cos.3\varphi-g\sin.3\varphi-3f\cos.\varphi+3g\sin.\varphi}{2\cos.\varphi}$$

$$=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-f-2f(\sin.\varphi)^2.$$

Deinde reperitur

$$\frac{f}{g}=\frac{\sin.5\varphi-2\sin.3\varphi+\sin.\varphi}{\cos.5\varphi-2\cos.3\varphi+\cos.\varphi}$$

et

$$a+f=\mathfrak{A}=2g(\sin.\varphi)^2\operatorname{tang}.\varphi-2f(\sin.\varphi)^2,$$

ergo

$$\frac{\mathfrak{A}}{2(\sin.\varphi)^2} = \frac{g\sin.\varphi - f\cos.\varphi}{\cos.\varphi};$$

ex quibus tandem oritur

$$f = \frac{\mathfrak{A}(\sin.\varphi - 2\sin.3\varphi + \sin.5\varphi)}{16(\sin.\varphi)^5},$$

$$g = \frac{\mathfrak{A}(\cos.\varphi - 2\cos.3\varphi + \cos.5\varphi)}{16(\sin.\varphi)^5}.$$

Ob

$$16(\sin.\varphi)^5 = \sin.5\varphi - 5\sin.3\varphi + 10\sin.\varphi$$

erit

$$a = \frac{\mathfrak{A}(9\sin.\varphi - 3\sin.3\varphi)}{16(\sin.\varphi)^5}$$

et

$$b = \frac{\mathfrak{A}(-\sin.2\varphi + \sin.2\varphi)}{16(\sin.\varphi)^5} = 0.$$

Est autem

$$3\sin.\varphi - \sin.3\varphi = 4(\sin.\varphi)^3,$$

ergo

$$a = \frac{3\mathfrak{A}}{4(\sin.\varphi)^2}.$$

Quocirca erit terminus generalis

$$\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^n z^n \frac{\sin.(n+1)\varphi - 2\sin.(n+3)\varphi + \sin.(n+5)\varphi}{16(\sin.\varphi)^5}$$
$$+ 3\mathfrak{A}p^n z^n \frac{(n+3)\sin.(n+1)\varphi - (n+1)\sin.(n+3)\varphi}{16(\sin.\varphi)^5}$$
$$= \frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{aligned}&\frac{(n+4)(n+5)}{1\cdot 2}\sin.(n+1)\varphi - \frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi\\ &\qquad + \frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi\end{aligned}\right\}.$$

222. Seriei ergo, quae oritur ex hac fractione

$$\frac{\mathfrak{A} + \mathfrak{B}pz}{(1 - 2pz\cos.\varphi + ppzz)^3},$$

terminus generalis erit hic

$$\frac{\mathfrak{A}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{c}\frac{(n+5)(n+4)}{1\cdot 2}\sin.(n+1)\varphi-\frac{2(n+1)(n+5)}{1\cdot 2}\sin.(n+3)\varphi\\ +\frac{(n+1)(n+2)}{1\cdot 2}\sin.(n+5)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^n z^n}{16(\sin.\varphi)^5}\left\{\begin{array}{c}\frac{(n+4)(n+3)}{1\cdot 2}\sin.n\varphi-\frac{2n(n+4)}{1\cdot 2}\sin.(n+2)\varphi\\ +\frac{n(n+1)}{1\cdot 2}\sin.(n+4)\varphi\end{array}\right\}.$$

Atque ulterius progrediendo seriei, quae oritur ex hac fractione

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{(1-2pz\cos.\varphi+ppzz)^4},$$

terminus generalis erit hic

$$\frac{\mathfrak{A}p^n z^n}{64(\sin.\varphi)^7}\left\{\begin{array}{l}\frac{(n+7)(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+1)\varphi-\frac{3(n+1)(n+7)(n+6)}{1\cdot 2\cdot 3}\sin.(n+3)\varphi\\ +\frac{3(n+1)(n+2)(n+7)}{1\cdot 2\cdot 3}\sin.(n+5)\varphi-\frac{(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3}\sin.(n+7)\varphi\end{array}\right\}$$

$$+\frac{\mathfrak{B}p^n z^n}{64(\sin.\varphi)^7}\left\{\begin{array}{l}\frac{(n+6)(n+5)(n+4)}{1\cdot 2\cdot 3}\sin.n\varphi-\frac{3n(n+6)(n+5)}{1\cdot 2\cdot 3}\sin.(n+2)\varphi\\ +\frac{3n(n+1)(n+6)}{1\cdot 2\cdot 3}\sin.(n+4)\varphi-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\sin.(n+6)\varphi\end{array}\right\}.$$

Ex his autem expressionibus facile intelligitur, quemadmodum formae terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expressionum penitius inspiciendam notari convenit esse[1])

$$\begin{aligned}
\sin.\varphi &= \sin.\varphi,\\
4(\sin.\varphi)^3 &= 3\sin.\varphi-\sin.3\varphi,\\
16(\sin.\varphi)^5 &= 10\sin.\varphi-5\sin.3\varphi+\sin.5\varphi,\\
64(\sin.\varphi)^7 &= 35\sin.\varphi-21\sin.3\varphi+7\sin.5\varphi-\sin.7\varphi,\\
256(\sin.\varphi)^9 &= 126\sin.\varphi-84\sin.3\varphi+36\sin.5\varphi-9\sin.7\varphi+\sin.9\varphi
\end{aligned}$$

etc.

1) Confer § 262. F. R.

223. Cum igitur hoc pacto omnes functiones fractae in fractiones partiales reales resolvi queant, simul omnium serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adiuncta sunt.

EXEMPLUM 1

Ex fractione

$$\frac{1}{(1-z)(1-zz)(1-z^3)} = \frac{1}{1-z-zz+z^4+z^5-z^6}$$

oritur ista series recurrens

$$1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\text{etc.},$$

cuius terminus generalis desideratur.

Fractio proposita secundum factores ordinata fit

$$=\frac{1}{(1-z)^3(1+z)(1+z+zz)},$$

quae resolvitur in has fractiones

$$\frac{1}{6(1-z)^3}+\frac{1}{4(1-z)^2}+\frac{17}{72(1-z)}+\frac{1}{8(1+z)}+\frac{2+z}{9(1+z+zz)}.$$

Harum prima $\frac{1}{6(1-z)^3}$ dat terminum generalem

$$\frac{(n+1)(n+2)}{1\cdot 2}\cdot\frac{1}{6}z^n = \frac{nn+3n+2}{12}z^n,$$

secunda $\frac{1}{4(1-z)^2}$ dat

$$\frac{n+1}{4}z^n,$$

tertia $\frac{17}{72(1-z)}$ dat

$$\frac{17}{72}z^n,$$

quarta $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n.$$

Quinta vero $\frac{2+z}{9(1+z+zz)}$ comparata cum forma (§ 218)

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

dat

$$p=-1,\quad \varphi=\frac{\pi}{3}=60^0,\quad \mathfrak{A}=+\frac{2}{9}\quad \text{et}\quad \mathfrak{B}=-\frac{1}{9},$$

unde oritur terminus generalis

$$\frac{2\sin.(n+1)\varphi-\sin.n\varphi}{9\sin.\varphi}(-1)^n z^n=\frac{4\sin.(n+1)\varphi-2\sin.n\varphi}{9\sqrt{3}}(-1)^n z^n$$

$$=\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}(-1)^n z^n.$$

Colligantur hae expressiones omnes in unam summam ac prodibit seriei propositae terminus generalis quaesitus

$$=\left(\frac{nn}{12}+\frac{n}{2}+\frac{47}{72}\right)z^n\pm\frac{1}{8}z^n\pm\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}z^n,$$

ubi signa superiora valent, si n numerus par, inferiora, sin impar. Ubi notandum est, si fuerit n numerus formae $3m$, fore

$$\frac{4\sin.\frac{(n+1)\pi}{3}-2\sin.\frac{n\pi}{3}}{9\sqrt{3}}=\pm\frac{2}{9};$$

si fuerit $n=3m+1$, erit haec expressio $=\mp\frac{1}{9}$; at si $n=3m+2$, erit ista expressio [iterum] $=\mp\frac{1}{9}$; [semper] prout n fuerit numerus vel par vel impar. Ex his natura seriei ita explicari potest, ut,

si fuerit	terminus generalis futurus sit
$n = 6m + 0$	$\left(\frac{nn}{12} + \frac{n}{2} + 1\right) z^n$
$n = 6m + 1$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$
$n = 6m + 2$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$
$n = 6m + 3$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{3}{4}\right) z^n$
$n = 6m + 4$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{2}{3}\right) z^n$
$n = 6m + 5$	$\left(\frac{nn}{12} + \frac{n}{2} + \frac{5}{12}\right) z^n$

Sic si fuerit $n = 50$, valet forma $n = 6m + 2$ eritque terminus seriei $= 234z^{50}$.

EXEMPLUM 2

Ex fractione

$$\frac{1 + z + zz}{1 - z - z^4 + z^5}$$

oritur haec series recurrens

$$1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \text{etc.},$$

cuius terminum generalem invenire oportet.

Fractio proposita ad hanc formam reducitur

$$\frac{1 + z + zz}{(1 - z)^2 (1 + z)(1 + zz)},$$

quae propterea resolvitur in has fractiones partiales

$$\frac{3}{4(1 - z)^2} + \frac{3}{8(1 - z)} + \frac{1}{8(1 + z)} + \frac{-1 + z}{4(1 + zz)}.$$

Harum prima $\frac{3}{4(1 - z)^2}$ dat terminum generalem

$$\frac{3(n + 1)}{4} z^n,$$

secunda $\frac{3}{8(1-z)}$ dat

$$\frac{3}{8}z^n,$$

tertia $\frac{1}{8(1+z)}$ dat

$$\frac{1}{8}(-1)^n z^n$$

et quarta $\frac{-1+z}{4(1+zz)}$ comparata cum forma

$$\frac{\mathfrak{A}+\mathfrak{B}pz}{1-2pz\cos.\varphi+ppzz}$$

dat

$$p=1,\quad \cos.\varphi=0 \quad \text{et} \quad \varphi=\frac{\pi}{2},\quad \mathfrak{A}=-\frac{1}{4},\quad \mathfrak{B}=+\frac{1}{4},$$

unde fit terminus generalis

$$=\left(-\frac{1}{4}\sin.\frac{(n+1)\pi}{2}+\frac{1}{4}\sin.\frac{n\pi}{2}\right)z^n.$$

Quare colligendo erit terminus generalis quaesitus

$$=\left(\frac{3}{4}n+\frac{9}{8}\right)z^n\pm\frac{1}{8}z^n-\frac{1}{4}\left(\sin.\frac{(n+1)\pi}{2}-\sin.\frac{n\pi}{2}\right)z^n.$$

Hinc

si fuerit	erit terminus generalis
$n=4m+0$	$\left(\frac{3}{4}n+1\right)z^n$
$n=4m+1$	$\left(\frac{3}{4}n+\frac{5}{4}\right)z^n$
$n=4m+2$	$\left(\frac{3}{4}n+\frac{3}{2}\right)z^n$
$n=4m+3$	$\left(\frac{3}{4}n+\frac{3}{4}\right)z^n$

Ita si $n=50$, valebit $n=4m+2$ eritque terminus $=39z^{50}$.

224. Proposita ergo serie recurrente, quoniam illa fractio, unde oritur, facile cognoscitur, eius terminus generalis secundum praecepta data reperietur. Ex lege autem seriei recurrentis, qua quisque terminus ex praecedentibus definitur, statim innotescit denominator fractionis huiusque factores praebe-

bunt formam termini generalis; per numeratorem enim tantum coefficientes determinantur. Sit nempe proposita haec series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

cuius lex progressionis, qua unusquisque terminus ex aliquot praecedentibus determinatur, praebeat hunc fractionis denominatorem

$$1 - \alpha z - \beta z^2 - \gamma z^3,$$

ita ut sit

$$D = \alpha C + \beta B + \gamma A, \quad E = \alpha D + \beta C + \gamma B, \quad F = \alpha E + \beta D + \gamma C \quad \text{etc.},$$

qui multiplicatores $+\alpha$, $+\beta$, $+\gamma$ a MOIVREO *scalam relationis*[1]) constituere dicuntur. Lex ergo progressionis posita est in scala relationis atque scala relationis statim praebet denominatorem fractionis, ex cuius resolutione proposita series recurrens oritur.

225. Ad terminum ergo generalem seu coefficientem potestatis indefinitae z^n inveniendum quaeri debent denominatoris $1 - \alpha z - \beta z^2 - \gamma z^3$ factores vel simplices vel duplices, si imaginarios vitare velimus. Sint primo factores simplices omnes inter se inaequales et reales hi

$$(1 - pz)(1 - qz)(1 - rz)$$

atque fractio generans seriem propositam resolvetur in

$$\frac{\mathfrak{A}}{1 - pz} + \frac{\mathfrak{B}}{1 - qz} + \frac{\mathfrak{C}}{1 - rz};$$

unde seriei terminus generalis erit

$$(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n)z^n.$$

---

1) Accuratius *indicem seu scalam relationis*; vide p. 27 libri, qui inscribitur *Miscellanea analytica*, nota p. 79 laudati. In dissertatione enim *De fractionibus algebraicis* eadem nota laudata tantum significatio *index relationis* invenitur. F. R.

Si duo factores fuerint aequales, nempe $q = p$, tum terminus generalis huiusmodi erit[1])

$$\left((\mathfrak{A}(n+1) + \mathfrak{B})p^n + \mathfrak{C}r^n\right)z^n,$$

et si insuper fuerit $r = q = p$, erit terminus generalis[2])

$$\left(\mathfrak{A}\frac{(n+1)(n+2)}{1\cdot 2} + \mathfrak{B}(n+1) + \mathfrak{C}\right)p^n z^n.$$

Quodsi vero denominator $1 - \alpha z - \beta z^2 - \gamma z^3$ duplicem habeat factorem, ut sit

$$= (1 - pz)(1 - 2qz\cos.\varphi + qqzz),$$

tum terminus generalis erit

$$= \left(\mathfrak{A}p^n + \frac{\mathfrak{B}\sin.(n+1)\varphi + \mathfrak{C}\sin.n\varphi}{\sin.\varphi}q^n\right)z^n.$$

Cum igitur positis pro n successive numeris 0, 1, 2 prodire debeant termini A, Bz, Cz^2, hinc valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ determinabuntur.

226. Sit scala relationis bimembris seu determinetur quisque terminus per duos praecedentes, ita ut sit

$$C = \alpha B - \beta A, \quad D = \alpha C - \beta B, \quad E = \alpha D - \beta C \quad \text{etc.},$$

atque manifestum est seriem hanc recurrentem, quae sit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n + Qz^{n+1} + \text{etc.},$$

oriri ex fractione, cuius denominator sit

$$1 - \alpha z + \beta zz.$$

Sint huius denominatoris factores

$$(1 - pz)(1 - qz);$$

erit

$$p + q = \alpha \quad \text{et} \quad pq = \beta$$

1) Editio princeps: $((\mathfrak{A}n + \mathfrak{B})p^n + \mathfrak{C}r^n)z^n$. Correxit F. R.
2) Editio princeps: $(\mathfrak{A}n^2 + \mathfrak{B}n + \mathfrak{C})p^n z^n$. Correxit F. R.

atque seriei terminus generalis erit

$$(\mathfrak{A}p^n + \mathfrak{B}q^n)z^n.$$

Hinc facto $n = 0$ erit

$$A = \mathfrak{A} + \mathfrak{B}$$

et facto $n = 1$ erit

$$B = \mathfrak{A}p + \mathfrak{B}q,$$

unde fit

$$Aq - B = \mathfrak{A}(q - p)$$

et

$$\mathfrak{A} = \frac{Aq - B}{q - p} \quad \text{et} \quad \mathfrak{B} = \frac{Ap - B}{p - q}.$$

Inventis autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ erit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \quad \text{et} \quad Q = \mathfrak{A}p^{n+1} + \mathfrak{B}q^{n+1}.$$

Tum vero erit

$$\mathfrak{A}\mathfrak{B} = \frac{BB - \alpha AB + \beta AA}{4\beta - \alpha\alpha}.$$

227. Hinc deduci potest modus quemvis terminum ex unico praecedente formandi, cum ad hoc per legem progressionis duo requirantur. Cum enim sit

$$P = \mathfrak{A}p^n + \mathfrak{B}q^n \quad \text{et} \quad Q = \mathfrak{A}p \cdot p^n + \mathfrak{B}q \cdot q^n,$$

erit

$$Pq - Q = \mathfrak{A}(q - p)p^n \quad \text{et} \quad Pp - Q = \mathfrak{B}(p - q)q^n.$$

Multiplicentur hae expressiones in se invicem eritque

$$P^2pq - (p + q)PQ + QQ + \mathfrak{A}\mathfrak{B}(p - q)^2p^nq^n = 0.$$

At est

$$p + q = \alpha, \quad pq = \beta, \quad (p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 4\beta \quad \text{et} \quad p^nq^n = \beta^n.$$

Quibus substitutis habebitur

$$\beta P^2 - \alpha PQ + QQ = (\beta AA - \alpha AB + BB)\beta^n$$

seu

$$\frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA} = \beta^n,$$

quae est insignis proprietas serierum recurrentium, quarum quisque terminus per duos praecedentes determinatur. At cognito quovis termino P erit sequens

$$Q=\tfrac{1}{2}\alpha P+\sqrt{\left(\left(\tfrac{1}{4}\alpha^2-\beta\right)P^2+\left(BB-\alpha AB+\beta AA\right)\beta^n\right)},$$

quae expressio, etsi speciem irrationalitatis prae se fert, tamen semper est rationalis, propterea quod termini irrationales in serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibusvis Pz^n et Qz^{n+1} commode assignari potest terminus multo magis remotus Xz^{2n}. Ponatur enim

$$X=fP^2+gPQ-h\mathfrak{A}\mathfrak{B}\beta^n.$$

Quoniam est

$$P=\mathfrak{A}p^n+\mathfrak{B}q^n \quad \text{et} \quad Q=\mathfrak{A}p\cdot p^n+\mathfrak{B}q\cdot q^n \quad \text{atque} \quad X=\mathfrak{A}p^{2n}+\mathfrak{B}q^{2n},$$

erit ut sequitur:

$$\begin{array}{rl} fP^2 = & f\mathfrak{A}^2p^{2n} + f\mathfrak{B}^2q^{2n} + 2f\mathfrak{A}\mathfrak{B}\beta^n \\ gPQ = & g\mathfrak{A}^2p\cdot p^{2n} + g\mathfrak{B}^2q\cdot q^{2n} + g\mathfrak{A}\mathfrak{B}\alpha\beta^n \\ -h\mathfrak{A}\mathfrak{B}\beta^n = & \qquad\qquad\qquad\qquad - h\mathfrak{A}\mathfrak{B}\beta^n \\ \hline X = & \mathfrak{A}p^{2n}+\mathfrak{B}q^{2n} \end{array}$$

Fiet ergo

$$f+gp=\frac{1}{\mathfrak{A}}, \quad f+gq=\frac{1}{\mathfrak{B}} \quad \text{et} \quad h=2f+g\alpha,$$

unde

$$g=\frac{\mathfrak{B}-\mathfrak{A}}{\mathfrak{A}\mathfrak{B}(p-q)} \quad \text{et} \quad f=\frac{\mathfrak{A}p-\mathfrak{B}q}{\mathfrak{A}\mathfrak{B}(p-q)}.$$

At est

$$\mathfrak{B}-\mathfrak{A}=\frac{\alpha A-2B}{p-q}, \quad \mathfrak{A}p-\mathfrak{B}q=\frac{\alpha B-2A\beta}{p-q}.$$

Ergo

$$f=\frac{\alpha B-2A\beta}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)} \quad \text{et} \quad g=\frac{\alpha A-2B}{\mathfrak{A}\mathfrak{B}(\alpha\alpha-4\beta)}$$

seu

$$f=\frac{2A\beta-\alpha B}{BB-\alpha AB+\beta AA} \quad \text{et} \quad g=\frac{2B-\alpha A}{BB-\alpha AB+\beta AA};$$

ideoque

$$h=\frac{(4\beta-\alpha\alpha)A}{BB-\alpha AB+\beta AA}.$$

Eritque ergo

$$X = \frac{(2A\beta - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \beta AA} - A\beta^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\beta A - (\alpha\alpha - 2\beta)B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \beta AA)} - \frac{2B\beta^n}{\alpha}.$$

His coniungendis per eliminationem termini β^n reperitur

$$X = \frac{(\beta A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \beta AA}.$$ [1])

229. Simili modo si statuantur termini sequentes

$$A + Bz + Cz^2 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \cdots + Xz^{2n} + Yz^{2n+1} + Zz^{2n+2} + \text{etc.},$$

erit

$$Z = \frac{(\beta A - \alpha B)Q^2 + 2BQR - ARR}{BB - \alpha AB + \beta AA}$$

et ob $R = \alpha Q - \beta P$ erit

$$Z = \frac{-\beta\beta AP^2 + 2\beta(\alpha A - B)PQ + (\alpha B - (\alpha\alpha - \beta)A)Q^2}{BB - \alpha AB + \beta AA}.$$

At est $Z = \alpha Y - \beta X$, ergo $Y = \frac{Z + \beta X}{\alpha}$; unde fit

$$Y = \frac{-\beta BP^2 + 2\beta APQ + (B - \alpha A)QQ}{BB - \alpha AB + \beta AA}.$$

Sic igitur porro ex X et Y definiri poterunt simili modo coefficientes potestatum z^{4n} et z^{4n+1} hincque ipsarum z^{8n}, z^{8n+1}, et ita porro.

1) Quae resolutio statim iam ex prima aequatione pro X inventa elicitur ponendo (§ 227) $\beta^n = \frac{QQ - \alpha PQ + \beta PP}{BB - \alpha AB + \beta AA}$. F. R.

EXEMPLUM

Sit proposita ista series recurrens

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \cdots + Pz^n + Qz^{n+1} + \text{etc.};$$

cuius cum quilibet coefficiens sit summa duorum praecedentium, erit denominator fractionis hanc seriem producentis

$$1 - z - zz;$$

ideoque

$$\alpha = 1, \quad \beta = -1 \quad \text{et} \quad A = 1, \quad B = 3,$$

unde fit

$$BB - \alpha AB + \beta AA = 5.$$

Ex quo orietur primum

$$Q = \frac{P + \sqrt{(5PP + 20(-1)^n)}}{2} = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

ubi signum superius valet, si n sit numerus par, inferius, si impar. Sic si $n = 4$, ob $P = 11$ erit

$$Q = \frac{11 + \sqrt{(5 \cdot 121 + 20)}}{2} = \frac{11 + 25}{2} = 18.$$

Si porro coefficiens termini z^{2n} sit X, erit

$$X = \frac{-4PP + 6PQ - QQ}{5};$$

ergo potestatis z^8 coefficiens erit

$$= \frac{-4 \cdot 121 + 6 \cdot 198 - 324}{5} = 76.$$

Cum autem sit

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

erit

$$QQ = \frac{3PP \pm 10 + P\sqrt{(5PP \pm 20)}}{2}$$

ideoque

$$X = \frac{-PP \mp 2 + P\sqrt{(5PP \pm 20)}}{2}.$$

Ex termino ergo seriei quocunque Pz^n obtinentur hi

$$\frac{P+\sqrt{(5PP\pm 20)}}{2}z^{n+1} \quad \text{et} \quad \frac{-PP\mp 2+P\sqrt{(5PP\pm 20)}}{2}z^{2n}.$$

230. Simili modo in seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri potest. Sit enim series huiusmodi recurrens

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n+Qz^{n+1}+Rz^{n+2}+\text{etc.},$$

cuius scala relationis sit α, $-\beta$, $+\gamma$ seu quae oriatur ex fractione, cuius denominator

$$=1-\alpha z+\beta z^2-\gamma z^3.$$

Quodsi iam termini P, Q, R eodem modo per factores huius denominatoris, qui sint

$$(1-pz)(1-qz)(1-rz),$$

exprimantur, ut sit

$$P=\mathfrak{A}p^n \quad +\mathfrak{B}q^n \quad +\mathfrak{C}r^n,$$

$$Q=\mathfrak{A}p\cdot p^n+\mathfrak{B}q\cdot q^n+\mathfrak{C}r\cdot r^n$$

et

$$R=\mathfrak{A}p^2\cdot p^n+\mathfrak{B}q^2\cdot q^n+\mathfrak{C}r^2\cdot r^n,$$

ob

$$p+q+r=\alpha,\quad pq+pr+qr=\beta \quad \text{et} \quad pqr=\gamma$$

reperietur haec proportio[1])

1) Quae proportio reperietur computando valores ipsorum $\mathfrak{A}p^n$, $\mathfrak{B}q^n$, $\mathfrak{C}r^n$ ex aequationibus

$$\mathfrak{A}p^n \quad +\mathfrak{B}q^n \quad +\mathfrak{C}r^n \quad =P,$$
$$\mathfrak{A}p^n\cdot p+\mathfrak{B}q^n\cdot q+\mathfrak{C}r^n\cdot r=Q,$$
$$\mathfrak{A}p^n\cdot p^2+\mathfrak{B}q^n\cdot q^2+\mathfrak{C}r^n\cdot r^2=R$$

atque simili modo valores ipsorum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ ex aequationibus correspondentibus

$$\begin{aligned}
&R^3 - 2\alpha QR^2 + (\alpha\alpha+\beta)Q^2R - (\alpha\beta-\gamma)Q^3 : \gamma^n \\
&\quad + \beta P - (\alpha\beta+3\gamma)PQ + (\alpha\gamma+\beta\beta)PQ^2 \\
&\quad + \alpha\gamma P^2 - 2\beta\gamma P^2Q \\
&\quad + \gamma\gamma P^3 \\
&= C^3 - 2\alpha BC^2 + (\alpha\alpha+\beta)B^2C - (\alpha\beta-\gamma)B^3 : 1. \\
&\quad + \beta A - (\alpha\beta+3\gamma)AB + (\alpha\gamma+\beta\beta)AB^2 \\
&\quad + \alpha\gamma A^2 - 2\beta\gamma A^2B \\
&\quad + \gamma\gamma A^3
\end{aligned}$$

Pendet ergo inventio termini R ex duobus praecedentibus P et Q a resolutione aequationis cubicae.

231. His de terminis generalibus serierum recurrentium notatis superest, ut earundem serierum summas investigemus. Ac primo quidem manifestum est summam seriei recurrentis in infinitum extensae aequalem esse fractioni, ex qua oritur; cuius fractionis cum denominator ex ipsa progressionis lege pateat, reliquum est, ut numeratorem definiamus. Sit itaque proposita haec series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \text{etc.},$$

cuius lex progressionis praebeat hunc denominatorem

$$1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4.$$

$$\begin{aligned}
\mathfrak{A} + \mathfrak{B} + \mathfrak{C} &= A, \\
\mathfrak{A}p + \mathfrak{B}q + \mathfrak{C}r &= B, \\
\mathfrak{A}p^2 + \mathfrak{B}q^2 + \mathfrak{C}r^2 &= C.
\end{aligned}$$

Solutiones inter se multiplicando obtinetur

$$\begin{aligned}
&(Pqr - Q(q+r) + R)(Prp - Q(r+p) + R)(Ppq - Q(p+q) + R) \\
&: (Aqr - B(q+r) + C)(Arp - B(r+p) + C)(Apq - B(p+q) + C) \\
&= p^n q^n r^n : 1 = \gamma^n : 1,
\end{aligned}$$

quae aequatio secundum potestates ipsorum R et C evoluta ope relationum $p+q+r=\alpha$, $qr+rp+pq=\beta$, $pqr=\gamma$ proportionem illam praebet. F. R.

Sumamus fractionem summae seriei in infinitum [extensae] aequalem esse

$$= \frac{a + bz + cz^2 + dz^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4};$$

ex qua cum series proposita oriri debeat, erit comparando

$$a = A,$$

$$b = B - \alpha A,$$

$$c = C - \alpha B + \beta A,$$

$$d = D - \alpha C + \beta B - \gamma A.$$

Hinc erit summa quaesita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 + (D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

232. Hinc facile intelligitur, quemadmodum seriei recurrentis summa ad datum terminum usque inveniri debeat. Quaeratur scilicet seriei modo assumptae summa ad terminum Pz^n atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \cdots + Pz^n.$$

Quoniam huius seriei summa in infinitum constat, quaeratur summa terminorum ultimum Pz^n in infinitum sequentium, qui sint

$$t = Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + Tz^{n+4} + \text{etc.};$$

haec series per z^{n+1} divisa dat seriem recurrentem propositae aequalem, cuius propterea summa erit

$$t = \frac{Qz^{n+1} + (R - \alpha Q)z^{n+2} + (S - \alpha R + \beta Q)z^{n+3} + (T - \alpha S + \beta R - \gamma Q)z^{n+4}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

Unde orietur summa quaesita

$$s = \frac{A+(B-\alpha A)z+(C-\alpha B+\beta A)z^2+(D-\alpha C+\beta B-\gamma A)z^3}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}$$

$$-\frac{Qz^{n+1}+(R-\alpha Q)z^{n+2}+(S-\alpha R+\beta Q)z^{n+3}+(T-\alpha S+\beta R-\gamma Q)z^{n+4}}{1-\alpha z+\beta z^2-\gamma z^3+\delta z^4}.$$

233. Quodsi ergo scala relationis fuerit bimembris α, $-\beta$, seriei

$$A+Bz+Cz^2+Dz^3+\cdots+Pz^n,$$

quae oritur ex fractione

$$\frac{A+(B-\alpha A)z}{1-\alpha z+\beta zz},$$

summa erit

$$\frac{A+(B-\alpha A)z-Qz^{n+1}-(R-\alpha Q)z^{n+2}}{1-\alpha z+\beta zz}.$$

At est ex natura seriei

$$R=\alpha Q-\beta P,$$

unde prodibit summa

$$\frac{A+(B-\alpha A)z-Qz^{n+1}+\beta Pz^{n+2}}{1-\alpha z+\beta zz}$$

EXEMPLUM

Sit proposita series

$$1+3z+4z^2+7z^3+\cdots+Pz^n,$$

ubi est

$$\alpha=1,\quad \beta=-1,\quad A=1,\quad B=3;$$

erit huius summa

$$\frac{1+2z-Qz^{n+1}-Pz^{n+2}}{1-z-zz}.$$

Posito vero $z=1$ erit summa seriei

$$1+3+4+7+11+\cdots+P$$
$$=P+Q-3.$$

Summa ergo termini ultimi et sequentis ternario excedit summam seriei. Quia vero est

$$Q = \frac{P + \sqrt{(5PP \pm 20)}}{2},$$

erit summa seriei

$$1 + 3 + 4 + 7 + 11 + \cdots + P$$
$$= \frac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}.$$

Ex solo ergo termino ultimo summa potest exhiberi.

CAPUT XIV

DE MULTIPLICATIONE AC DIVISIONE ANGULORUM

234. Sit angulus vel arcus in circulo, cuius radius $=1$, quicunque $=z$, eius sinus $=x$, cosinus $=y$ et tangens $=t$; erit

$$xx+yy=1 \quad \text{et} \quad t=\frac{x}{y}.$$

Cum igitur, uti supra [§ 129] vidimus, tam sinus quam cosinus angulorum z, $2z$, $3z$, $4z$, $5z$ etc. constituant seriem recurrentem, cuius scala relationis est $2y$, -1, primum sinus horum arcuum ita se habebunt:

$$\begin{aligned}
\sin. 0z &= 0,\\
\sin. 1z &= x,\\
\sin. 2z &= 2xy,\\
\sin. 3z &= 4xy^2 - x,\\
\sin. 4z &= 8xy^3 - 4xy,\\
\sin. 5z &= 16xy^4 - 12xy^2 + x,\\
\sin. 6z &= 32xy^5 - 32xy^3 + 6xy,\\
\sin. 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x,\\
\sin. 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 8xy.
\end{aligned}$$

Hinc concluditur fore

$$\sin. nz = x \left\{ \begin{array}{c} 2^{n-1}y^{n-1} - (n-2)2^{n-3}y^{n-3} \\ + \frac{(n-3)(n-4)}{1\cdot 2}2^{n-5}y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3}2^{n-7}y^{n-7} \\ + \frac{(n-5)(n-6)(n-7)(n-8)}{1\cdot 2\cdot 3\cdot 4}2^{n-9}y^{n-9} - \text{etc.} \end{array} \right\}.$$

235. Si ponamus arcum $nz = s$, erit

$$\sin. nz = \sin. s = \sin. (\pi - s) = \sin. (2\pi + s) = \sin. (3\pi - s) \text{ etc.};$$

hi enim sinus omnes sunt inter se aequales. Hinc obtinemus plures valores pro x, qui erunt

$$\sin. \frac{s}{n}, \quad \sin. \frac{\pi - s}{n}, \quad \sin. \frac{2\pi + s}{n}, \quad \sin. \frac{3\pi - s}{n}, \quad \sin. \frac{4\pi + s}{n} \quad \text{etc.},$$

qui ergo omnes aequationi inventae aeque conveniunt. Tot autem prodibunt diversi pro x valores, quot numerus n continet unitates, qui propterea erunt radices aequationis inventae. Cavendum ergo est, ne valores aequales pro iisdem habeantur, quod fiet, dum alternae tantum expressiones assumantur. Cognitis igitur radicibus aequationis a posteriori, earum comparatio cum terminis aequationis notatu dignas praebebit proprietates. Quoniam autem ad hoc aequatio, in qua tantum x tamquam incognita insit, requiritur, pro y suus valor $\sqrt{(1 - xx)}$ substitui debet; unde duplex operatio instituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar; quia arcuum $-z$, $+z$, $+3z$, $+5z$ etc. differentia est $2z$ huiusque cosinus $= 1 - 2xx$, erit progressionis sinuum scala relationis haec $2 - 4xx$, -1. Hinc erit

$$\begin{aligned}
\sin. -z &= -x, \\
\sin. z &= x, \\
\sin. 3z &= 3x - 4x^3, \\
\sin. 5z &= 5x - 20x^3 + 16x^5, \\
\sin. 7z &= 7x - 56x^3 + 112x^5 - 64x^7, \\
\sin. 9z &= 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9.
\end{aligned}$$

Ergo

$$\sin. nz = nx - \frac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5$$
$$-\frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.},$$

siquidem n fuerit numerus impar. Huiusque aequationis radices sunt

$$\sin. z,\quad \sin.\left(\frac{2\pi}{n}+z\right),\quad \sin.\left(\frac{4\pi}{n}+z\right),\quad \sin.\left(\frac{6\pi}{n}+z\right),\quad \sin.\left(\frac{8\pi}{n}+z\right)\ \text{etc.},$$

quarum numerus est n.

237. Huius ergo aequationis

$$0 = 1 - \frac{nx}{\sin. nz} + \frac{n(nn-1)x^3}{1\cdot 2\cdot 3\sin. nz} - \frac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\sin. nz} + \cdots \pm \frac{2^{n-1}x^n}{\sin. nz}$$

(ubi signum superius valet, si n unitate deficiat a multiplo quaternarii, contra inferius) factores sunt

$$\left(1-\frac{x}{\sin. z}\right)\left(1-\frac{x}{\sin.\left(\frac{2\pi}{n}+z\right)}\right)\left(1-\frac{x}{\sin.\left(\frac{4\pi}{n}+z\right)}\right)\ \text{etc.},$$

ex quibus concluditur fore

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin.\left(\frac{2\pi}{n}+z\right)} + \frac{1}{\sin.\left(\frac{4\pi}{n}+z\right)} + \frac{1}{\sin.\left(\frac{6\pi}{n}+z\right)} + \text{etc.},$$

donec habeantur n termini. Tum vero productum omnium erit

$$\mp\frac{2^{n-1}}{\sin. nz} = \frac{1}{\sin. z\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\ \text{etc.}}$$

seu

$$\sin. nz = \mp 2^{n-1}\sin. z\sin.\left(\frac{2\pi}{n}+z\right)\sin.\left(\frac{4\pi}{n}+z\right)\sin.\left(\frac{6\pi}{n}+z\right)\ \text{etc.}$$

Et, quia terminus penultimus deest, erit

$$0 = \sin. z + \sin.\left(\frac{2\pi}{n}+z\right) + \sin.\left(\frac{4\pi}{n}+z\right) + \sin.\left(\frac{6\pi}{n}+z\right) + \text{etc.}$$

EXEMPLUM 1

Si ergo fuerit $n = 3$, prodibunt hae aequalitates

$$0 = \sin. z + \sin. (120 + z) + \sin. (240 + z)$$
$$= \sin. z + \sin. (60 - z) - \sin. (60 + z),$$

$$\frac{3}{\sin. 3z} = \frac{1}{\sin. z} + \frac{1}{\sin. (120 + z)} + \frac{1}{\sin. (240 + z)}$$
$$= \frac{1}{\sin. z} + \frac{1}{\sin. (60 - z)} - \frac{1}{\sin. (60 + z)},$$

$$\sin. 3z = -4 \sin. z \sin. (120 + z) \sin. (240 + z)$$
$$= 4 \sin. z \sin. (60 - z) \sin. (60 + z).$$

Erit ergo, uti iam supra [§ 131] notavimus,

$$\sin. (60 + z) = \sin. z + \sin. (60 - z)$$

et

$$3 \operatorname{cosec.} 3z = \operatorname{cosec.} z + \operatorname{cosec.} (60 - z) - \operatorname{cosec.} (60 + z).$$

EXEMPLUM 2

Ponamus esse $n = 5$ atque prodibunt hae aequationes

$$0 = \sin. z + \sin. \left(\frac{2\pi}{5} + z\right) + \sin. \left(\frac{4\pi}{5} + z\right) + \sin. \left(\frac{6\pi}{5} + z\right) + \sin. \left(\frac{8\pi}{5} + z\right)$$

seu

$$0 = \sin. z + \sin. \left(\frac{2\pi}{5} + z\right) + \sin. \left(\frac{\pi}{5} - z\right) - \sin. \left(\frac{\pi}{5} + z\right) - \sin. \left(\frac{2\pi}{5} - z\right)$$

seu

$$0 = \sin. z + \sin. \left(\frac{\pi}{5} - z\right) - \sin. \left(\frac{\pi}{5} + z\right)$$
$$- \sin. \left(\frac{2\pi}{5} - z\right) + \sin. \left(\frac{2\pi}{5} + z\right).$$

Deinde erit

$$\frac{5}{\sin. 5z} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{\pi}{5} - z\right)} - \frac{1}{\sin. \left(\frac{\pi}{5} + z\right)}$$
$$- \frac{1}{\sin. \left(\frac{2\pi}{5} - z\right)} + \frac{1}{\sin. \left(\frac{2\pi}{5} + z\right)},$$

$$\sin. 5z = 16 \sin. z \sin. \left(\frac{\pi}{5} - z\right) \sin. \left(\frac{\pi}{5} + z\right)$$
$$\sin. \left(\frac{2\pi}{5} - z\right) \sin. \left(\frac{2\pi}{5} + z\right).$$

EXEMPLUM 3

Hoc modo, si ponamus $n = 2m + 1$, erit

$$0 = \sin. z + \sin. \left(\frac{\pi}{n} - z\right) - \sin. \left(\frac{\pi}{n} + z\right)$$
$$- \sin. \left(\frac{2\pi}{n} - z\right) + \sin. \left(\frac{2\pi}{n} + z\right)$$
$$+ \sin. \left(\frac{3\pi}{n} - z\right) - \sin. \left(\frac{3\pi}{n} + z\right)$$
$$\vdots$$
$$\pm \sin. \left(\frac{m\pi}{n} - z\right) \mp \sin. \left(\frac{m\pi}{n} + z\right),$$

ubi signa superiora valent, si m sit numerus impar, inferiora, si sit par. Altera aequatio erit haec

$$\frac{n}{\sin. nz} = \frac{1}{\sin. z} + \frac{1}{\sin. \left(\frac{\pi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{\pi}{n} + z\right)}$$
$$- \frac{1}{\sin. \left(\frac{2\pi}{n} - z\right)} + \frac{1}{\sin. \left(\frac{2\pi}{n} + z\right)}$$
$$+ \frac{1}{\sin. \left(\frac{3\pi}{n} - z\right)} - \frac{1}{\sin. \left(\frac{3\pi}{n} + z\right)}$$
$$\vdots$$
$$\pm \frac{1}{\sin. \left(\frac{m\pi}{n} - z\right)} \mp \frac{1}{\sin. \left(\frac{m\pi}{n} + z\right)},$$

quae ad cosecantes commode transfertur. Tertio habetur hoc productum

$$\begin{aligned}\sin. nz = 2^{2m} \sin. z \sin.\left(\frac{\pi}{n} - z\right) \sin.\left(\frac{\pi}{n} + z\right)\\ \sin.\left(\frac{2\pi}{n} - z\right) \sin.\left(\frac{2\pi}{n} + z\right)\\ \sin.\left(\frac{3\pi}{n} - z\right) \sin.\left(\frac{3\pi}{n} + z\right)\\ \vdots\\ \sin.\left(\frac{m\pi}{n} - z\right) \sin.\left(\frac{m\pi}{n} + z\right).\end{aligned}$$

238. Sit n nunc numerus par, et quoniam est

$$y = \sqrt{(1 - xx)} \quad \text{et} \quad \cos. 2z = 1 - 2xx,$$

ita ut seriei sinuum sit scala relationis ut ante $2 - 4xx, \ -1$, erit

$$\begin{aligned}&\sin. 0z = 0,\\ &\sin. 2z = 2x\sqrt{(1 - xx)},\\ &\sin. 4z = (4x - 8x^3)\sqrt{(1 - xx)},\\ &\sin. 6z = (6x - 32x^3 + 32x^5)\sqrt{(1 - xx)},\\ &\sin. 8z = (8x - 80x^3 + 192x^5 - 128x^7)\sqrt{(1 - xx)}\end{aligned}$$

et generaliter

$$\sin. nz = \left\{\begin{array}{l} nx - \dfrac{n(nn-4)}{1\cdot 2\cdot 3}x^3 + \dfrac{n(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 \\ - \dfrac{n(nn-4)(nn-16)(nn-36)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \cdots \pm 2^{n-1}x^{n-1}\end{array}\right\}\sqrt{(1-xx)}$$

denotante n numerum quemcunque parem.

239. Ad aequationem hanc rationalem efficiendam sumantur utrinque quadrata ac prodibit huiusmodi aequatio

$$(\sin. nz)^2 = nnxx + Px^4 + Qx^6 + \cdots - 2^{2n-2}x^{2n}$$

seu

$$x^{2n} - \cdots - \frac{nn}{2^{2n-2}}xx + \frac{1}{2^{2n-2}}(\sin. nz)^2 = 0,$$

cuius aequationis radices erunt tam affirmativae quam negativae, scilicet

$$\pm \sin. z, \quad \pm \sin.\left(\frac{\pi}{n} - z\right), \quad \pm \sin.\left(\frac{2\pi}{n} + z\right), \quad \pm \sin.\left(\frac{3\pi}{n} - z\right), \quad \pm \sin.\left(\frac{4\pi}{n} + z\right) \text{ etc.}$$

sumendo omnino n huiusmodi expressiones. Cum igitur ultimus terminus sit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\sin. nz = \pm 2^{n-1} \sin. z \sin.\left(\frac{\pi}{n} - z\right) \sin.\left(\frac{2\pi}{n} + z\right) \sin.\left(\frac{3\pi}{n} - z\right) \cdots;$$

ubi quibus casibus utrumvis signum valeat, ex casibus particularibus erit dispiciendum.

EXEMPLUM

Substituendo autem pro n successive numeros 2, 4, 6 etc. et eligendo n sinus diversos erit

$$\sin. 2z = 2 \sin. z \sin.\left(\frac{\pi}{2} - z\right),$$

$$\sin. 4z = 8 \sin. z \sin.\left(\frac{\pi}{4} - z\right) \sin.\left(\frac{\pi}{4} + z\right) \sin.\left(\frac{2\pi}{4} - z\right),$$

$$\sin. 6z = 32 \sin. z \sin.\left(\frac{\pi}{6} - z\right) \sin.\left(\frac{\pi}{6} + z\right) \sin.\left(\frac{2\pi}{6} - z\right) \sin.\left(\frac{2\pi}{6} + z\right) \sin.\left(\frac{3\pi}{6} - z\right),$$

$$\sin. 8z = 128 \sin. z \sin.\left(\frac{\pi}{8} - z\right) \sin.\left(\frac{\pi}{8} + z\right) \sin.\left(\frac{2\pi}{8} - z\right) \sin.\left(\frac{2\pi}{8} + z\right) \sin.\left(\frac{3\pi}{8} - z\right) \sin.\left(\frac{3\pi}{8} + z\right) \sin.\left(\frac{4\pi}{8} - z\right).$$

240. Patet ergo fore generatim

$$\begin{aligned}\sin. nz = 2^{n-1} \sin. z \sin. \left(\frac{\pi}{n} - z\right) \sin. \left(\frac{\pi}{n} + z\right)\\ \sin. \left(\frac{2\pi}{n} - z\right) \sin. \left(\frac{2\pi}{n} + z\right)\\ \sin. \left(\frac{3\pi}{n} - z\right) \sin. \left(\frac{3\pi}{n} + z\right)\\ \vdots\\ \sin. \left(\frac{\pi}{2} - z\right),\end{aligned}$$

si n fuerit numerus par. Quodsi autem haec cum superiori, ubi n erat numerus impar, comparetur, tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive n fuerit numerus par sive impar,

$$\begin{aligned}\sin. nz = 2^{n-1} \sin. z \sin. \left(\frac{\pi}{n} - z\right) \sin. \left(\frac{\pi}{n} + z\right)\\ \sin. \left(\frac{2\pi}{n} - z\right) \sin. \left(\frac{2\pi}{n} + z\right)\\ \sin. \left(\frac{3\pi}{n} - z\right) \sin. \left(\frac{3\pi}{n} + z\right)\\ \text{etc.},\end{aligned}$$

donec tot habeantur factores, quot numerus n continet unitates.

241. Expressiones istae, quibus sinus angulorum multiplorum per factores exponuntur, non parum utilitatis afferre possunt ad logarithmos sinuum angulorum multiplorum inveniendos itemque ad plures expressiones sinuum per factores, quales supra (§ 184) dedimus, reperiendas. Erit autem

$$\sin. z = 1 \sin. z,$$

$$\sin. 2z = 2 \sin. z \sin. \left(\frac{\pi}{2} - z\right),$$

$$\sin. 3z = 4 \sin. z \sin. \left(\frac{\pi}{3} - z\right) \sin. \left(\frac{\pi}{3} + z\right),$$

$$\sin.4z = 8\sin.z\sin.\left(\frac{\pi}{4}-z\right)\sin.\left(\frac{\pi}{4}+z\right)$$
$$\sin.\left(\frac{2\pi}{4}-z\right),$$

$$\sin.5z = 16\sin.z\sin.\left(\frac{\pi}{5}-z\right)\sin.\left(\frac{\pi}{5}+z\right)$$
$$\sin.\left(\frac{2\pi}{5}-z\right)\sin.\left(\frac{2\pi}{5}+z\right),$$

$$\sin.6z = 32\sin.z\sin.\left(\frac{\pi}{6}-z\right)\sin.\left(\frac{\pi}{6}+z\right)$$
$$\sin.\left(\frac{2\pi}{6}-z\right)\sin.\left(\frac{2\pi}{6}+z\right)$$
$$\sin.\left(\frac{3\pi}{6}-z\right)$$

etc.

242. Cum deinde sit $\frac{\sin.2nz}{\sin.nz}=2\cos.nz$, cosinus angulorum multiplorum simili modo per factores exprimentur:

$$\cos.z = 1\sin.\left(\frac{\pi}{2}-z\right),$$

$$\cos.2z = 2\sin.\left(\frac{\pi}{4}-z\right)\sin.\left(\frac{\pi}{4}+z\right),$$

$$\cos.3z = 4\sin.\left(\frac{\pi}{6}-z\right)\sin.\left(\frac{\pi}{6}+z\right)$$
$$\sin.\left(\frac{3\pi}{6}-z\right),$$

$$\cos.4z = 8\sin.\left(\frac{\pi}{8}-z\right)\sin.\left(\frac{\pi}{8}+z\right)$$
$$\sin.\left(\frac{3\pi}{8}-z\right)\sin.\left(\frac{3\pi}{8}+z\right),$$

$$\cos.5z = 16\sin.\left(\frac{\pi}{10}-z\right)\sin.\left(\frac{\pi}{10}+z\right)$$
$$\sin.\left(\frac{3\pi}{10}-z\right)\sin.\left(\frac{3\pi}{10}+z\right)$$
$$\sin.\left(\frac{5\pi}{10}-z\right)$$

et generaliter

$$\cos. nz = 2^{n-1} \sin. \left(\frac{\pi}{2n} - z\right) \sin. \left(\frac{\pi}{2n} + z\right)$$
$$\sin. \left(\frac{3\pi}{2n} - z\right) \sin. \left(\frac{3\pi}{2n} + z\right)$$
$$\sin. \left(\frac{5\pi}{2n} - z\right) \sin. \left(\frac{5\pi}{2n} + z\right)$$
$$\text{etc.},$$

quoad tot habeantur factores, quot numerus n continet unitates.

243. Eaedem expressiones prodibunt ex consideratione cosinuum arcuum multiplorum. Si enim fuerit $\cos. z = y$, erit [§ 129], ut sequitur:

$$\begin{aligned}
\cos. 0z &= 1,\\
\cos. 1z &= y,\\
\cos. 2z &= 2y^2 - 1,\\
\cos. 3z &= 4y^3 - 3y,\\
\cos. 4z &= 8y^4 - 8y^2 + 1,\\
\cos. 5z &= 16y^5 - 20y^3 + 5y,\\
\cos. 6z &= 32y^6 - 48y^4 + 18y^2 - 1,\\
\cos. 7z &= 64y^7 - 112y^5 + 56y^3 - 7y
\end{aligned}$$

et generaliter

$$\cos. nz = 2^{n-1} y^n - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1 \cdot 2} 2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3} 2^{n-7} y^{n-6}$$
$$+ \frac{n(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3 \cdot 4} 2^{n-9} y^{n-8} - \text{etc.},$$

cuius aequationis, cum sit

$$\cos. nz = \cos. (2\pi - nz) = \cos. (2\pi + nz) = \cos. (4\pi \pm nz) = \cos. (6\pi \pm nz) \text{ etc.},$$

erunt radices ipsius y hae

$$\cos. z, \quad \cos. \left(\frac{2\pi}{n} \pm z\right), \quad \cos. \left(\frac{4\pi}{n} \pm z\right), \quad \cos. \left(\frac{6\pi}{n} \pm z\right) \quad \text{etc.},$$

quarum formularum tot diversae sunt pro y eligendae, quot dantur; dantur autem tot, quot n continet unitates.

244. Primum igitur patet ob terminum secundum deficientem excepto casu $n=1$ fore summam harum radicum omnium $=0$. Erit ergo

$$\begin{aligned}0 = \cos. z &+ \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right)\\ &+ \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right)\\ &+ \text{etc.}\end{aligned}$$

sumendo tot terminos, quot n continet unitates. Haec autem aequalitas sponte se offert, si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares unitate exclusa eritque ob $\cos. v = -\cos.(\pi - v)$

$$0 = \cos. z - \cos.\left(\frac{\pi}{3} - z\right) - \cos.\left(\frac{\pi}{3} + z\right),$$

$$\begin{aligned}0 = \cos. z &- \cos.\left(\frac{\pi}{5} - z\right) - \cos.\left(\frac{\pi}{5} + z\right)\\ &+ \cos.\left(\frac{2\pi}{5} - z\right) + \cos.\left(\frac{2\pi}{5} + z\right),\end{aligned}$$

$$\begin{aligned}0 = \cos. z &- \cos.\left(\frac{\pi}{7} - z\right) - \cos.\left(\frac{\pi}{7} + z\right)\\ &+ \cos.\left(\frac{2\pi}{7} - z\right) + \cos.\left(\frac{2\pi}{7} + z\right)\\ &- \cos.\left(\frac{3\pi}{7} - z\right) - \cos.\left(\frac{3\pi}{7} + z\right)\end{aligned}$$

et generaliter, si fuerit n numerus impar quicunque, erit

$$\begin{aligned}0 = \cos. z &- \cos.\left(\frac{\pi}{n} - z\right) - \cos.\left(\frac{\pi}{n} + z\right)\\ &+ \cos.\left(\frac{2\pi}{n} - z\right) + \cos.\left(\frac{2\pi}{n} + z\right)\\ &- \cos.\left(\frac{3\pi}{n} - z\right) - \cos.\left(\frac{3\pi}{n} + z\right)\\ &+ \cos.\left(\frac{4\pi}{n} - z\right) + \cos.\left(\frac{4\pi}{n} + z\right)\\ &- \text{etc.}\end{aligned}$$

sumendo tot terminos, quot numerus n continet unitates. Oportet autem n esse numerum imparem unitate maiorem, uti iam monuimus.

245. Quod ad productum ex omnibus attinet, variae quidem prodeunt expressiones, prout n fuerit numerus vel impar vel impariter par vel pariter par. Omnes autem comprehenduntur in expressione generali (§ 242) inventa, si singuli sinus in cosinus transmutentur. Erit scilicet

$$\cos. z = 1 \cos. z,$$

$$\cos. 2z = 2 \cos.\left(\frac{\pi}{4}+z\right)\cos.\left(\frac{\pi}{4}-z\right),$$

$$\cos. 3z = 4 \cos.\left(\frac{2\pi}{6}+z\right)\cos.\left(\frac{2\pi}{6}-z\right)\cos. z,$$

$$\cos. 4z = 8 \cos.\left(\frac{3\pi}{8}+z\right)\cos.\left(\frac{3\pi}{8}-z\right)\cos.\left(\frac{\pi}{8}+z\right)\cos.\left(\frac{\pi}{8}-z\right),$$

$$\cos. 5z = 16 \cos.\left(\frac{4\pi}{10}+z\right)\cos.\left(\frac{4\pi}{10}-z\right)\cos.\left(\frac{2\pi}{10}+z\right)\cos.\left(\frac{2\pi}{10}-z\right)\cos. z$$

et generaliter

$$\begin{aligned}\cos. nz = 2^{n-1} &\cos.\left(\frac{n-1}{2n}\pi+z\right)\cos.\left(\frac{n-1}{2n}\pi-z\right)\\ &\cos.\left(\frac{n-3}{2n}\pi+z\right)\cos.\left(\frac{n-3}{2n}\pi-z\right)\\ &\cos.\left(\frac{n-5}{2n}\pi+z\right)\cos.\left(\frac{n-5}{2n}\pi-z\right)\\ &\cos.\left(\frac{n-7}{2n}\pi+z\right)\cos.\left(\frac{n-7}{2n}\pi-z\right)\\ &\text{etc.}\end{aligned}$$

sumptis tot factoribus, quot numerus n continet unitates.

246. Sit n numerus impar atque aequatio incipiatur ab unitate; erit

$$0 = 1 \mp \frac{ny}{\cos. nz} \pm \text{etc.},$$

ubi signum superius valet, si n fuerit numerus impar formae $4m+1$, inferius, si $n=4m-1$. Hinc erit

$$+\frac{1}{\cos. z}=\frac{1}{\cos. z},$$

$$-\frac{3}{\cos. 3z}=\frac{1}{\cos. z}-\frac{1}{\cos.\left(\frac{\pi}{3}-z\right)}-\frac{1}{\cos.\left(\frac{\pi}{3}+z\right)},$$

$$+\frac{5}{\cos. 5z}=\frac{1}{\cos. z}-\frac{1}{\cos.\left(\frac{\pi}{5}-z\right)}-\frac{1}{\cos.\left(\frac{\pi}{5}+z\right)}+\frac{1}{\cos.\left(\frac{2\pi}{5}-z\right)}+\frac{1}{\cos.\left(\frac{2\pi}{5}+z\right)}$$

et generaliter posito $n=2m+1$ erit

$$\begin{aligned}\frac{n}{\cos. nz}=\frac{2m+1}{\cos.(2m+1)z}=&\frac{1}{\cos.\left(\frac{m}{n}\pi+z\right)}+\frac{1}{\cos.\left(\frac{m}{n}\pi-z\right)}\\&-\frac{1}{\cos.\left(\frac{m-1}{n}\pi+z\right)}-\frac{1}{\cos.\left(\frac{m-1}{n}\pi-z\right)}\\&+\frac{1}{\cos.\left(\frac{m-2}{n}\pi+z\right)}+\frac{1}{\cos.\left(\frac{m-2}{n}\pi-z\right)}\\&-\frac{1}{\cos.\left(\frac{m-3}{n}\pi+z\right)}-\frac{1}{\cos.\left(\frac{m-3}{n}\pi-z\right)}\\&+\text{etc.}\end{aligned}$$

sumendis tot terminis, quot n continet unitates.

247. Cum ergo sit $\frac{1}{\cos. v}=\sec. v$, hinc pro secantibus insignes proprietates deducuntur; erit nempe

$$\sec. z=\sec. z,$$

$$\begin{aligned}3\sec. 3z=&\sec.\left(\frac{\pi}{3}+z\right)+\sec.\left(\frac{\pi}{3}-z\right)\\&-\sec.\left(\frac{0\pi}{3}+z\right),\end{aligned}$$

$$\begin{aligned}5 \sec. 5z = {} & \sec.\left(\frac{2\pi}{5}+z\right)+\sec.\left(\frac{2\pi}{5}-z\right)\\ & -\sec.\left(\frac{\pi}{5}+z\right)-\sec.\left(\frac{\pi}{5}-z\right)\\ & +\sec.\left(\frac{0\pi}{5}+z\right),\end{aligned}$$

$$\begin{aligned}7 \sec. 7z = {} & \sec.\left(\frac{3\pi}{7}+z\right)+\sec.\left(\frac{3\pi}{7}-z\right)\\ & -\sec.\left(\frac{2\pi}{7}+z\right)-\sec.\left(\frac{2\pi}{7}-z\right)\\ & +\sec.\left(\frac{\pi}{7}+z\right)+\sec.\left(\frac{\pi}{7}-z\right)\\ & -\sec.\left(\frac{0\pi}{7}+z\right)\end{aligned}$$

et generaliter posito $n = 2m + 1$ erit

$$\begin{aligned}n \sec. nz = {} & \sec.\left(\frac{m}{n}\pi+z\right)+\sec.\left(\frac{m}{n}\pi-z\right)\\ & -\sec.\left(\frac{m-1}{n}\pi+z\right)-\sec.\left(\frac{m-1}{n}\pi-z\right)\\ & +\sec.\left(\frac{m-2}{n}\pi+z\right)+\sec.\left(\frac{m-2}{n}\pi-z\right)\\ & -\sec.\left(\frac{m-3}{n}\pi+z\right)-\sec.\left(\frac{m-3}{n}\pi-z\right)\\ & +\sec.\left(\frac{m-4}{n}\pi+z\right)+\sec.\left(\frac{m-4}{n}\pi-z\right)\\ & \vdots\\ & \pm \sec. z.\end{aligned}$$

248. Pro cosecantibus autem erit ex § 237

$$\operatorname{cosec.} z = \operatorname{cosec.} z,$$

$$3 \operatorname{cosec.} 3z = \operatorname{cosec.} z + \operatorname{cosec.}\left(\frac{\pi}{3}-z\right)-\operatorname{cosec.}\left(\frac{\pi}{3}+z\right),$$

$$5 \operatorname{cosec.} 5z = \operatorname{cosec.} z + \operatorname{cosec.}\left(\frac{\pi}{5} - z\right) - \operatorname{cosec.}\left(\frac{\pi}{5} + z\right)$$
$$- \operatorname{cosec.}\left(\frac{2\pi}{5} - z\right) + \operatorname{cosec.}\left(\frac{2\pi}{5} + z\right),$$

$$7 \operatorname{cosec.} 7z = \operatorname{cosec.} z + \operatorname{cosec.}\left(\frac{\pi}{7} - z\right) - \operatorname{cosec.}\left(\frac{\pi}{7} + z\right)$$
$$- \operatorname{cosec.}\left(\frac{2\pi}{7} - z\right) + \operatorname{cosec.}\left(\frac{2\pi}{7} + z\right)$$
$$+ \operatorname{cosec.}\left(\frac{3\pi}{7} - z\right) - \operatorname{cosec.}\left(\frac{3\pi}{7} + z\right)$$

et generaliter ponendo $n = 2m + 1$ erit

$$n \operatorname{cosec.} nz = \operatorname{cosec.} z + \operatorname{cosec.}\left(\frac{\pi}{n} - z\right) - \operatorname{cosec.}\left(\frac{\pi}{n} + z\right)$$
$$- \operatorname{cosec.}\left(\frac{2\pi}{n} - z\right) + \operatorname{cosec.}\left(\frac{2\pi}{n} + z\right)$$
$$+ \operatorname{cosec.}\left(\frac{3\pi}{n} - z\right) - \operatorname{cosec.}\left(\frac{3\pi}{n} + z\right)$$
$$\vdots$$
$$\mp \operatorname{cosec.}\left(\frac{m\pi}{n} - z\right) \pm \operatorname{cosec.}\left(\frac{m\pi}{n} + z\right),$$

ubi signa superiora valent, si m fuerit numerus par, inferiora, si m sit impar.

249. Cum sit, uti supra [§ 133] vidimus,

$$\cos. nz \pm \sqrt{-1} \cdot \sin. nz = (\cos. z \pm \sqrt{-1} \cdot \sin. z)^n,$$

erit

$$\cos. nz = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^n + (\cos. z - \sqrt{-1} \cdot \sin. z)^n}{2}$$

et

$$\sin. nz = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^n - (\cos. z - \sqrt{-1} \cdot \sin. z)^n}{2\sqrt{-1}},$$

ergo

$$\operatorname{tang.} nz = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^n - (\cos. z - \sqrt{-1} \cdot \sin. z)^n}{(\cos. z + \sqrt{-1} \cdot \sin. z)^n \sqrt{-1} + (\cos. z - \sqrt{-1} \cdot \sin. z)^n \sqrt{-1}}.$$

Ponamus

$$\text{tang.}\, z = \frac{\sin. z}{\cos. z} = t;$$

erit

$$\text{tang.}\, nz = \frac{(1+t\sqrt{-1})^n - (1-t\sqrt{-1})^n}{(1+t\sqrt{-1})^n\sqrt{-1} + (1-t\sqrt{-1})^n\sqrt{-1}},$$

unde oriuntur tangentes angulorum multiplorum sequentes

$$\text{tang.}\, z = t,$$

$$\text{tang.}\, 2z = \frac{2t}{1-tt},$$

$$\text{tang.}\, 3z = \frac{3t-t^3}{1-3tt},$$

$$\text{tang.}\, 4z = \frac{4t-4t^3}{1-6tt+t^4},$$

$$\text{tang.}\, 5z = \frac{5t-10t^3+t^5}{1-10tt+5t^4}$$

et generaliter

$$\text{tang.}\, nz = \frac{nt - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}t^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}t^5 - \text{etc.}}{1 - \frac{n(n-1)}{1\cdot 2}tt + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}t^4 - \text{etc.}}.$$

Cum iam sit

$$\text{tang.}\, nz = \text{tang.}\, (\pi + nz) = \text{tang.}\, (2\pi + nz) = \text{tang.}\, (3\pi + nz) \text{ etc.},$$

erunt valores ipsius t seu radices aequationis hae

$$\text{tang.}\, z, \quad \text{tang.}\left(\frac{\pi}{n} + z\right), \quad \text{tang.}\left(\frac{2\pi}{n} + z\right), \quad \text{tang.}\left(\frac{3\pi}{n} + z\right) \quad \text{etc.},$$

quarum numerus est n.

250. Quodsi aequatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{\text{tang.}\, nz} - \frac{n(n-1)tt}{1\cdot 2} + \frac{n(n-1)(n-2)t^3}{1\cdot 2\cdot 3\ \text{tang.}\, nz} + \text{etc.}$$

Ex comparatione ergo coefficientium cum radicibus erit

$$n \cot. nz = \cot. z + \cot. \left(\frac{\pi}{n} + z\right) + \cot. \left(\frac{2\pi}{n} + z\right) + \cot. \left(\frac{3\pi}{n} + z\right)$$

$$+ \cot. \left(\frac{4\pi}{n} + z\right) + \cdots + \cot. \left(\frac{n-1}{n}\pi + z\right).$$

Deinde erit summa quadratorum harum cotangentium omnium

$$= \frac{nn}{(\sin. nz)^2} - n$$

similique modo ulteriores potestates possunt definiri. Ponendo autem loco n numeros definitos erit

$$\cot. z \quad = \cot. z,$$

$$2 \cot. 2z = \cot. z + \cot. \left(\frac{\pi}{2} + z\right),$$

$$3 \cot. 3z = \cot. z + \cot. \left(\frac{\pi}{3} + z\right) + \cot. \left(\frac{2\pi}{3} + z\right),$$

$$4 \cot. 4z = \cot. z + \cot. \left(\frac{\pi}{4} + z\right) + \cot. \left(\frac{2\pi}{4} + z\right)$$

$$+ \cot. \left(\frac{3\pi}{4} + z\right),$$

$$5 \cot. 5z = \cot. z + \cot. \left(\frac{\pi}{5} + z\right) + \cot. \left(\frac{2\pi}{5} + z\right)$$

$$+ \cot. \left(\frac{3\pi}{5} + z\right) + \cot. \left(\frac{4\pi}{5} + z\right).$$

251. Quia vero est $\cot. v = -\cot.(\pi - v)$, erit

$$\cot. z = \cot. z,$$

$$2\cot. 2z = \cot. z - \cot.\left(\frac{\pi}{2} - z\right),$$

$$3\cot. 3z = \cot. z - \cot.\left(\frac{\pi}{3} - z\right) + \cot.\left(\frac{\pi}{3} + z\right),$$

$$4\cot. 4z = \cot. z - \cot.\left(\frac{\pi}{4} - z\right) + \cot.\left(\frac{\pi}{4} + z\right) - \cot.\left(\frac{2\pi}{4} - z\right),$$

$$5\cot. 5z = \cot. z - \cot.\left(\frac{\pi}{5} - z\right) + \cot.\left(\frac{\pi}{5} + z\right) - \cot.\left(\frac{2\pi}{5} - z\right) + \cot.\left(\frac{2\pi}{5} + z\right)$$

et generaliter

$$n\cot. nz = \cot. z - \cot.\left(\frac{\pi}{n} - z\right) + \cot.\left(\frac{\pi}{n} + z\right) - \cot.\left(\frac{2\pi}{n} - z\right) + \cot.\left(\frac{2\pi}{n} + z\right) - \cot.\left(\frac{3\pi}{n} - z\right) + \cot.\left(\frac{3\pi}{n} + z\right) - \text{etc.},$$

donec tot habeantur termini, quot numerus n continet unitates.

252. Incipiamus aequationem inventam a potestate summa, ubi primum distinguendi sunt casus, quibus n est vel numerus par vel impar. Sit n numerus impar, $n = 2m + 1$; erit

$$t - \text{tang.}\, z = 0,$$

$$t^3 - 3tt\,\text{tang.}\, 3z - 3t + \text{tang.}\, 3z = 0,$$

$$t^5 - 5t^4\,\text{tang.}\, 5z - 10t^3 + 10tt\,\text{tang.}\, 5z + 5t - \text{tang.}\, 5z = 0$$

et generaliter

$$t^n - nt^{n-1}\,\text{tang.}\,nz - \cdots \mp \text{tang.}\,nz = 0,$$

ubi signum superius $-$ valet, si m sit numerus par, inferius $+$, si m sit numerus impar. Erit ergo ex coefficiente secundi termini

$$\text{tang.}\,z = \text{tang.}\,z,$$

$$3\,\text{tang.}\,3z = \text{tang.}\,z + \text{tang.}\left(\frac{\pi}{3} + z\right) + \text{tang.}\left(\frac{2\pi}{3} + z\right),$$

$$5\,\text{tang.}\,5z = \text{tang.}\,z + \text{tang.}\left(\frac{\pi}{5} + z\right) + \text{tang.}\left(\frac{2\pi}{5} + z\right) + \text{tang.}\left(\frac{3\pi}{5} + z\right) + \text{tang.}\left(\frac{4\pi}{5} + z\right)$$

etc.

253. Cum igitur sit $\text{tang.}\,v = -\text{tang.}\,(\pi - v)$, anguli recto maiores ad angulos recto minores reducuntur eritque

$$\text{tang.}\,z = \text{tang.}\,z,$$

$$3\,\text{tang.}\,3z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{3} - z\right) + \text{tang.}\left(\frac{\pi}{3} + z\right),$$

$$5\,\text{tang.}\,5z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{5} - z\right) + \text{tang.}\left(\frac{\pi}{5} + z\right) - \text{tang.}\left(\frac{2\pi}{5} - z\right) + \text{tang.}\left(\frac{2\pi}{5} + z\right),$$

$$7\,\text{tang.}\,7z = \text{tang.}\,z - \text{tang.}\left(\frac{\pi}{7} - z\right) + \text{tang.}\left(\frac{\pi}{7} + z\right) - \text{tang.}\left(\frac{2\pi}{7} - z\right) + \text{tang.}\left(\frac{2\pi}{7} + z\right) - \text{tang.}\left(\frac{3\pi}{7} - z\right) + \text{tang.}\left(\frac{3\pi}{7} + z\right)$$

et generaliter, si $n = 2m + 1$, erit

$$n \text{ tang. } nz = \text{tang. } z - \text{tang.}\left(\frac{\pi}{n} - z\right) + \text{tang.}\left(\frac{\pi}{n} + z\right)$$
$$- \text{tang.}\left(\frac{2\pi}{n} - z\right) + \text{tang.}\left(\frac{2\pi}{n} + z\right)$$
$$- \text{tang.}\left(\frac{3\pi}{n} - z\right) + \text{tang.}\left(\frac{3\pi}{n} + z\right)$$
$$\vdots$$
$$- \text{tang.}\left(\frac{m\pi}{n} - z\right) + \text{tang.}\left(\frac{m\pi}{n} + z\right).$$

254. Tum vero productum ex his tangentibus omnibus erit $= \text{tang. } nz$, propterea quod per signorum negativorum numerum alternatim parem et imparem superior signorum ambiguitas tollitur. Sic erit

$$\text{tang. } z \;= \text{tang. } z,$$
$$\text{tang. } 3z = \text{tang. } z \text{ tang.}\left(\frac{\pi}{3} - z\right) \text{tang.}\left(\frac{\pi}{3} + z\right),$$
$$\text{tang. } 5z = \text{tang. } z \text{ tang.}\left(\frac{\pi}{5} - z\right) \text{tang.}\left(\frac{\pi}{5} + z\right)$$
$$\text{tang.}\left(\frac{2\pi}{5} - z\right) \text{tang.}\left(\frac{2\pi}{5} + z\right)$$

et generaliter, si $n = 2m + 1$, erit

$$\text{tang. } nz = \text{tang. } z \text{ tang.}\left(\frac{\pi}{n} - z\right) \text{tang.}\left(\frac{\pi}{n} + z\right)$$
$$\text{tang.}\left(\frac{2\pi}{n} - z\right) \text{tang.}\left(\frac{2\pi}{n} + z\right)$$
$$\text{tang.}\left(\frac{3\pi}{n} - z\right) \text{tang.}\left(\frac{3\pi}{n} + z\right)$$
$$\vdots$$
$$\text{tang.}\left(\frac{m\pi}{n} - z\right) \text{tang.}\left(\frac{m\pi}{n} + z\right).$$

255. Sit iam n numerus par atque incipiendo a potestate summa erit

$$tt + 2t \cot. 2z - 1 = 0,$$

$$t^4 + 4t^3 \cot. 4z - 6tt - 4t \cot. 4z + 1 = 0$$

et generaliter, si $n = 2m$, erit

$$t^n + nt^{n-1} \cot. nz - \cdots \mp 1 = 0,$$

ubi signum superius — valet, si m sit numerus impar, inferius +, si m sit par. Comparando ergo radices cum coefficiente secundi termini erit

$$-2 \cot. 2z = \operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{2} + z\right),$$

$$-4 \cot. 4z = \operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{4} + z\right) + \operatorname{tang.}\left(\frac{2\pi}{4} + z\right) + \operatorname{tang.}\left(\frac{3\pi}{4} + z\right),$$

$$-6 \cot. 6z = \operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{2\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{3\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{4\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{5\pi}{6} + z\right)$$

etc.

256. Cum sit $\operatorname{tang.} v = -\operatorname{tang.}(\pi - v)$, sequentes formabuntur aequationes

$$2 \cot. 2z = -\operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{2} - z\right),$$

$$4 \cot. 4z = -\operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{4} - z\right) - \operatorname{tang.}\left(\frac{\pi}{4} + z\right) + \operatorname{tang.}\left(\frac{2\pi}{4} - z\right),$$

$$6 \cot. 6z = -\operatorname{tang.} z + \operatorname{tang.}\left(\frac{\pi}{6} - z\right) - \operatorname{tang.}\left(\frac{\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{2\pi}{6} - z\right) - \operatorname{tang.}\left(\frac{2\pi}{6} + z\right) + \operatorname{tang.}\left(\frac{3\pi}{6} - z\right)$$

et generaliter, si $n = 2m$, erit

$$
\begin{aligned}
n \cot. nz = -\tan g. z &+ \text{tang.}\left(\frac{\pi}{n} - z\right) - \text{tang.}\left(\frac{\pi}{n} + z\right) \\
&+ \text{tang.}\left(\frac{2\pi}{n} - z\right) - \text{tang.}\left(\frac{2\pi}{n} + z\right) \\
&+ \text{tang.}\left(\frac{3\pi}{n} - z\right) - \text{tang.}\left(\frac{3\pi}{n} + z\right) \\
&\vdots \\
&+ \text{tang.}\left(\frac{m\pi}{n} - z\right).
\end{aligned}
$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus destruitur eritque idcirco

$$1 = \text{tang.}\, z \,\text{tang.}\left(\frac{\pi}{2} - z\right),$$

$$
\begin{aligned}
1 = \text{tang.}\, z \,\text{tang.}\left(\frac{\pi}{4} - z\right) &\text{tang.}\left(\frac{\pi}{4} + z\right) \\
\text{tang.}\left(\frac{2\pi}{4} - z\right)&,
\end{aligned}
$$

$$
\begin{aligned}
1 = \text{tang.}\, z \,\text{tang.}\left(\frac{\pi}{6} - z\right) &\text{tang.}\left(\frac{\pi}{6} + z\right) \\
\text{tang.}\left(\frac{2\pi}{6} - z\right) &\text{tang.}\left(\frac{2\pi}{6} + z\right) \\
\text{tang.}\left(\frac{3\pi}{6} - z\right)&
\end{aligned}
$$

etc.

Harum vero aequationum ratio statim sponte in oculos incurrit, cum perpetuo bini anguli reperiantur, quorum alter est alterius complementum ad rectum. Huiusmodi ergo binorum angulorum tangentes productum dant $= 1$ ideoque omnium productum unitati debet esse aequale.

258. Quoniam sinus et cosinus angulorum progressionem arithmeticam constituentium seriem recurrentem praebent, per caput praecedens summa huiusmodi sinuum et cosinuum quotcunque exhiberi poterit. Sint anguli in arithmetica progressione

$$a, \quad a + b, \quad a + 2b, \quad a + 3b, \quad a + 4b, \quad a + 5b \quad \text{etc.}$$

et quaeratur primo summa sinuum horum angulorum in infinitum progredientium; ponatur ergo

$$s = \sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \text{etc.},$$

et quia haec series est recurrens, cuius scala relationis est $2 \cos. b$, -1, orietur haec series ex evolutione fractionis, cuius denominator est

$$1 - 2z \cos. b + zz$$

posito $z = 1$. Ipsa vero fractio erit

$$= \frac{\sin. a + z(\sin. (a + b) - 2 \sin. a \cos. b)}{1 - 2z \cos. b + zz};$$

quare facto $z = 1$ erit

$$s = \frac{\sin. a + \sin. (a + b) - 2 \sin. a \cos. b}{2 - 2 \cos. b} = \frac{\sin. a - \sin. (a - b)}{2(1 - \cos. b)}$$

ob

$$2 \sin. a \cos. b = \sin. (a + b) + \sin. (a - b).$$

Cum autem sit

$$\sin. f - \sin. g = 2 \cos. \frac{f + g}{2} \sin. \frac{f - g}{2},$$

erit

$$\sin. a - \sin. (a - b) = 2 \cos. \left(a - \frac{1}{2} b\right) \sin. \frac{1}{2} b;$$

at

$$1 - \cos. b = 2 \left(\sin. \frac{1}{2} b\right)^2,$$

unde erit

$$s = \frac{\cos. (a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b}.$$

259. Hinc itaque summa quotcunque sinuum, quorum arcus in arithmetica progressione incedunt, assignari poterit. Quaeratur nempe summa huius progressionis

$$\sin. a + \sin. (a + b) + \sin. (a + 2b) + \sin. (a + 3b) + \cdots + \sin. (a + nb).$$

Quia summa huius progressionis in infinitum continuatae est

$$\frac{\cos. (a - \frac{1}{2} b)}{2 \sin. \frac{1}{2} b},$$

considerentur termini ultimum sequentes in infinitum hi

$$\sin.(a+(n+1)b)+\sin.(a+(n+2)b)+\sin.(a+(n+3)b)+\text{etc.};$$

quia horum sinuum summa est

$$=\frac{\cos.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b},$$

si haec a priori subtrahatur, remanebit summa quaesita. Scilicet, si fuerit

$$s=\sin.a+\sin.(a+b)+\sin.(a+2b)+\cdots+\sin.(a+nb),$$

erit

$$s=\frac{\cos.(a-\frac{1}{2}b)-\cos.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b}=\frac{\sin.(a+\frac{1}{2}nb)\sin.\frac{1}{2}(n+1)b}{\sin.\frac{1}{2}b}.$$

260. Pari modo si consideretur summa cosinuum atque ponatur

$$s=\cos.a+\cos.(a+b)+\cos.(a+2b)+\cos.(a+3b)+\text{etc. in infinitum},$$

erit

$$s=\frac{\cos.a+z(\cos.(a+b)-2\cos.a\cos.b)}{1-2z\cos.b+zz}$$

posito $z=1$. Quare ob

$$2\cos.a\cos.b=\cos.(a-b)+\cos.(a+b)$$

fiet

$$s=\frac{\cos.a-\cos.(a-b)}{2(1-\cos.b)}.$$

At est

$$\cos.f-\cos.g=2\sin.\frac{f+g}{2}\sin.\frac{g-f}{2};$$

unde erit

$$\cos.a-\cos.(a-b)=-2\sin.\left(a-\frac{1}{2}b\right)\sin.\frac{1}{2}b,$$

et ob

$$1-\cos.b=2\left(\sin.\frac{1}{2}b\right)^2$$

erit

$$s=-\frac{\sin.(a-\frac{1}{2}b)}{2\sin.\frac{1}{2}b}.$$

Quare, cum simili modo sit huius seriei

$$\cos.(a+(n+1)b)+\cos.(a+(n+2)b)+\cos.(a+(n+3)b)+\text{etc.}$$

summa

$$=-\frac{\sin.(a+(n+\frac{1}{2})b)}{2\sin.\frac{1}{2}b},$$

si haec ab illa subtrahatur, relinquetur summa huius seriei

$$s = \cos. a + \cos. (a + b) + \cos. (a + 2b) + \cos. (a + 3b) + \cdots + \cos. (a + nb)$$

eritque

$$s = \frac{-\sin. (a - \frac{1}{2} b) + \sin. (a + (n + \frac{1}{2}) b)}{2 \sin. \frac{1}{2} b} = \frac{\cos. (a + \frac{1}{2} nb) \sin. \frac{1}{2} (n + 1) b}{\sin. \frac{1}{2} b}.$$

261. Plurimae aliae quaestiones circa sinus et tangentes ex principiis allatis resolvi possent; cuiusmodi sunt, si quadrata altioresve potestates sinuum tangentiumve summari deberent; verum quia haec ex reliquis aequationum superiorum coefficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque sinuum cosinuumque potestatem per singulos sinus cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

262. Ad hoc expediendum iuvabit ex praecedentibus haec lemmata deprompsisse

$$2 \sin. a \sin. z = \cos. (a - z) - \cos. (a + z),$$
$$2 \cos. a \sin. z = \sin. (a + z) - \sin. (a - z),$$
$$2 \sin. a \cos. z = \sin. (a + z) + \sin. (a - z),$$
$$2 \cos. a \cos. z = \cos. (a - z) + \cos. (a + z).$$

Hinc igitur primum potestates sinuum reperiuntur:

$$\sin. z = \sin. z,$$
$$2 (\sin. z)^2 = 1 - \cos. 2z,$$
$$4 (\sin. z)^3 = 3 \sin. z - \sin. 3z,$$
$$8 (\sin. z)^4 = 3 - 4 \cos. 2z + \cos. 4z,$$
$$16 (\sin. z)^5 = 10 \sin. z - 5 \sin. 3z + \sin. 5z,$$
$$32 (\sin. z)^6 = 10 - 15 \cos. 2z + 6 \cos. 4z - \cos. 6z,$$
$$64 (\sin. z)^7 = 35 \sin. z - 21 \sin. 3z + 7 \sin. 5z - \sin. 7z,$$
$$128 (\sin. z)^8 = 35 - 56 \cos. 2z + 28 \cos. 4z - 8 \cos. 6z + \cos. 8z,$$
$$256 (\sin. z)^9 = 126 \sin. z - 84 \sin. 3z + 36 \sin. 5z - 9 \sin. 7z + \sin. 9z$$

etc.

Lex, qua hi coefficientes progrediuntur, ex unciis binomii elevati intelligitur, nisi quod numerus absolutus in potestatibus paribus semissis tantum sit eius, quem unciae praebent.

263. Pari modo potestates cosinuum definientur:

$$\begin{aligned}
\cos. z &= \cos. z,\\
2(\cos. z)^2 &= 1 + \cos. 2z,\\
4(\cos. z)^3 &= 3\cos. z + \cos. 3z,\\
8(\cos. z)^4 &= 3 + 4\cos. 2z + \cos. 4z,\\
16(\cos. z)^5 &= 10\cos. z + 5\cos. 3z + \cos. 5z,\\
32(\cos. z)^6 &= 10 + 15\cos. 2z + 6\cos. 4z + \cos. 6z,\\
64(\cos. z)^7 &= 35\cos. z + 21\cos. 3z + 7\cos. 5z + \cos. 7z
\end{aligned}$$

etc.

Hic ratione legis progressionis eadem sunt monenda, quae circa sinus notavimus.

CAPUT XV

DE SERIEBUS EX EVOLUTIONE FACTORUM ORTIS

264. Sit propositum productum ex factoribus numero sive finitis sive infinitis constans huiusmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\varepsilon z)(1+\zeta z) \text{ etc.},$$

quod, si per multiplicationem actualem evolvatur, det

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.},$$

atque manifestum est coefficientes A, B, C, D, E etc. ita formari ex numeris α, β, γ, δ, ε, ζ etc., ut sit

$A = \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \text{etc.} =$ summae singulorum,

$B =$ summae factorum ex binis diversis,

$C =$ summae factorum ex ternis diversis,

$D =$ summae factorum ex quaternis diversis,

$E =$ summae factorum ex quinis diversis

etc.,

donec perveniatur ad productum ex omnibus.

265. Quodsi ergo ponatur $z = 1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum serie occurret.

266. Si ponatur $z = -1$, productum hoc

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis vel ternis vel quinis vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis vel quaternis vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro $\alpha, \beta, \gamma, \delta$ etc. numeri primi omnes

$$2, 3, 5, 7, 11, 13 \text{ etc.}$$

atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13) \text{ etc.} = P$$

aequabitur unitati cum serie omnium numerorum vel primorum ipsorum vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+\text{etc.},$$

in qua serie omnes occurrunt numeri naturales exceptis potestatibus iisque, qui per quamvis potestatem sunt divisibiles. Desunt scilicet numeri 4, 8, 9, 12, 16, 18 etc., quoniam sunt vel potestates ut 4, 8, 9, 16 etc., vel per potestates divisibiles ut 12, 18 etc.

268. Simili modo res se habebit, si pro $\alpha, \beta, \gamma, \delta$ etc. potestates quaecunque numerorum primorum substituantur, scilicet si ponamus

$$P = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right) \text{ etc.}$$

Erit enim multiplicatione instituta

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{etc.},$$

in quibus fractionibus omnes occurrunt numeri praeter illos, qui vel ipsi sunt potestates vel per potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

269. Si numeri $\alpha, \beta, \gamma, \delta$ etc. negative capiantur, ut ante (§ 266) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{ etc.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{14^n} + \frac{1}{15^n} - \text{etc.},$$

ubi iterum ut ante omnes occurrunt numeri praeter potestates ac divisibiles per potestates. Verum ipsi numeri primi et qui ex ternis, quinis numerove imparibus constant, signum habent praefixum $-$, qui autem ex binis vel quaternis vel senis vel numero paribus formantur, signum habent $+$. Sic in hac serie occurret terminus $\frac{1}{30^n}$, quia est $30 = 2 \cdot 3 \cdot 5$ neque adeo potestatem complectitur; habebit vero hic terminus $\frac{1}{30^n}$ signum $-$, quia 30 est productum ex tribus numeris primis.

270. Consideremus iam hanc expressionem

$$\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \varepsilon z) \text{ etc.}},$$

quae per divisionem actualem evoluta praebeat hanc seriem

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.},$$

atque manifestum est coefficientes A, B, C, D, E etc. sequenti modo ex numeris $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. componi, ut sit

A = summae singulorum,

B = summae factorum ex binis,

C = summae factorum ex ternis,

D = summae factorum ex quaternis

etc.

non exclusis factoribus iisdem.

271. Posito ergo $z = 1$ ista expressio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve in se multiplicandis oriuntur non exclusis aequalibus. Hoc ergo differt ista numerorum series ab illa, quae § 265 prodiit, quod ibi factores tantum diversi sumi debebant, hic autem idem factor bis pluriesve occurrere possit. Hic scilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta$ etc. provenire possunt.

272. Hanc ob rem series semper ex terminorum numero infinito constat, sive factorum numerus fuerit infinitus sive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.},$$

ubi omnes numeri adsunt, qui ex binario solo per multiplicationem oriuntur, seu omnes binarii potestates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \text{etc.},$$

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 et 3 per multiplicationem originem trahunt, seu qui alios divisores praeter 2 et 3 non habent.

273. Si igitur pro α, β, γ, δ etc. unitas per singulos omnes numeros primos scribatur ac ponatur

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.},$$

ubi omnes numeri, tam primi quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi vel ex primis per multiplicationem oriundi, manifestum est hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum potestates quaecunque accipiantur. Si enim ponatur

$$P = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.},$$

ubi omnes numeri naturales nullo excepto occurrunt. Quodsi autem in factoribus ubique signum $+$ statuatur, ut sit

$$P = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \text{etc.},$$

ubi numeri primi habent signum $-$; qui sunt producti ex duobus primis, sive iisdem sive diversis, signum habent $+$; et generatim, quorum numerorum numerus factorum primorum est par, signum habent $+$, qui autem ex factoribus primis numero imparibus constant, habent signum $-$. Sic terminus $\frac{1}{240^n}$ ob $240 = 2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 5$ habebit signum $+$. Cuius legis ratio percipitur ex § 270, si ponatur $z = -1$.

275. Si haec cum superioribus conferantur, nascentur binae series, quarum productum unitati aequatur. Sit enim

$$P = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q = \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.};$$

erit

$$P = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\text{ etc.},$$

$$Q = 1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\text{ etc.}$$

(§ 269) atque manifestum est fore $PQ = 1$.

276. Sin autem ponatur

$$P = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

erit

$$P = 1-\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}+\text{ etc.},$$

$$Q = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\text{ etc.}$$

similique modo habebitur $PQ = 1$. Cognita ergo alterius seriei summa simul alterius innotescet.

277. Vicissim porro ex cognitis summis harum serierum assignari poterunt valores factorum infinitorum. Sit nimirum

$$M = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\text{ etc.},$$

$$N = 1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{6^{2n}}+\frac{1}{7^{2n}}+\text{ etc.}$$

eritque

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}}.$$

Hinc per divisionem nascitur

$$\frac{M}{N} = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

denique vero erit

$$\frac{MM}{N} = \frac{2^n+1}{2^n-1}\cdot\frac{3^n+1}{3^n-1}\cdot\frac{5^n+1}{5^n-1}\cdot\frac{7^n+1}{7^n-1}\cdot\frac{11^n+1}{11^n-1}\cdot\text{etc.}$$

Ex cognitis ergo M et N praeter valores horum productorum summae harum serierum habebuntur:

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \text{etc.},$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \text{etc.},$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{etc.},$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \text{etc.},$$

ex quarum combinatione multae aliae deduci possunt.

EXEMPLUM 1

Sit $n = 1$, et quoniam supra [§ 123] demonstravimus esse

$$l\,\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \text{etc.},$$

erit posito $x = 1$

$$l\frac{1}{1-1} = l\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}$$

At logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{etc.} = \infty.$$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$ fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \text{etc.}$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

unde fit

$$\infty = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\text{etc.}$$

et

$$0 = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot\text{etc.}$$

Deinde per summationem serierum supra [§ 167] traditam erit

$$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{etc.} = \frac{\pi\pi}{6}.$$

Hinc obtinentur istae summae serierum:

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{etc.},$$

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \text{etc.},$$

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \text{etc.}$$

Denique pro factoribus orietur

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.}$$

seu

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\frac{169}{168}\cdot\text{etc.}$$

et ob $\frac{M}{N} = \infty$ seu $\frac{N}{M} = 0$ habebitur

$$\infty = \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{14}{13}\cdot\frac{18}{17}\cdot\frac{20}{19}\cdot\text{etc.}$$

seu

$$0 = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\text{etc.}$$

atque

$$\infty = \frac{3}{1}\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{14}{12}\cdot\frac{18}{16}\cdot\frac{20}{18}\cdot\text{etc.}$$

seu

$$0 = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\text{etc.},$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summae autem ex numeratoribus et denominatoribus cuiusque fractionis constanter praebent numeros primos 3, 5, 7, 11, 13, 17, 19 etc.

EXEMPLUM 2

Sit $n = 2$ eritque ex superioribus [§ 167]

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \text{etc.} = \frac{\pi\pi}{6},$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \text{etc.} = \frac{\pi^4}{90}.$$

Hinc primo istae series summantur:

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{etc.},$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \text{etc.},$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} - \text{etc.}$$

Deinde valores sequentium productorum innotescunt:

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \text{etc.},$$

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{etc.},$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \text{etc.}$$

seu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \text{etc.}$$

et

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \text{etc.}$$

sive

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{etc.}$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{etc.}$$

In his fractionibus numeratores unitate superant denominatores, simul vero sumpti praebent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2 etc.

EXEMPLUM 3

Quia ex superioribus [§ 167] valores ipsius M tantum, si n sit numerus par, assignare licet, ponamus $n = 4$ eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \frac{\pi^4}{90},$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \frac{\pi^8}{9450}.$$

Hinc primo sequentes series summantur:

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} - \text{etc.},$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} + \text{etc.},$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} + \text{etc.}$$

Deinde etiam valores sequentium productorum obtinentur:

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{etc.},$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \text{etc.},$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \text{etc.}$$

et

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \text{etc.}$$

seu

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \text{etc.}$$

In his factoribus numeratores unitate superant denominatores, simul vero sumpti praebent biquadrata numerorum primorum imparium 3, 5, 7, 11 etc.

278. Quoniam hic summam seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

ad factores reduximus, ad logarithmos commode progredi licebit. Nam cum sit

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

erit

$$lM = -l\left(1-\frac{1}{2^n}\right) - l\left(1-\frac{1}{3^n}\right) - l\left(1-\frac{1}{5^n}\right) - l\left(1-\frac{1}{7^n}\right) - \text{etc.}$$

Hinc sumendis logarithmis hyperbolicis erit

$$\begin{aligned} lM = &+ 1\left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{etc.}\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Quodsi insuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.},$$

ut sit

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}},$$

fiet logarithmis hyperbolicis sumendis

$$\begin{aligned} lN = &+ 1\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \text{etc.}\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Ex his coniunctis fiet

$$
\begin{aligned}
lM-\frac{1}{2}lN = &+1\left(\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\text{etc.}\right)\\
&+\frac{1}{3}\left(\frac{1}{2^{3n}}+\frac{1}{3^{3n}}+\frac{1}{5^{3n}}+\frac{1}{7^{3n}}+\frac{1}{11^{3n}}+\text{etc.}\right)\\
&+\frac{1}{5}\left(\frac{1}{2^{5n}}+\frac{1}{3^{5n}}+\frac{1}{5^{5n}}+\frac{1}{7^{5n}}+\frac{1}{11^{5n}}+\text{etc.}\right)\\
&+\frac{1}{7}\left(\frac{1}{2^{7n}}+\frac{1}{3^{7n}}+\frac{1}{5^{7n}}+\frac{1}{7^{7n}}+\frac{1}{11^{7n}}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}
$$

279. Si $n=1$, erit

$$M = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc.} = l\infty$$

et

$$N=\frac{\pi\pi}{6};$$

hincque erit

$$
\begin{aligned}
l.\,l\infty-\frac{1}{2}l\frac{\pi\pi}{6} = &+1\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}\right)\\
&+\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}+\text{etc.}\right)\\
&+\frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{11^5}+\text{etc.}\right)\\
&+\frac{1}{7}\left(\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{11^7}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}
$$

Verum hae series praeter primam non solum summas habent finitas, sed etiam cunctae simul sumptae summam efficiunt finitam eamque satis parvam; unde necesse est, ut seriei primae

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}$$

summa sit infinite magna. Quantitate scilicet satis parva deficiet a logarithmo hyperbolico seriei

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$$

280. Sit $n=2$; erit

$$M=\frac{\pi\pi}{6} \quad \text{et} \quad N=\frac{\pi^4}{90},$$

unde fit

$$\begin{aligned} 2\,l\pi - l6 = {} & 1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right) \\ & +\frac{1}{2}\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right) \\ & +\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right) \\ & \text{etc.,} \end{aligned}$$

$$\begin{aligned} 4\,l\pi - l90 = {} & +1\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right) \\ & +\frac{1}{2}\left(\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{11^8}+\text{etc.}\right) \\ & +\frac{1}{3}\left(\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{5^{12}}+\frac{1}{7^{12}}+\frac{1}{11^{12}}+\text{etc.}\right) \\ & \text{etc.,} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\,l\frac{5}{2} = {} & 1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right) \\ & +\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right) \\ & +\frac{1}{5}\left(\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\frac{1}{11^{10}}+\text{etc.}\right) \\ & \text{etc.} \end{aligned}$$

281. Quanquam lex, qua numeri primi progrediuntur, non constat, tamen harum serierum altiorum potestatum summae non difficulter proxime assignari poterunt. Sit enim haec series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{etc.}$$

et

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{etc.};$$

erit

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \text{etc.}$$

et ob

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \text{etc.}$$

erit

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \text{etc.}$$

seu

$$S = (M-1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \text{etc.}$$

et ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.}$$

erit

$$S = (M-1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{49^n} - \text{etc.}\ ^{1)}$$

Hinc ob datam summam M [§ 168] valor ipsius S commode invenitur, siquidem n fuerit numerus mediocriter magnus.

282. Inventis autem summis altiorum potestatum etiam summae potestatum minorum ex formulis inventis exhiberi possunt. Atque hac methodo sequentes prodierunt summae seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \text{etc.}:$$

1) Editio princeps: $S = \cdots + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{45^n} - \text{etc.}$ Correxit F. R.

Si sit	erit summa seriei[1])
$n = 2$	0,452247420041065
$n = 4$	0,076993139764247
$n = 6$	0,017070086850637
$n = 8$	0,004061405366518
$n = 10$	0,000993603574437
$n = 12$	0,000246026470035
$n = 14$	0,000061244396725
$n = 16$	0,000015282026219
$n = 18$	0,000003817278703
$n = 20$	0,000000953961124
$n = 22$	0,000000238450446
$n = 24$	0,000000059608185
$n = 26$	0,000000014901555
$n = 28$	0,000000003725334
$n = 30$	0,000000000931327
$n = 32$	0,000000000232831
$n = 34$	0,000000000058208
$n = 36$	0,000000000014552

Reliquae summae parium potestatum in ratione quadrupla decrescunt.

283. Haec autem seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}$$

1) In editione principe quinque ultimae figurae harum serierum ita se habent:
($n = 2$) 41222; ($n = 4$) 64252; ($n = 6$) 50639; ($n = 8$) 66515; ($n = 10$) 73633; ($n = 12$) 70033; ($n = 18$) 78702; ($n = 20$) 61123; ($n = 24$) 08184; ($n = 28$) 25333; ($n = 30$) 31323; ($n = 32$) 32830; ($n = 34$) 58207; ($n = 36$) 14551. Correxit F. R.

in productum infinitum conversio etiam directe institui potest hoc modo. Sit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.};$$

subtrahe

$$\frac{1}{2^n}A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{2^n}\right)A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{etc.} = B.$$

Sic sublati sunt omnes termini per 2 divisibiles. Subtrahe

$$\frac{1}{3^n}B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{3^n}\right)B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{etc.} = C.$$

Sic insuper sublati sunt omnes termini per 3 divisibiles. Subtrahe

$$\frac{1}{5^n}C = \frac{1}{5^n} + \frac{1}{25^n} + \frac{1}{35^n} + \frac{1}{55^n} + \text{etc.};$$

erit

$$\left(1 - \frac{1}{5^n}\right)C = 1 + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \text{etc.}$$

Sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11 reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro B, C, D, E etc. valoribus restitutis tandem orietur

$$A\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{etc.} = 1,$$

unde seriei propositae summa erit

$$A = \frac{1}{\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right) \text{etc.}}$$

seu

$$A = \frac{2^n}{2^n - 1} \cdot \frac{3^n}{3^n - 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n - 1} \cdot \text{etc.}$$

284. Haec methodus iam commode adhiberi poterit ad alias series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (§ 175) summas harum serierum

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\frac{1}{11^n}+\frac{1}{13^n}-\text{etc.},$$

si n fuerit numerus impar. Summa enim est $=N\pi^n$ et valores ipsius N loco citato dedimus. Notandum autem est, cum hic tantum numeri impares occurrunt, eos, qui sint formae $4m+1$, habere signum $+$, reliquos formae $4m-1$ signum $-$. Sit igitur

$$A=1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\frac{1}{11^n}+\frac{1}{13^n}-\frac{1}{15^n}+\text{etc.}$$

Addatur

$$\frac{1}{3^n}A=\frac{1}{3^n}-\frac{1}{9^n}+\frac{1}{15^n}-\frac{1}{21^n}+\frac{1}{27^n}-\text{etc.};$$

erit

$$\left(1+\frac{1}{3^n}\right)A=1+\frac{1}{5^n}-\frac{1}{7^n}-\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}-\text{etc.}=B.$$

Subtrahatur

$$\frac{1}{5^n}B=\frac{1}{5^n}+\frac{1}{25^n}-\frac{1}{35^n}-\frac{1}{55^n}+\text{etc.};$$

erit

$$\left(1-\frac{1}{5^n}\right)B=1-\frac{1}{7^n}-\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}-\text{etc.}=C,$$

ubi iam numeri per 3 et 5 divisibiles desunt. Addatur

$$\frac{1}{7^n}C=\frac{1}{7^n}-\frac{1}{49^n}-\frac{1}{77^n}+\text{etc.};$$

erit

$$\left(1+\frac{1}{7^n}\right)C=1-\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}-\text{etc.}=D.$$

Sic etiam numeri per 7 divisibiles sunt sublati. Addatur

$$\frac{1}{11^n}D=\frac{1}{11^n}-\frac{1}{121^n}+\text{etc.};$$

erit

$$\left(1+\frac{1}{11^n}\right)D=1+\frac{1}{13^n}+\frac{1}{17^n}-\text{etc.}=E.$$

Sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus tandem prodibit

$$A\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\left(1-\frac{1}{13^n}\right) \text{ etc.} = 1$$

seu

$$A = \frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n+1}\cdot\frac{11^n}{11^n+1}\cdot\frac{13^n}{13^n-1}\cdot\frac{17^n}{17^n-1}\cdot \text{etc.},$$

ubi in numeratoribus occurrunt potestates omnium numerorum primorum, quae in denominatoribus insunt unitate sive auctae sive minutae, prout numeri primi fuerint formae $4m-1$ vel $4m+1$.

285. Posito ergo $n=1$ ob $A=\frac{\pi}{4}$ [§ 175] erit

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot \text{etc.}$$

Supra [§ 277] autem invenimus esse

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{3^2}{2\cdot4}\cdot\frac{5^2}{4\cdot6}\cdot\frac{7^2}{6\cdot8}\cdot\frac{11^2}{10\cdot12}\cdot\frac{13^2}{12\cdot14}\cdot\frac{17^2}{16\cdot18}\cdot\frac{19^2}{18\cdot20}\cdot \text{etc.}$$

Dividatur secunda per primam et orietur

$$\frac{2\pi}{3} = \frac{4}{3}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot \text{etc.}$$

seu

$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot \text{etc.},$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares unitate differentes a numeratoribus. Quodsi haec denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{18}\cdot\frac{24}{22}\cdot \text{etc.}$$

seu

$$2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot \text{etc.},$$

quae fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17 etc. quemque in duas partes unitate differentes dispescendo et partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hae expressiones cum WALLISIANA[1]) comparentur

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11}\text{ etc.}$$

seu

$$\frac{4}{\pi}=\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\text{ etc.},$$

cum sit [§ 277]

$$\frac{\pi\pi}{8}=\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\frac{13\cdot 13}{12\cdot 14}\cdot\text{ etc.},$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3}=\frac{9\cdot 9}{8\cdot 10}\cdot\frac{15\cdot 15}{14\cdot 16}\cdot\frac{21\cdot 21}{20\cdot 22}\cdot\frac{25\cdot 25}{24\cdot 26}\cdot\text{ etc.},$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit iam $n=3$; erit $A=\frac{\pi^3}{32}$ [§ 175], unde fit

$$\frac{\pi^3}{32}=\frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3+1}\cdot\frac{11^3}{11^3+1}\cdot\frac{13^3}{13^3-1}\cdot\frac{17^3}{17^3-1}\cdot\text{ etc.}$$

At ex serie [§ 167]

$$\frac{\pi^6}{945}=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{ etc.}$$

fit [§ 277]

$$\frac{\pi^6}{945}=\frac{2^6}{2^6-1}\cdot\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{ etc.}$$

seu

$$\frac{\pi^6}{960}=\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{ etc.},$$

1) Vide notam p. 197. F. R.

quae per primam divisa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot \text{etc.}$$

Haec vero denuo per primam divisa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot \text{etc.}$$

seu

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \text{etc.},$$

quae fractiones formantur ex cubis numerorum primorum imparium quemque in duas partes unitate differentes dispescendo ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novae series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit [§ 285]

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \text{etc.},$$

erit

$$\frac{\pi}{6} = \frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right) \text{etc.}},$$

unde per evolutionem haec series nascetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi ratio signorum ita est comparata, ut binarius habeat —, numeri primi formae $4m-1$ signum — et numeri primi formae $4m+1$ signum +; numeri autem compositi ea habent signa, quae ipsis ratione multiplicationis ex primis conveniunt. Sic patebit signum fractionis $\frac{1}{60}$ ob

$$60 = \overset{-}{2} \cdot \overset{-}{2} \cdot \overset{-}{3} \cdot \overset{+}{5},$$

quod erit —.

Simili modo porro erit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

unde orietur haec series

$$\frac{\pi}{2}=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

ubi binarius habet signum $+$, numeri primi formae $4m-1$ signum $-$, numeri primi formae $4m+1$ signum $+$; et numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit secundum regulas multiplicationis.

289. Cum deinde sit [§ 285]

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

erit per evolutionem

$$\frac{\pi}{2}=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}-\text{etc.},$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formae $4m-1$ signum habeant $+$, numeri primi formae $4m+1$ signum $-$, unde simul numerorum compositorum signa definiuntur.

Binae porro series hinc formari possunt, ubi omnes numeri occurrunt. Erit scilicet

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\text{etc.},$$

ubi binarius signum habet $+$, numeri primi formae $4m-1$ signum $+$, numeri vero primi formae $4m+1$ signum $-$.

Tum vero etiam erit

$$\frac{\pi}{3}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{\pi}{3}=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\text{etc.},$$

ubi binarius habet signum $-$, numeri primi formae $4m-1$ signum $+$ et numeri primi formae $4m+1$ signum $-$.

290. Possunt hinc etiam innumerabiles aliae signorum conditiones exhiberi, ita ut seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8}\ \text{etc.}$$

summa assignari queat. Cum scilicet sit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}},$$

multiplicetur haec expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=2$; erit

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}}$$

et

$$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}+\text{etc.},$$

ubi binarius signum habet $+$, ternarius $+$, reliqui numeri primi omnes formae $4m-1$ signum $-$, at numeri primi formae $4m+1$ signum $+$; unde pro numeris compositis ratio signorum intelligitur.

Simili modo cum sit

$$\pi = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$; erit

$$\frac{3\pi}{2} = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\text{etc.},$$

ubi binarius habet signum $+$, numeri primi formae $4m-1$ signum $+$ et numeri primi formae $4m+1$ praeter quinarium signum $-$.

291. Possunt etiam innumerabiles huiusmodi series exhiberi, quarum summa sit $=0$. Cum enim sit [§ 277]

$$0 = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\text{etc.},$$

erit

$$0 = \frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde, ut supra vidimus, oritur

$$0 = 1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

ubi omnes numeri primi signum habent $-$ compositorumque numerorum signa regulam multiplicationis sequuntur.

Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$; erit pariter

$$0 = \frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi binarius habet signum $+$, reliqui numeri primi omnes signum $-$.

Simili modo quoque erit

$$0 = \frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{etc.}},$$

unde oritur ista series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \text{etc.},$$

ubi omnes numeri primi praeter 3 et 5 habent signum $-$.

In genere autem notandum est, quoties omnes numeri primi exceptis tantum aliquibus habeant signum $-$, summa seriei fore $= 0$, contra autem, quoties omnes numeri primi exceptis tantum aliquibus habeant signum $+$, tum summam seriei fore infinite magnam.

292. Supra etiam (§ 176) summam dedimus seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} - \text{etc.},$$

si fuerit n numerus impar. Erit ergo

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \text{etc.},$$

quae addita dat

$$B = \left(1 + \frac{1}{2^n}\right)A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \text{etc.}$$

Addatur

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} + \text{etc.};$$

erit

$$C = \left(1 + \frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} - \text{etc.}$$

Subtrahatur

$$\frac{1}{7^n} C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{etc.};$$

erit

$$D = \left(1 - \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{etc.}$$

Ex his tandem fiet

$$A\left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right) \text{ etc.} = 1,$$

ubi numeri primi unitate excedentes multipla senarii habent signum $-$, deficientes autem signum $+$. Eritque

$$A = \frac{2^n}{2^n + 1} \cdot \frac{5^n}{5^n + 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \text{etc.}$$

293. Consideremus casum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$, eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{etc.},$$

ubi in numeratoribus post 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate discrepant suntque omnes per 6 divisibiles. Cum iam sit [§ 277]

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \cdot \text{etc.},$$

erit hac expressione per illam divisa

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{etc.},$$

ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{etc.},$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \text{etc.},$$

quarum haec per illam divisa dat

$$\frac{4}{3}=\frac{6}{4}\cdot\frac{6}{8}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{18}{16}\cdot\frac{18}{20}\cdot\frac{24}{22}\cdot\text{etc.}$$

seu

$$\frac{4}{3}=\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{12}{11}\cdot\text{etc.},$$

ubi singulae fractiones ex numeris primis 5, 7, 11 etc. formantur singulos numeros primos in duas partes unitate differentes dispescendo et partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra [§ 285] vidimus esse

$$\frac{\pi}{4}=\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\text{etc.}$$

seu

$$\frac{\pi}{3}=\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\text{etc.},$$

si superiores $\frac{\pi}{2\sqrt{3}}$ et $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2}=\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{14}{15}\cdot\frac{16}{15}\cdot\text{etc.},$$

$$\frac{2}{\sqrt{3}}=\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\frac{30}{29}\cdot\text{etc.}$$

In priori expressione fractiones formantur ex numeris primis formae $12m+6\pm1$, in posteriore ex numeris primis formae $12m\pm1$, singulos in duas partes unitate discrepantes dispescendo et partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc seriem supra (§ 179) inventam, quae ita progrediebatur

$$\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\text{etc.}=A;$$

erit

$$\frac{1}{3}A=\frac{1}{3}+\frac{1}{9}-\frac{1}{15}-\frac{1}{21}+\frac{1}{27}+\frac{1}{33}-\text{etc.}$$

Subtrahatur [eritque]

$$\left(1-\frac{1}{3}\right)A = 1-\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{13}+\frac{1}{17}+\frac{1}{19}-\text{etc.} = B.$$

Addatur

$$\frac{1}{5}B = \frac{1}{5}-\frac{1}{25}-\frac{1}{35}+\frac{1}{55}-\text{etc.};$$

erit

$$\left(1+\frac{1}{5}\right)B = 1-\frac{1}{7}+\frac{1}{11}-\frac{1}{13}+\frac{1}{17}+\frac{1}{19}-\text{etc.} = C.$$

Sicque progrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right)\text{ etc.} = 1,$$

ubi signa ita se habent, ut numerorum primorum formae $8m+1$ vel $8m+3$ signa sint $-$, numerorum primorum vero formae $8m+5$ vel $8m+7$ signa sint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\text{etc.},$$

ubi omnes denominatores vel divisibiles sunt per 8 vel tantum sunt numeri impariter pares. Cum igitur sit [§ 285]

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.},$$

$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\text{etc.},$$

ergo

$$\frac{\pi\pi}{8} = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{11\cdot 11}{10\cdot 12}\cdot\frac{13\cdot 13}{12\cdot 14}\cdot\text{etc.},$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.},$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{10}{9}\cdot\frac{11}{12}\cdot\text{etc.},$$

quae fractiones formantur ex numeris primis singulos in duas partes unitate discrepantes dispescendo et partes pares (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquae series, quas supra pro expressione arcuum circularium invenimus (§ 179 et sq.) in factores transformari possunt, qui ex numeris primis constituantur. Sicque multae aliae insignes proprietates tam huiusmodi factorum quam serierum infinitarum erui poterunt. Quoniam vero praecipuas hic iam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.

CAPUT XVI

DE PARTITIONE NUMERORUM[1)]

297. Proposita sit ista expressio

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z)(1+x^{\varepsilon}z) \text{ etc.};$$

quae cuiusmodi induat formam, si per multiplicationem evolvatur, inquiramus. Ponamus prodire

$$1+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}$$

atque manifestum est P fore summam potestatum

$$x^{\alpha}+x^{\beta}+x^{\gamma}+x^{\delta}+x^{\varepsilon}+\text{etc.}$$

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1) Confer hoc cum capite L. EULERI Commentationes 158, 191, 394 (indicis ENESTROEMIANI): *Observationes analyticae variae de combinationibus*, Comment. acad. sc. Petrop. **13** (1741/3), 1751, p. 64, *De partitione numerorum*, Novi comment. acad. sc. Petrop. **3** (1750/1), 1753, p. 125, *De partitione numerorum in partes tam numero quam specie datas*, Novi comment. acad. sc. Petrop. **14** (1769): I, 1770, p. 168; *LEONHARDI EULERI Opera omnia*, series I, vol. 2, p. 163 et 254 (vide etiam Prooemium huius voluminis p. XVIII—XX), vol. 3, p. 131 (vide etiam Prooemium huius voluminis p. XIX—XX).

Vide porro epistolam a PH. NAUDÉ minore (1684—1745) ad EULERUM datam a. d. IV. Calendas Septembres 1740; *LEONHARDI EULERI Opera omnia*, series III. Qua epistola NAUDÉ haec duo problemata EULERO proposuerat:

Invenire, quot variis modis datus numerus produci queat ex additione aliquot numerorum integrorum inter se inaequalium, quorum numerus detur.

Invenire, quot variis modis datus numerus m partiri possit in μ partes tam aequales quam inaequales, sive invenire, quot variis modis datus numerus m per additionem μ numerorum integrorum, sive aequalium sive inaequalium, produci queat.

Resolutionem horum problematum EULERUS primum dedit in Commentatione 158 supra laudata. F. R.

Deinde Q est summa factorum ex binis potestatibus diversis seu Q erit aggregatum plurium potestatum ipsius x, quarum exponentes sunt summae duorum terminorum diversorum huius seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta \text{ etc.}$$

Simili modo R erit aggregatum potestatum ipsius x, quarum exponentes sunt summae trium terminorum diversorum. Atque S erit aggregatum potestatum ipsius x, quarum exponentes sunt summae quatuor terminorum diversorum eiusdem seriei $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., et ita porro.

298. Singulae hae potestates ipsius x, quae in valoribus litterarum P, Q, R, S etc. insunt, unitatem pro coefficiente habebunt, siquidem earum exponentes unico modo ex $\alpha, \beta, \gamma, \delta$ etc. formari queant; sin autem eiusdem potestatis exponens pluribus modis possit esse summa duorum, trium pluriumve terminorum seriei $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., tum etiam potestas illa coefficientem habebit, qui unitatem toties in se complectatur. Sic si in valore ipsius Q reperiatur Nx^n, indicio hoc erit numerum n esse N diversis modis summam duorum terminorum diversorum seriei $\alpha, \beta, \gamma, \delta$ etc. Atque si in evolutione factorum propositorum occurrat terminus $Nx^n z^m$, eius coefficiens N indicabit, quot variis modis numerus n possit esse summa m terminorum diversorum seriei $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc.

299. Quodsi ergo productum propositum

$$(1+x^{\alpha}z)(1+x^{\beta}z)(1+x^{\gamma}z)(1+x^{\delta}z) \text{ etc.}$$

per multiplicationem veram evolvatur, ex expressione resultante statim apparebit, quot variis modis datus numerus possit esse summa tot terminorum diversorum seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \text{ etc.},$$

quot quis voluerit. Scilicet, si quaeratur, quot variis modis numerus n possit esse summa m terminorum illius seriei diversorum, in expressione evoluta quaeri debet terminus $x^n z^m$ eiusque coefficiens indicabit numerum quaesitum.

300. Quo haec fiant planiora, sit propositum hoc productum ex factoribus constans infinitis

$$(1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z) \text{ etc.},$$

quod per multiplicationem actualem evolutum dat

$$\begin{aligned}
1 &+ z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.})\\
&+ z^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \text{etc.})\\
&+ z^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \text{etc.})\\
&+ z^4(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \text{etc.})\\
&+ z^5(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \text{etc.})\\
&+ z^6(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \text{etc.})\\
&+ z^7(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \text{etc.})\\
&+ z^8(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \text{etc.})\\
&\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus ex dato terminorum diversorum huius seriei

$$1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8 \ \text{etc.}$$

numero oriri queat. Sic si quaeratur, quot variis modis numerus 35 possit esse summa septem terminorum diversorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc., quaeratur in serie z^7 multiplicante potestas x^{35} eiusque coefficiens 15 indicabit numerum propositum 35 quindecim variis modis esse summam septem terminorum seriei 1, 2, 3, 4, 5, 6, 7, 8 etc.

301. Quodsi autem ponatur $z=1$ et similes potestates ipsius x in unam summam coniiciantur seu, quod eodem redit, si evolvatur haec expressio infinita

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.},$$

quo facto orietur haec series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \text{etc.},$$

ubi quivis coefficiens indicat, quot variis modis exponens potestatis ipsius x coniunctae ex terminis diversis seriei 1, 2, 3, 4, 5, 6, 7 etc. per additionem emergere possit. Sic apparet numerum 8 sex modis per additionem diversorum numerorum produci, qui sunt

$8 = 8$	$8 = 5 + 3$
$8 = 7 + 1$	$8 = 5 + 2 + 1$
$8 = 6 + 2$	$8 = 4 + 3 + 1$

ubi notandum est numerum propositum ipsum simul computari debere, quia numerus terminorum non definitur ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quisque numerus per additionem diversorum numerorum producatur. Conditio autem diversitatis omittetur, si factores illos in denominatorem transponamus. Sit igitur proposita haec expressio

$$\frac{1}{(1-x^{\alpha}z)(1-x^{\beta}z)(1-x^{\gamma}z)(1-x^{\delta}z)(1-x^{\varepsilon}z)\text{ etc.}},$$

quae per divisionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \text{etc.}$$

Atque manifestum est fore P aggregatum potestatum ipsius x, quarum exponentes contineantur in hac serie

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta \text{ etc.}$$

Deinde Q erit aggregatum potestatum ipsius x, quarum exponentes sint summae duorum terminorum huius seriei, sive eorundem sive diversorum. Tum erit R summa potestatum ipsius x, quarum exponentes ex additione trium terminorum illius seriei oriantur, et S summa potestatum, quarum exponentes ex additione quatuor terminorum in illa serie contentorum formantur, et ita porro.

303. Si igitur tota expressio per singulos terminos explicetur et termini similes coniunctim exprimantur, intelligetur, quot variis modis propositus

numerus n per additionem m terminorum, sive diversorum sive non diversorum, seriei

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta \text{ etc.}$$

produci queat. Quaeratur scilicet in expressione evoluta terminus $x^n z^m$ eiusque coefficiens, qui sit N, ita ut totus terminus sit $= Nx^n z^m$, atque coefficiens N indicabit, quot variis modis numerus n per additionem m terminorum in serie α, β, γ, δ, ε etc. contentorum produci queat. Hoc igitur pacto quaestio priori, quam ante sumus contemplati, similis resolvetur.

304. Accommodemus haec ad casum inprimis notatu dignum sitque proposita haec expressio

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z) \text{ etc.}},$$

quae per divisionem evoluta dabit

$$\begin{aligned}
1 &+ z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.})\\
&+ z^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \text{etc.})\\
&+ z^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \text{etc.})\\
&+ z^4(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \text{etc.})\\
&+ z^5(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \text{etc.})\\
&+ z^6(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \text{etc.})\\
&+ z^7(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \text{etc.})\\
&+ z^8(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \text{etc.})\\
&\qquad\text{etc.}
\end{aligned}$$

Ex his ergo seriebus statim definire licet, quot variis modis propositus numerus per additionem ex dato terminorum huius seriei

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \text{ etc.}$$

nume o produci queat. Sic si quaeratur, quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cuius coefficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z = 1$ atque similes potestates ipsius x coniunctim exprimantur, haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{ etc.}}$$

evolvetur in hanc seriem

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \text{etc.};$$

in qua quilibet coefficiens indicat, quot variis modis exponens potestatis adiunctae per additionem produci queat ex numeris integris, sive aequalibus sive inaequalibus. Scilicet ex termino $11x^6$ cognoscitur numerum 6 undecim modis per additionem numerorum integrorum produci posse, qui sunt

$6 = 6$	$6 = 3 + 1 + 1 + 1$
$6 = 5 + 1$	$6 = 2 + 2 + 2$
$6 = 4 + 2$	$6 = 2 + 2 + 1 + 1$
$6 = 4 + 1 + 1$	$6 = 2 + 1 + 1 + 1 + 1$
$6 = 3 + 3$	$6 = 1 + 1 + 1 + 1 + 1 + 1$
$6 = 3 + 2 + 1$	

ubi quoque notari debet ipsum numerum propositum, cum in serie numerorum 1, 2, 3, 4, 5, 6 etc. proposita contineatur, unum modum praebere.

306. His in genere expositis diligentius inquiramus in modum hanc compositionum multitudinem inveniendi. Ac primo quidem consideremus eam ex numeris integris compositionem, in qua numeri tantum diversi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem proposita haec expressio

$$Z = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z)\text{ etc.},$$

quae evoluta et secundum potestates ipsius z digesta praebeat

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \text{etc.},$$

ubi methodus desideratur has ipsius x functiones P, Q, R, S, T etc. expedite inveniendi; hoc enim pacto quaestioni propositae convenientissime satisfiet.

307. Patet autem, si loco z ponatur xz, prodire

$$(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z) \text{ etc.} = \frac{Z}{1+xz}.$$

Ergo posito xz loco z valor producti, qui erat Z, abibit in $\frac{Z}{1+xz}$; sicque, cum sit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \text{etc.},$$

erit

$$\frac{Z}{1+xz} = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.}$$

Multiplicetur ergo actu per $1 + xz$ atque prodibit

$$\begin{aligned} Z = 1 &+ Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.} \\ &+ \quad xz + Px^2z^2 + Qx^3z^3 + Rx^4z^4 + \text{etc.}, \end{aligned}$$

qui valor ipsius Z cum superiori comparatus dabit

$$P = \frac{x}{1-x}, \quad Q = \frac{Px^2}{1-x^2}, \quad R = \frac{Qx^3}{1-x^3}, \quad S = \frac{Rx^4}{1-x^4} \quad \text{etc.}$$

Sequentes ergo pro P, Q, R, S etc. obtinentur valores:

$$P = \frac{x}{1-x},$$

$$Q = \frac{x^3}{(1-x)(1-x^2)},$$

$$R = \frac{x^6}{(1-x)(1-x^2)(1-x^3)},$$

$$S = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)},$$

$$T = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

etc.

308. Sic igitur seorsim unamquamque seriem potestatum ipsius x exhibere possumus, ex qua definire licet, quot variis modis propositus numerus ex dato partium integrarum numero per additionem formari possit. Manifestum autem

porro est has singulas series esse recurrentes, quia ex evolutione functionis fractae ipsius x nascuntur. Prima scilicet expressio

$$P=\frac{x}{1-x}$$

dat seriem geometricam

$$x+x^2+x^3+x^4+x^5+x^6+x^7+\text{etc.},$$

ex qua quidem manifestum est quemvis numerum semel in serie numerorum integrorum contineri.

309. Expressio secunda

$$\frac{x^3}{(1-x)(1-xx)}$$

dat hanc seriem

$$x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+4x^{10}+\text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot modis exponens ipsius x in duas partes inaequales dispertiri possit. Sic terminus $4x^9$ indicat numerum 9 quatuor modis in duas partes inaequales secari posse. Quodsi hanc seriem per x^3 dividamus, prodibit series, quam praebet ista fractio

$$\frac{1}{(1-x)(1-x^2)},$$

quae erit

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+\text{etc.},$$

cuius terminus generalis sit $=Nx^n$; atque ex genesi huius seriei intelligitur coefficientem N indicare, quot variis modis exponens n ex numeris 1 et 2 per additionem nasci queat. Cum igitur prioris seriei terminus generalis sit $=Nx^{n+3}$, deducitur hinc istud theorema:

Quot variis modis numerus n per additionem ex numeris 1 *et* 2 *produci potest, totidem variis modis numerus $n+3$ in duas partes inaequales secari poterit.*

310. Expressio tertia

$$\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$

in seriem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicat, quot variis modis exponens potestatis x adiunctae in tres partes inaequales dispertiri possit. Quodsi autem haec fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}$$

evolvatur, prodibit haec series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \text{etc.};$$

cuius terminus generalis si ponatur $= Nx^n$, coefficiens N indicabit, quot variis modis numerus n ex numeris 1, 2, 3 per additionem produci possit. Cum igitur prioris seriei terminus generalis sit Nx^{n+6}, sequetur hinc istud theorema:

Quot variis modis numerus n per additionem ex numeris 1, 2, 3 *produci potest, totidem variis modis numerus $n + 6$ in tres partes inaequales secari poterit.*

311. Expressio quarta

$$\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

in seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + \text{etc.},$$

in qua cuiusvis termini coefficiens indicabit, quot variis modis exponens potestatis x adiunctae in quatuor partes inaequales dispertiri possit. Quodsi autem haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

evolvatur, prodibit superior series per x^{10} divisa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \text{etc.},$$

cuius terminum generalem ponamus $= Nx^n$; atque hinc patebit coefficientem N indicare, quot variis modis numerus n per additionem oriri possit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris seriei terminus generalis futurus sit $= Nx^{n+10}$, deducitur hoc theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, *totidem variis modis numerus* $n+10$ *in quatuor partes inaequales secari poterit.*

312. Generaliter ergo, si haec expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur eiusque terminus generalis fuerit

$$= Nx^n,$$

coefficiens N indicabit, quot variis modis numerus n per additionem produci possit ex his numeris 1, 2, 3, 4, ... m. Quodsi autem haec expressio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

in seriem evolvatur, erit eius terminus generalis

$$= Nx^{n+\frac{m(m+1)}{2}}$$

atque hic coefficiens N indicat, quot variis modis numerus $n+\frac{m(m+1)}{1\cdot 2}$ in m partes inaequales secari possit, unde hoc habetur theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, ... *m, totidem modis numerus* $n+\frac{m(m+1)}{1\cdot 2}$ *in m partes inaequales secari poterit.*

313. Ex posita partitione numerorum in partes inaequales perpendamus quoque partitionem in partes, ubi aequalitas partium non excluditur; quae partitio ex hac expressione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}}.$$

Ponamus evolutione per divisionem instituta prodire

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \text{etc.}$$

Perspicuum autem est, si loco z ponatur xz, prodire

$$\frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{ etc.}} = (1-xz)Z.$$

Facta ergo in serie evoluta eadem mutatione fiet

$$(1-xz)Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \text{etc.}$$

Multiplicetur ergo superior series pariter per $(1-xz)$ eritque

$$\begin{aligned}(1-xz)Z = 1 + Pz + {} & Qz^2 + Rz^3 + Sz^4 + \text{etc.}\\ - xz - {} & Pxz^2 - Qxz^3 - Rxz^4 - \text{etc.}\end{aligned}$$

Comparatione ergo instituta orietur

$$P = \frac{x}{1-x}, \quad Q = \frac{Px}{1-x^2}, \quad R = \frac{Qx}{1-x^3}, \quad S = \frac{Rx}{1-x^4} \quad \text{etc.},$$

unde pro P, Q, R, S etc. sequentes valores proveniunt:

$$P = \frac{x}{1-x},$$

$$Q = \frac{x^2}{(1-x)(1-x^2)},$$

$$R = \frac{x^3}{(1-x)(1-x^2)(1-x^3)},$$

$$S = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$

etc.

314. Expressiones istae a superioribus aliter non discrepant, nisi quod numeratores hic minores habeant exponentes quam casu praecedente. Atque hanc ob rem series, quae per evolutionem nascuntur, ratione coefficientium omnino convenient, quae convenientia iam ex comparatione § 300 et 304

perspicitur, nunc vero demum eius ratio intelligitur. Hinc ergo omnino similia theoremata consequentur, quae sunt:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, *totidem modis numerus* $n+2$ *in duas partes dispertiri poterit.*

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, *totidem modis numerus* $n+3$ *in tres partes dispertiri poterit.*

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, 4, *totidem modis numerus* $n+4$ *in quatuor partes dispertiri poterit.*

Atque generaliter habebitur hoc theorema:

Quot variis modis numerus n per additionem produci potest ex numeris 1, 2, 3, ... m, *totidem modis numerus* $n+m$ *in m partes dispertiri poterit.*

315. Sive ergo quaeratur, quot modis datus numerus in m partes inaequales, sive in m partes aequalibus non exclusis dispertiri possit, utraque quaestio resolvetur, si cognoscatur, quot modis quisque numerus per additionem produci possit ex numeris 1, 2, 3, 4, ... m, quemadmodum hoc patebit ex sequentibus theorematis, quae ex superioribus sunt derivata:

Numerus n tot modis in m partes inaequales dispertiri potest, quot modis numerus $n-\frac{m(m+1)}{2}$ *per additionem produci potest ex numeris* 1, 2, 3, 4, ... m.

Numerus n tot modis in m partes, sive aequales sive inaequales, dispertiri potest, quot modis numerus $n-m$ *per additionem produci potest ex numeris* 1, 2, 3, ... m.

Hinc porro sequuntur haec theoremata:

Numerus n totidem modis in m partes inaequales secari potest, quot modis numerus $n-\frac{m(m-1)}{2}$ *in m partes, sive aequales sive inaequales, dispertitur.*

Numerus n totidem modis in m partes, sive inaequales sive aequales, secari potest, quot modis numerus $n+\frac{m(m-1)}{2}$ *in m partes inaequales dispertiri potest.*

316. Per formationem autem serierum recurrentium inveniri poterit, quot variis modis datus numerus n per additionem produci possit ex numeris 1, 2, 3, ... m. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

atque series recurrens continuari debebit usque ad terminum Nx^n, cuius coefficiens N indicabit, quot modis numerus n per additionem produci possit ex numeris 1, 2, 3, 4, ... m. At vero hic solvendi modus non parum habebit difficultatis, si numeri m et n sint modice magni; scala enim relationis, quam praebet denominator per multiplicationem evolutus, ex pluribus terminis constat, unde operosum erit seriem ad plures terminos continuare.

317. Haec autem disquisitio minus erit molesta, si casus simpliciores primum expediantur; ex his enim facile erit ad casus magis compositos progredi. Sit seriei, quae ex hac fractione oritur

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)},$$

terminus generalis $= Nx^n$; at seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

ortae terminus generalis sit Mx^n, ubi coefficiens M indicabit, quot variis modis numerus $n-m$ per additionem produci possit ex numeris 1, 2, 3, ... m. Subtrahatur posterior expressio a priori ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^{m-1})}$$

atque manifestum est seriei hinc ortae terminum generalem futurum esse $(N-M)x^n$; quare coefficiens $N-M$ indicabit, quot variis modis numerus n per additionem produci possit ex numeris 1, 2, 3, ... $m-1$.

318. Hinc ergo sequentem regulam nanciscimur:

Sit L numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... $m-1$,

sit M numerus modorum, quibus numerus $n-m$ per additionem produci potest ex numeris 1, 2, 3, ... m,

sitque N numerus modorum, quibus numerus n per additionem produci potest ex numeris 1, 2, 3, ... m;

his positis erit, ut vidimus,

$$L = N - M$$

ideoque

$$N = L + M.$$

Quodsi ergo iam invenerimus, quot variis modis numeri n et $n - m$ per additionem produci queant, ille ex numeris $1, 2, 3, \ldots m - 1$, hic vero ex numeris $1, 2, 3, \ldots m$, hinc addendo cognoscemus, quot variis modis numerus n per additionem produci queat ex numeris $1, 2, 3, \ldots m$. Ope huius theorematis a casibus simplicioribus, qui nihil habent difficultatis, continuo ad magis compositos progredi licebit hocque modo tabula hic annexa[1]) est computata, cuius usus ita se habet:

Si quaeratur, quot variis modis numerus 50 in 7 partes inaequales dispertiri possit, sumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem suprema numerus romanus VII; atque numerus in angulo positus 522 indicabit modorum numerum quaesitum.

Sin autem quaeratur, quot variis modis numerus 50 in 7 partes, sive aequales sive inaequales, dispertiri possit, in prima columna verticali sumatur numerus $50 - 7 = 43$, cui in columna septima respondebit numerus quaesitus 8946.

1) Confer tabulam correspondentem, quae continetur in Commentatione 191 nota 1 pag. 313 laudata. Confer imprimis paragraphum 33 huius Commentationis 191, ubi expositum est, quomodo haec tabula per solam continuam additionem ratione satis perspicua construi possit. F. R.

TABULA

n	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	136	164	186	201	212	219
17	1	9	33	72	119	163	201	230	252	267	278
18	1	10	37	84	141	199	248	288	318	340	355
19	1	10	40	94	164	235	300	352	393	423	445
20	1	11	44	108	192	282	364	434	488	530	560
21	1	11	48	120	221	331	436	525	598	653	695
22	1	12	52	136	255	391	522	638	732	807	863
23	1	12	56	150	291	454	618	764	887	984	1060
24	1	13	61	169	333	532	733	919	1076	1204	1303
25	1	13	65	185	377	612	860	1090	1291	1455	1586
26	1	14	70	206	427	709	1009	1297	1549	1761	1930
27	1	14	75	225	480	811	1175	1527	1845	2112	2331
28	1	15	80	249	540	931	1367	1801	2194	2534	2812
29	1	15	85	270	603	1057	1579	2104	2592	3015	3370
30	1	16	91	297	674	1206	1824	2462	3060	3590	4035

n	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
31	1	16	96	321	748	1360	2093	2857	3589	4242	4802
32	1	17	102	351	831	1540	2400	3319	4206	5013	5708[1])
33	1	17	108	378	918	1729	2738	3828	4904	5888	6751
34	1	18	114	411	1014	1945	3120	4417	5708	6912	7972
35	1	18	120	441	1115	2172	3539	5066	6615	8070	9373
36	1	19	127	478	1226	2432	4011	5812	7657	9418	11004
37	1	19	133	511	1342	2702	4526	6630	8824	10936	12866
38	1	20	140	551	1469	3009	5102	7564	10156	12690	15021
39	1	20	147	588	1602	3331	5731	8588	11648	14663	17475
40	1	21	154	632	1747	3692	6430	9749	13338	16928	20298
41	1	21	161	672	1898	4070	7190	11018	15224	19466	23501
42	1	22	169	720	2062	4494	8033	12450	17354	22367	27169
43	1	22	176	764	2233	4935	8946	14012	19720	25608	31316
44	1	23	184	816	2418	5427	9953	15765	22380	29292	36043
45	1	23	192	864	2611	5942	11044	17674	25331	33401	41373
46	1	24	200	920	2818	6510	12241	19805	28629	38047	47420
47	1	24	208	972	3034	7104	13534	22122	32278	43214	54218
48	1	25	217	1033	3266	7760	14950	24699	36347	49037	61903
49	1	25	225	1089	3507	8442	16475	27493	40831	55494	70515
50	1	26	234	1154	3765	9192	18138	30588	45812	62740	80215
51	1	26	243	1215	4033	9975	19928	33940	51294	70760	91058
52	1	27	252	1285	4319	10829	21873	37638	57358	79725	103226
53	1	27	261	1350	4616	11720	23961	41635	64015	89623	116792
54	1	28	271	1425	4932	12692	26226	46031	71362	100654	131970
55	1	28	280	1495	5260	13702	28652	50774	79403	112804	148847
56	1	29	290	1575	5608	14800	31275	55974	88252	126299	167672
57	1	29	300	1650	5969	15944	34082	61575	97922	141136	188556
58	1	30	310	1735	6351	17180	37108	67696	108527	157564	211782
59	1	30	320	1815	6747	18467	40340	74280	120092	175586	237489
60	1	31	331	1906	7166	19858	43819	81457	132751	195491	266006
61	1	31	341	1991	7599	21301	47527	89162	146520	217280	297495
62	1	32	352	2087	8056	22856	51508	97539	161554	241279	332337
63	1	32	363	2178	8529	24473	55748	106522	177884	267507	370733
64	1	33	374	2280	9027	26207	60289	116263	195666	296320	413112
65	1	33	385	2376	9542	28009	65117	126692	214944	327748	459718
66	1	34	397	2484	10083	29941	70281	137977	235899	362198	511045
67	1	34	408	2586	10642	31943	75762	150042	258569	399705	567377
68	1	35	420	2700	11229	34085	81612	163069	283161	440725	629281
69	1	35	432	2808	11835	36308	87816	176978	309729	485315	697097

1) Editio princeps: 5788. Correxit F. R.

319. Series huius tabulae verticales, etsi sunt recurrentes, tamen ingentem habent connexionem cum numeris naturalibus, trigonalibus, pyramidalibus et sequentibus, quam paucis exponere operae pretium erit. Quoniam enim ex fractione

$$\frac{1}{(1-x)(1-xx)}$$

oritur series

$$1+x+2x^2+2x^3+3x^4+3x^5+\text{etc.}$$

ac proinde ex fractione

$$\frac{x}{(1-x)(1-xx)}$$

haec

$$x+x^2+2x^3+2x^4+3x^5+3x^6+\text{etc.},$$

si duae hae series addantur, nascitur ista

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+\text{etc.},$$

quae per divisionem oritur ex fractione

$$\frac{1+x}{(1-x)(1-xx)}=\frac{1}{(1-x)^2};$$

unde patet seriei postremae terminos numericos seriem numerorum naturalium constituere. Hinc ex serie tabulae secunda addendo binos terminos proveniet series numerorum naturalium posito $x=1$:

$$1+1+2+2+3+3+4+4+5+\ \ 5+\ \ 6+\ \ 6+\text{etc.}$$
$$1+2+3+4+5+6+7+8+9+10+11+12+\text{etc.}$$

Vicissim ergo ex serie numerorum naturalium superior invenitur subtrahendo quemque terminum seriei superioris a termino inferioris sequente.

320. Series verticalis tertia oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)}.$$

Cum autem sit

$$\frac{1}{(1-x)^3}=\frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)},$$

manifestum est, si primo seriei illius terni termini addantur, tum bini huius novae seriei denuo addantur, prodire debere numeros trigonales; id quod ex schemate sequente apparebit:

$$1+1+2+3+4+5+7+8+10+12+14+16+19+\text{etc.}$$
$$1+2+4+6+9+12+16+20+25+30+36+42+49+\text{etc.}$$
$$1+3+6+10+15+21+28+36+45+55+66+78+91+\text{etc.}$$

Vicissim autem apparet, quomodo ex serie trigonalium erui debeat series superior.

321. Simili modo, quia series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)},$$

erit

$$\frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}=\frac{1}{(1-x)^4}.$$

Si in serie quarta primum quaterni termini addantur, tum in serie resultante terni, denique in hac bini, prodibit series numerorum pyramidalium, uti ex sequenti calculo videre licet:

$$1+1+2+3+5+6+9+11+15+18+23+27+\text{etc.}$$
$$1+2+4+7+11+16+23+31+41+53+67+83+\text{etc.}$$
$$1+3+7+13+22+34+50+70+95+125+161+203+\text{etc.}$$
$$1+4+10+20+35+56+84+120+165+220+286+364+\text{etc.}$$

Simili autem modo series quinta deducet ad numeros pyramidales secundi ordinis, sexta ad tertii ordinis, et ita porro.

322. Vicissim igitur ex numeris figuratis illae ipsae series, quae in tabula occurrunt, formari poterunt per operationes, quae ex inspectione calculi sequentis sponte elucebunt.

$1+$	$2+$	$3+$	$4+$	$5+$	$6+$	$7+$	$8+$	$9+$	$10+$	etc.	
$1+$	$1+$	$2+$	$2+$	$3+$	$3+$	$4+$	$4+$	$5+$	$5+$	etc.	II
$1+$	$3+$	$6+$	$10+$	$15+$	$21+$	$28+$	$36+$	$45+$	$55+$	etc.	
$1+$	$2+$	$4+$	$6+$	$9+$	$12+$	$16+$	$20+$	$25+$	$30+$	etc.	
$1+$	$1+$	$2+$	$3+$	$4+$	$5+$	$7+$	$8+$	$10+$	$12+$	etc.	III
$1+$	$4+$	$10+$	$20+$	$35+$	$56+$	$84+$	$120+$	$165+$	$220+$	etc.	
$1+$	$3+$	$7+$	$13+$	$22+$	$34+$	$50+$	$70+$	$95+$	$125+$	etc.	
$1+$	$2+$	$4+$	$7+$	$11+$	$16+$	$23+$	$31+$	$41+$	$53+$	etc.	
$1+$	$1+$	$2+$	$3+$	$5+$	$6+$	$9+$	$11+$	$15+$	$18+$	etc.	IV
$1+$	$5+$	$15+$	$35+$	$70+$	$126+$	$210+$	$330+$	$495+$	$715+$	etc.	
$1+$	$4+$	$11+$	$24+$	$46+$	$80+$	$130+$	$200+$	$295+$	$420+$	etc.	
$1+$	$3+$	$7+$	$14+$	$25+$	$41+$	$64+$	$95+$	$136+$	$189+$	etc.	
$1+$	$2+$	$4+$	$7+$	$12+$	$18+$	$27+$	$38+$	$53+$	$71+$	etc.	
$1+$	$1+$	$2+$	$3+$	$5+$	$7+$	$10+$	$13+$	$18+$	$23+$	etc.	V

etc.

In his ordinibus primae series sunt numeri figurati, unde subtrahendo quemvis terminum seriei secundae a termino primae sequente formatur series secunda. Tum seriei tertiae bini termini coniunctim subtrahantur a termino sequente seriei secundae sicque oritur series tertia. Hocque modo subtrahendo ulterius summam trium, quatuor et ita porro terminorum a termino superioris seriei sequente formabuntur reliquae series, donec perveniatur ad seriem, quae incipit ab $1+1+2+$ etc., haecque erit series in tabula exhibita.

323. Series verticales tabulae omnes similiter incipiunt continuoque plures habent terminos communes; ex quo intelligitur in infinitum has series inter se fore congruentes. Prodibit autem series, quae oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}};$$

quae cum sit recurrens, primum denominator spectari debet, ut hinc scala

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relationis habeatur. Quodsi autem factores denominatoris continuo in se multiplicentur, prodibit

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.};$$

quae series si attentius consideretur, aliae potestates ipsius x adesse non deprehenduntur, nisi quarum exponentes continenatur in hac formula $\frac{3nn \pm n}{2}$, atque si n sit numerus impar, potestates erunt negativae, affirmativae autem, si n fuerit numerus par.[1])

324. Cum igitur scala relationis sit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0 \text{ etc.},$$

series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7) \text{ etc.}}$$

oriunda erit haec

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1) Haec celebris series apud Eulerum primum in Commentatione 158 (exhib. 6. Apr. 1741) nota 1 p. 313 laudata invenitur. Evolutionis demonstrationem Eulerus dedit in Commentatione 244 (indicis Enestroemiani): *Demonstratio theorematis circa ordinem in summis divisorum observatum*, Novi comment. acad. sc. Petrop. **5** (1754/5), 1760, p. 75; *Leonhardi Euleri Opera omnia*, series I, vol. 2, p. 390.

Series illa eo magis digna est, quae consideretur, quod iam exemplum praebet illarum functionum, quas centum fere abhinc annos C. G. J. Jacobi ut fundamenta theoriae functionum ellipticarum in analysin introduxit et hoc charactere ϑ significavit; confer C. G. J. Jacobi, *Elementarer Beweis einer merkwürdigen analytischen Formel etc.*, Journal f. d. reine u. angew. Mathem. **21**, 1840, p. 13; *Ges. Werke* **6**, 1891, p. 281.

Formula Euleriana

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - \cdots$$

invenitur etiam apud Jacobi in libro, qui inscribitur *Fundamenta nova theoriae functionum ellipticarum*, Regiomonti 1829, § 66, formula (6), *Ges. Werke* **1**, 1881, p. 237.

Notandum autem est series, quarum exponentes seriem arithmeticam secundi ordinis formant, iam sexaginta annis ante Eulerum inveniri apud Jac. Bernoulli et G. Leibniz; confer G. Eneström, *Jakob Bernoulli und die Jacobischen Thetafunktionen*, Biblioth. Mathem. 9_3, 1908—1909, p. 206. De cetero vide notam adiectam ad paragraphum 36 illius Commentationis 158; *Leonhardi Euleri Opera omnia*, series I, vol. 2, p. 191. F. R.

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11}$$
$$+ 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20}$$
$$+ 792x^{21} + 1002x^{22} + 1255x^{23} + 1575x^{24} + \text{etc.}$$ [1])

In hac ergo serie coefficiens quisque indicat, quot variis modis exponens ipsius x per additionem ex numeris integris oriri queat. Sic numerus 7 quindecim modis per additionem oriri potest:

$7 = 7$	$7 = 4 + 2 + 1$	$7 = 3 + 1 + 1 + 1 + 1$
$7 = 6 + 1$	$7 = 4 + 1 + 1 + 1$	$7 = 2 + 2 + 2 + 1$
$7 = 5 + 2$	$7 = 3 + 3 + 1$	$7 = 2 + 2 + 1 + 1 + 1$
$7 = 5 + 1 + 1$	$7 = 3 + 2 + 2$	$7 = 2 + 1 + 1 + 1 + 1 + 1$
$7 = 4 + 3$	$7 = 3 + 2 + 1 + 1$	$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$

325. Quodsi autem hoc productum

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \text{ etc.}$$

evolvatur, sequens prodibit series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + 10x^{10} + \text{etc.},$$

in qua quisque coefficiens indicat, quot variis modis exponens ipsius x per additionem numerorum inaequalium oriri possit. Sic numerus 9 octo variis modis per additionem ex numeris inaequalibus formari potest:

$9 = 9$	$9 = 6 + 2 + 1$
$9 = 8 + 1$	$9 = 5 + 4$
$9 = 7 + 2$	$9 = 5 + 3 + 1$
$9 = 6 + 3$	$9 = 4 + 3 + 2$

1) Editio princeps: $1 + x + 2x^2 + \cdots + 1002x^{22} + 1250x^{23} + 1570x^{24}$ etc.
Correxit F. R.

326. Ut comparationem inter has formas instituamus, sit

$$P = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6) \text{ etc.}$$

et

$$Q = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) \text{ etc.};$$

erit

$$PQ = (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12}) \text{ etc.};$$

qui factores cum omnes in P contineantur, dividatur P per PQ; erit

$$\frac{1}{Q} = (1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \text{ etc.}$$

ideoque

$$Q = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9) \text{ etc.}};$$

quae fractio si evolvatur, prodibit series, in qua quisque coefficiens indicabit, quot variis modis exponens ipsius x per additionem ex numeris imparibus produci possit. Cum igitur haec expressio aequalis sit illi, quam in paragrapho praecedente contemplati sumus, sequitur hinc istud theorema:

Quot modis datus numerus per additionem formari potest ex omnibus numeris integris inter se inaequalibus, totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus, sive aequalibus sive inaequalibus.

327. Cum igitur, ut ante vidimus, sit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \text{etc.},$$

erit scribendo xx loco x

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \text{etc.}$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{etc.}};$$

erit ergo series Q pariter recurrens atque ex serie $\frac{1}{P}$ oritur hanc per

$$1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - \text{etc.}$$

multiplicando. Nempe, cum sit ex § 324

$$\frac{1}{P} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.},$$

si is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{etc.},$$

fiet

$$\begin{aligned} &1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{etc.} \\ &\qquad - x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{etc.} \\ &\qquad\qquad - x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{etc.} \end{aligned}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{etc.} = Q.$$

Hinc ergo, si formatio numerorum per additionem numerorum, sive aequalium sive inaequalium, constet, deducetur formatio numerorum per additionem numerorum inaequalium hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Restant in hoc genere casus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoscenda. Consideretur nempe haec expressio

$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32}) \text{ etc.},$$

in qua exponentes ipsius x in ratione dupla progrediuntur. Haec expressio si evolvatur, reperietur quidem haec series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.},$$

quoniam vero dubium esse potest, utrum haec series in infinitum hac lege geometrica progrediatur, hanc ipsam seriem investigemus. Sit igitur

$$P = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \text{ etc.}$$

ac ponatur series per evolutionem oriunda

$$P = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \text{etc.}$$

Patet autem, si loco x scribatur xx, tum prodire productum

$$(1+xx)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) \text{ etc.} = \frac{P}{1+x}.$$

Facta ergo in serie eadem substitutione erit

$$\frac{P}{1+x} = 1 + \alpha x^2 + \beta x^4 + \gamma x^6 + \delta x^8 + \varepsilon x^{10} + \zeta x^{12} + \text{etc.}$$

Multiplicetur ergo per $1+x$ eritque

$$P = 1 + x + \alpha x^2 + \alpha x^3 + \beta x^4 + \beta x^5 + \gamma x^6 + \gamma x^7 + \delta x^8 + \delta x^9 + \text{etc.};$$

qui valor ipsius P si cum superiori comparetur, habebitur

$$\alpha = 1, \quad \beta = \alpha, \quad \gamma = \alpha, \quad \delta = \beta, \quad \varepsilon = \beta, \quad \zeta = \gamma, \quad \eta = \gamma \quad \text{etc.};$$

erunt ergo omnes coefficientes $= 1$ ideoque productum propositum P evolutum dabit seriem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

329. Cum igitur hic omnes ipsius x potestates singulaeque semel occurrant, ex forma producti

$$(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32}) \text{ etc.}$$

sequitur omnem numerum integrum ex terminis progressionis geometricae duplae

$$1, \ 2, \ 4, \ 8, \ 16, \ 32 \ \text{etc.}$$

diversis per additionem formari posse hocque unico modo.

Nota est haec proprietas in praxi ponderandi. Si enim habeantur pondera 1, 2, 4, 8, 16, 32 etc. librarum, his solis ponderibus omnia onera ponderari poterunt, nisi partes librae requirant.[1]) Sic his decem ponderibus, nempe

1) Id quod iam docuit LEONARDO PISANO in libro, qui inscribitur *Liber abbaci* (1202), ed. B. BONCOMPAGNI, Roma 1857, p. 297. Vide porro M. STIFEL (1486—1567), *Die Coss CHRISTOFFS RUDOLFFS*, Königsperg 1553, fol. 11r, et FR. v. SCHOOTEN (1615—1660), *Exercitationum mathematicarum libri quinque*, Lugd. Batav. 1657, lib. V sectio VIII, p. 410. Confer denique G. ENESTRÖM, *Über die ältere Geschichte der Zerfällung ganzer Zahlen in Summen kleinerer Zahlen,* Biblioth. Mathem. 13_3, 1912—1913, p. 352. F. R.

1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔,

omnia pondera usque ad 1024 ℔ librari possunt, et si unum pondus 1024 ℔ addatur, omnibus oneribus usque ad 2048 ℔ ponderandis sufficient.

330. Ostendi autem insuper solet in praxi ponderandi paucioribus ponderibus, quae scilicet in ratione geometrica tripla progrediantur, nempe

$$1, \ 3, \ 9, \ 27, \ 81 \text{ etc.}$$

librarum, pariter omnia onera ponderari posse, nisi opus sit fractionibus. In hac autem praxi pondera non solum uni lanci, sed ambabus, uti necessitas exigit, imponi debent.[1] Nititur ergo ista praxis hoc fundamento, quod ex terminis progressionis geometricae triplae 1, 3, 9, 27, 81 etc. diversis semper sumendis per additionem ac subtractionem omnes omnino numeri produci queant; erit scilicet

$1 = 1$	$5 = 9 - 3 - 1$	$9 = 9$
$2 = 3 - 1$	$6 = 9 - 3$	$10 = 9 + 1$
$3 = 3$	$7 = 9 - 3 + 1$	$11 = 9 + 3 - 1$
$4 = 3 + 1$	$8 = 9 - 1$	$12 = 9 + 3$

etc.

331. Ad hanc veritatem ostendendam considero hoc productum infinitum

$$(x^{-1} + 1 + x^{1})(x^{-3} + 1 + x^{3})(x^{-9} + 1 + x^{9})(x^{-27} + 1 + x^{27}) \text{ etc.} = P,$$

quod evolutum alias non dabit potestates ipsius x, nisi quarum exponentes formari possint ex numeris 1, 3, 9, 27, 81 etc., sive addendo sive subtrahendo. Num vero omnes potestates prodeant singulaeque semel, sic exploro. Sit

$$P = \text{etc.} + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^{1} + \beta x^{2} + \gamma x^{3} + \delta x^{4} + \varepsilon x^{5} + \text{etc.}$$

Manifestum vero est, si x^3 loco x scribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^{1}} = \text{etc.} + bx^{-6} + ax^{-3} + 1 + \alpha x^{3} + \beta x^{6} + \gamma x^{9} + \text{etc.}$$

1) Etiam hoc invenitur apud LEONARDO PISANO; vide notam praecedentem. F. R.

Hinc igitur reperitur

$$P = \text{etc.} + ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 + \beta x^5 + \beta x^6 + \beta x^7 + \text{etc.},$$

quae expressio cum assumpta comparata dabit

$$\alpha = 1, \quad \beta = \alpha, \quad \gamma = \alpha, \quad \delta = \alpha, \quad \varepsilon = \beta, \quad \zeta = \beta \quad \text{etc.}$$

et

$$a = 1, \quad b = a, \quad c = a, \quad d = a, \quad e = b \quad \text{etc.}$$

Hinc itaque erit

$$\begin{aligned} P = 1 + x &+ x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{etc.} \\ &+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \text{etc.}, \end{aligned}$$

unde patet omnes ipsius x potestates, tam affirmativas quam negativas, hic occurrere atque adeo omnes numeros ex terminis progressionis geometricae triplae vel addendo vel subtrahendo formari posse et unumquemque numerum unico tantum modo.

CAPUT XVII

DE USU SERIERUM RECURRENTIUM IN RADICIBUS AEQUATIONUM INDAGANDIS

332. Indicavit Vir Celeb. DANIEL BERNOULLI[1]) insignem usum serierum recurrentium in investigandis radicibus aequationum cuiusvis gradus in Comment. Acad. Petropol. Tomo III, ubi ostendit, quemadmodum cuiusque aequationis algebraicae, quotcunque fuerit dimensionum, valores radicum veris proximi ope serierum recurrentium assignari queant. Quae inventio cum saepenumero maximam afferat utilitatem, eam hic diligentius explicare constitui, ut intelligatur, quibus casibus adhiberi possit. Interdum enim praeter expectationem evenit, ut nulla aequationis radix ope huius methodi cognosci queat. Quocirca, ut vis huius methodi clarius perspiciatur, ex proprietatibus serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis series recurrens ex evolutione cuiusdam fractionis rationalis oritur, sit ista fractio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}},$$

unde oriatur sequens series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \text{etc.},$$

1) D. BERNOULLI, *Observationes de seriebus recurrentibus*, Comment. acad. sc. Petrop. 3 (1728), 1732, p. 85. F. R.

cuius coefficientes A, B, C, D etc. ita determinantur, ut sit

$$\begin{aligned} A &= a, \\ B &= \alpha A + b, \\ C &= \alpha B + \beta A + c, \\ D &= \alpha C + \beta B + \gamma A + d, \\ E &= \alpha D + \beta C + \gamma B + \delta A + e \\ &\text{etc.} \end{aligned}$$

Terminus autem generalis seu coefficiens potestatis z^n invenitur ex resolutione fractionis propositae in fractiones simplices, quarum denominatores sint factores denominatoris

$$1 - \alpha z - \beta zz - \gamma z^3 - \text{etc.},$$

uti cap. XIII est ostensum.

334. Forma autem termini generalis potissimum pendet ab indole factorum simplicium denominatoris, utrum sint reales an imaginarii, et utrum sint inter se inaequales an eorum bini pluresve aequales. Quos varios casus ut ordine percurramus, ponamus primum omnes denominatoris factores simplices cum reales esse tum inter se inaequales. Sint ergo factores simplices denominatoris omnes

$$(1 - pz)(1 - qz)(1 - rz)(1 - sz) \text{ etc.},$$

ex quibus fractio proposita in sequentes fractiones simplices resolvatur

$$\frac{\mathfrak{A}}{1 - pz} + \frac{\mathfrak{B}}{1 - qz} + \frac{\mathfrak{C}}{1 - rz} + \frac{\mathfrak{D}}{1 - sz} + \text{etc.}$$

Quibus cognitis erit seriei recurrentis terminus generalis [§ 215]

$$= z^n (\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n + \mathfrak{D}s^n + \text{etc.}),$$

quem statuamus $= Pz^n$; sit scilicet P coefficiens potestatis z^n sequentiumque Q, R etc., ita ut series recurrens fiat

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + Rz^{n+2} + \text{etc.}$$

335. Ponamus iam n esse numerum maximum seu seriem recurrentem ad plurimos terminos esse continuatam. Quoniam numerorum inaequalium potestates eo magis fiunt inaequales, quo fuerint altiores, tanta erit diversitas in potestatibus $\mathfrak{A}p^n$, $\mathfrak{B}q^n$, $\mathfrak{C}r^n$ etc., ut ea, quae oritur ex maximo numerorum p, q, r etc., reliquas magnitudine longe superet prae eaque reliquae penitus evanescant, si n fuerit numerus plane infinite magnus. Cum igitur numeri p, q, r etc. sint inter se inaequales, ponamus inter eos p esse maximum. Ac propterea, si n sit numerus infinitus, fiet

$$P = \mathfrak{A}p^n;$$

sin autem n sit numerus vehementer magnus, erit tantum proxime $P = \mathfrak{A}p^n$. Simili vero modo erit

$$Q = \mathfrak{A}p^{n+1}$$

ideoque

$$\frac{Q}{P} = p.$$

Unde patet, si series recurrens iam longe fuerit producta, coefficientem cuiusque termini per praecedentem divisum proxime esse exhibiturum valorem maximae litterae p.

336. Si igitur in fractione proposita

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

denominator habeat omnes factores simplices reales et inter se inaequales, ex serie recurrente inde orta cognosci poterit unus factor simplex, is scilicet $1 - pz$, in quo littera p omnium maximum habet valorem. Neque in hoc negotio coefficientes numeratoris a, b, c, d etc. in computum ingrediuntur, sed quicunque ii statuantur, tamen denique idem verus valor litterae maximae p invenitur. Verus quidem valor ipsius p tum demum innotescit, quando series in infinitum fuerit continuata; interim tamen, si iam plures eius termini fuerint formati, eo propius valor ipsius p cognoscetur, quo maior fuerit terminorum numerus et quo magis littera ista p excedat reliquas q, r, s etc. Perinde vero est, utrum haec maxima littera p fuerit signo + an signo — affecta, quoniam eius potestates aeque increscunt.

337. Quemadmodum nunc haec investigatio ad inventionem radicum aequationis cuiusvis algebraicae accommodari possit, satis est perspicuum. Ex factoribus enim denominatoris

$$1 - \alpha z - \beta zz - \gamma z^3 - \delta z^4 - \text{etc.}$$

cognitis facile assignantur radices aequationis huius

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.} = 0,$$

ita ut, si factor fuerit $1 - pz$, huius aequationis radix una futura sit $z = \frac{1}{p}$. Cum igitur ex serie recurrente reperiatur maximus numerus p, indidem obtinebitur minima radix aequationis

$$1 - \alpha z - \beta z^2 - \gamma z^3 - \text{etc.} = 0.$$

Vel si ponatur $z = \frac{1}{x}$, ut prodeat haec aequatio

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

eiusdem methodi ope eruitur maxima huius aequationis radix $x = p$.

338. Si igitur proponatur aequatio haec

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0,$$

quae omnes radices habeat reales et inter se inaequales, harum radicum maxima sequenti modo reperietur. Formetur ex coefficientibus huius aequationis fractio

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}.$$

Hincque formetur series recurrens assumendo pro arbitrio numeratorem seu, quod eodem redit, assumendo pro lubitu terminos initiales. Quae sit

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + \text{etc.}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximae x pro aequatione proposita eo propius, quo maior fuerit numerus n.

EXEMPLUM 1

Sit proposita ista aequatio

$$xx - 3x - 1 = 0,$$

cuius maximam radicem inveniri oporteat.

Formetur fractio

$$\frac{a + bz}{1 - 3z - zz},$$

unde positis duobus primis terminis 1, 2 orietur ista series recurrens

$$1,\ 2,\ 7,\ 23,\ 76,\ 251,\ 829,\ 2738 \text{ etc.}$$

Erit ergo

$$\frac{2738}{829}$$

proxime aequalis radici aequationis propositae maximae. Valor autem huius fractionis in partibus decimalibus expressus est

$$3,3027744;$$

aequationis vero radix maxima est

$$= \frac{3 + \sqrt{13}}{2} = 3,3027756,$$

quae inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse maiores et minores.

EXEMPLUM 2

Proposita sit ista aequatio

$$3x - 4x^3 = \frac{1}{2},$$

cuius radices exhibent sinus trium arcuum, quorum triplorum sinus est $= \frac{1}{2}$.

Aequatione perducta ad hanc formam

$$0 = 1 - 6x * + 8x^3$$

quaeratur huius, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{z}$. Formetur ergo haec fractio

$$\frac{a+bx+cxx}{1-6x*+8x^3},$$

ex qua sumendis pro lubitu tribus terminis initialibus 0, 0, 1, quia hoc modo calculus facillime expeditur, orietur haec series recurrens omittendis potestatibus ipsius x, quia tantum coefficientibus opus est,

$$0,\ 0,\ 1,\ 6,\ 36,\ 208,\ 1200,\ 6912,\ 39808,\ 229248.$$

Erit ergo proxime aequationis radix minima

$$=\frac{39808}{229248}=\frac{311}{1791}=0{,}1736460\text{[1]},$$

quae propterea esse deberet sinus anguli 10^0; hic autem ex tabulis est 0,1736482, qui superat radicem inventam parte $\frac{22}{10000000}$.

Facilius autem haec eadem radix inveniri potest ponendo $x=\frac{1}{2}y$, ut prodeat aequatio

$$1-3y*+y^3=0,$$

ex qua simili modo tractata oritur series

$$0,\ 0,\ 1,\ 3,\ 9,\ 26,\ 75,\ 216,\ 622,\ 1791,\ 5157 \text{ etc.}$$

Erit ergo proxime aequationis radix minima

$$y=\frac{1791}{5157}=\frac{199}{573}=0{,}3472949,$$

unde fit

$$x=\frac{1}{2}y=0{,}1736475\text{[2]},$$

qui valor fere ter propius accedit quam praecedens.

---

1) Editio princeps: $\frac{311}{1791}=0{,}1736515$, *quae propterea esse deberet sinus anguli* 10^0; *hic autem ex tabulis est* 0,1736482, *quem superat radix inventa parte* $\frac{33}{10000000}$. Correxit F. R.

2) Editio princeps: $x=\frac{1}{2}y=0{,}1736479$, *qui valor decies propius accedit quam praecedens.* Correxit F. R.

EXEMPLUM 3

Si desideretur eiusdem aequationis propositae

$$0 = 1 - 6x * + 8x^3$$

radix maxima, ponatur $x = \frac{y}{2}$ eritque

$$y^3 * - 3y + 1 = 0.$$

Cuius aequationis radix maxima reperietur per seriem recurrentem, cuius scala relationis est 0, 3, -1, unde ergo oritur sumptis tribus terminis initialibus pro arbitrio

$$1,\ 1,\ 1,\ 2,\ 2,\ 5,\ 4,\ 13,\ 7,\ 35,\ 8,\ 98,\ -11 \text{ etc.};$$

in qua serie cum ad terminos negativos perveniatur, id indicio est maximam radicem esse negativam; est enim

$$x = -\sin. 70^0 = -0{,}9396926.$$

Quare huius ratio in terminis initialibus est habenda hoc modo

$$1,\ -2,\ +4,\ -7,\ +14,\ -25,\ +49,\ -89,\ +172,\ -316,\ +605 \text{ etc.},$$

ex qua erit

$$y = \frac{-605}{316} \quad \text{et} \quad x = \frac{-605}{632} = -0{,}957,$$

quae a veritate vehementer abludit.

339. Ratio huius dissensus potissimum est, quod aequationis propositae radices sint

$$\sin. 10^0,\ \sin. 50^0 \quad \text{et} \quad -\sin. 70^0,$$

quarum binae maximae tam parum a se invicem discrepant, ut in potestatibus, ad quas seriem continuavimus, secunda radix $\sin. 50^0$ adhuc notabilem teneat rationem ad radicem maximam ideoque prae ea non evanescat. Hincque etiam saltus pendet, quod alternatim valores inventi fiant nimis magni et nimis parvi. Sic sumendo

$$y = \frac{-316}{172}$$

fit

$$x = \frac{-158}{172} = \frac{-79}{86} = -0{,}919.$$

Nam quoniam potestates radicis maximae alternatim fiunt affirmativae et negativae, alternatim quoque potestates secundae radicis adduntur et tolluntur; quamobrem, quo haec discrepantia fiat insensibilis, series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri potest transmutando aequationem ope idoneae substitutionis in aliam formam, cuius radices sibi non amplius sint tam vicinae. Sic si in aequatione

$$0 = 1 - 6x + 8x^3,$$

cuius radices sunt $-\sin. 70^0$, $+\sin. 50^0$, $+\sin. 10^0$, ponatur $x = y - 1$, aequationis

$$0 = 8y^3 - 24yy + 18y - 1$$

radices erunt $1 - \sin. 70^0$, $1 + \sin. 50^0$, $1 + \sin. 10^0$ ideoque eius radix minima erit $1 - \sin. 70^0$, cum tamen haec $\sin. 70^0$ esset radix maxima aequationis praecedentis, atque $1 + \sin. 50^0$ nunc est radix maxima, cum $\sin. 50^0$ ante esset media. Atque hoc modo quaevis radix per substitutionem in maximam minimamve radicem novae aequationis transmutari ideoque per methodum hic traditam inveniri poterit. Quia praeterea in hoc exemplo radix $1 - \sin. 70^0$ multo minor est quam binae reliquae, etiam facile per seriem recurrentem proxime cognoscetur.

EXEMPLUM 4

Invenire radicem minimam aequationis

$$0 = 8y^3 - 24yy + 18y - 1,$$

quae ab unitate subtracta relinquet sinum anguli 70^0.

Ponatur $y = \frac{1}{2}z$, ut sit

$$0 = z^3 - 6zz + 9z - 1,$$

cuius radix minima invenietur per seriem recurrentem, cuius scala relationis est 9, -6, $+1$; pro radice autem maxima invenienda scala relationis sumi deberet 6, -9, $+1$. Pro minima ergo formetur haec series

1, 1, 1, 4, 31, 256, 2122, 17593, 145861 etc.

Erit ergo proxime

$$z = \frac{17593}{145861} = 0{,}12061483$$

et

$$y = 0{,}06030741$$

atque

$$\sin. 70^0 = 1 - y = 0{,}93969258,$$

quae a veritate ne in ultima quidem figura discrepat. Ex hoc ergo exemplo intelligitur, quantam utilitatem idonea transformatio aequationis ope substitutionis ad inventionem radicum afferat et quod hoc pacto methodus tradita non solum ad maximas minimasve radices adstringatur, sed etiam omnes radices exhibere queat.

341. Cognita ergo iam quacunque aequationis propositae radice proxime ita ut verbi gratia numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ seu $x = y + k$ hocque modo prodibit aequatio, cuius radix minima erit $= x - k$; quae igitur si per series recurrentes indagetur, quod facillime fiet, quia haec radix multo minor erit quam ceterae, si ea ad k addatur, habebitur radix vera ipsius x pro aequatione proposita. Hoc vero artificium tam late patet, ut, etiamsi aequatio contineat radices imaginarias, usum suum retineat.

342. Imprimis autem sine hoc artificio radix cognosci nequit, cui datur alia aequalis, sed signo contrario affecta. Scilicet, si aequatio, cuius maxima radix p, eadem radicem habeat $-p$, tum, etiamsi series recurrens in infinitum

continuetur, tamen radix haec p nunquam obtinebitur. Sit, ut hoc exemplo illustremus, proposita aequatio

$$x^3 - x^2 - 5x + 5 = 0,$$

cuius maxima radix est $\sqrt{5}$, praeter quam vero inest quoque $-\sqrt{5}$. Si igitur modo ante praescripto pro radice maxima invenienda utamur atque seriem recurrentem formemus ex scala relationis 1, $+5$, -5, erit haec

$$1,\ 2,\ 3,\ 8,\ 13,\ 38,\ 63,\ 188,\ 313,\ 938,\ 1563 \text{ etc.},$$

ubi ad nullam rationem constantem pervenitur. Termini vero alterni rationem aequabilem induunt; quorum si quisque per praecedentem dividatur, reperietur quadratum maximae radicis; sic enim est proxime

$$5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}.$$

Quoties ergo termini tantum alterni sese ad rationem constantem componunt, toties quadratum radicis quaesitae proxime obtinetur. Ipsa autem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$, unde fit

$$1 - 3y - 5yy - y^3 = 0,$$

cuius radix minima cognoscetur ex serie

$$1,\ 1,\ 1,\ 9,\ 33,\ 145,\ 609,\ 2585,\ 10945 \text{ etc.};$$

erit enim proxime

$$= \frac{2585}{10945} = 0{,}2361;$$

at 2,2361 est proxime $= \sqrt{5}$, quae est radix maxima aequationis.

343. Quanquam numerator fractionis, ex qua series recurrens formatur, a nostro arbitrio pendet, tamen idonea eius constitutio plurimum confert, ut valor radicis cito vero proxime exhibeatur. Cum enim assumptis ut supra factoribus denominatoris (§ 334) sit terminus generalis seriei recurrentis

$$= z^n(\mathfrak{A}p^n + \mathfrak{B}q^n + \mathfrak{C}r^n + \text{etc.}),$$

isti coefficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. per numeratorem fractionis determinantur, unde fieri potest, ut $\mathfrak{A}$ sive magnum sive parvum valorem obtineat; priori casu radix maxima p cito reperitur, posteriore vero tarde. Quin etiam numerator ita accipi potest, ut $\mathfrak{A}$ prorsus evanescat, quo casu, etiamsi series in infinitum continuetur, tamen nunquam radicem maximam p praebebit. Hoc autem evenit, si numerator ita accipiatur, ut ipse eundem habeat factorem $1-pz$; sic enim ex computo penitus tolletur. Sic si proponatur aequatio

$$x^3 - 6xx + 10x - 3 = 0,$$

cuius maxima radix est $= 3$, indeque formetur fractio

$$\frac{1-3z}{1-6z+10z^2-3z^3},$$

ut seriei recurrentis sit scala relationis $6, -10, +3$, [series erit haec]

$$1, \ 3, \ 8, \ 21, \ 55, \ 144, \ 377 \text{ etc.},$$

cuius termini prorsus non convergunt ad rationem $1:3$. Eadem enim series oritur ex fractione

$$\frac{1}{1-3z+zz}$$

ac propterea maximam radicem aequationis

$$x^2 - 3x + 1 = 0$$

exhibet.

344. Quin etiam numerator ita assumi potest, ut per seriem recurrentem quaevis radix aequationis reperiatur, quod fiet, si numerator fuerit productum ex omnibus factoribus denominatoris praeter eum, cui respondet radix, quam velimus. Sic si in priori exemplo sumatur numerator $1-3z+zz$, fractio

$$\frac{1-3z+zz}{1-6z+10z^2-3z^3}$$

dabit hanc seriem recurrentem

$$1, \ 3, \ 9, \ 27, \ 81, \ 243 \text{ etc.},$$

quae, cum sit geometrica, statim monstrat radicem $x = 3$. Fractio enim illa aequalis est huic simplici

$$\frac{1}{1-3z}.$$

Hinc apparet, si termini initiales, quos pro lubitu assumere licet, ita accipiantur, ut progressionem geometricam constituant, cuius exponens aequetur uni radici aequationis, tum totam seriem recurrentem fore geometricam ideoque eam ipsam radicem esse exhibituram, etiamsi neque sit maxima neque minima.

345. Ne igitur, dum quaerimus radicem vel maximam vel minimam, praeter expectationem nobis alia radix per seriem recurrentem exhibeatur, eiusmodi numerator debet eligi, qui cum denominatore nullum factorem habeat communem, quod fiet, si pro numeratore unitas accipiatur, unde terminus primus seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix aequationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic proposita aequatione

$$y^3 * - 3y + 1 = 0,$$

cuius radix maxima desideratur, ex scala relationis $0, +3, -1$ incipiendo ab unitate sequens oritur series recurrens

$$1, -0, +3, -1, +9, -6, +28, -27, +90, -109, +297, -417, \\ +1000, -1548, +3417, -5644 \text{ etc.}^{1)},$$

quae ad rationem constantem convergit ostenditque radicem maximam esse negativam atque proxime

$$y = \frac{-5644}{3417} = -1{,}651741,$$

1) Editio princeps: $1 - 0 + 3 - 1 + \cdots + 297 - 517 + 1000 - 1848 + 3517 - 6544 + \text{etc.}$, *quae manifesto ad rationem constantem convergit, ostenditque radicem maximam esse negativam, atque proxime* $y = \frac{-6544}{3517} = -1{,}860676$, *quae esse debebat* $= -1{,}86793852$. Correxit F. R.

quae esse debebat $= -1,8793852$. Ratio autem supra [§ 330] est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima simulque sit affirmativa.

346. His probe perpensis, quae cum in genere tum ad exempla allata monuimus, summa utilitas huius methodi ad investigandas aequationum radices luculenter perspicietur. Artificia vero, quibus operatio contrahi eoque promptior reddi queat, satis quoque sunt indicata, ita ut nihil insuper addendum esset, nisi casus, quibus aequatio vel radices habet aequales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

habere factorem $(1-pz)^2$ reliquis factoribus existentibus $1-qz$, $1-rz$ etc. Seriei ergo recurrentis hinc natae terminus generalis erit

$$= z^n((n+1)\mathfrak{A}p^n + \mathfrak{B}p^n + \mathfrak{C}q^n + \text{etc.});$$

quae cuiusmodi valorem sit adeptura, si n fuerit numerus vehementer magnus, duo casus sunt distinguendi, alter, quo p est numerus maior reliquis q, r etc., alter, quo p non praebet radicem maximam. Casu priori, quo p simul est radix maxima, ob coefficientem $n+1$ reliqui termini $\mathfrak{B}p^n$, $\mathfrak{C}q^n$ etc. non tam cito prae eo evanescent quam ante; sin autem q fuerit $> p$, tum quoque tarde terminus $(n+1)\mathfrak{A}p^n$ prae $\mathfrak{C}q^n$ evanescet ideoque investigatio radicis maximae admodum evadet molesta.

EXEMPLUM 1

Sit proposita aequatio

$$x^3 - 3xx + 4 = 0,$$

cuius maxima radix 2 bis occurrit.

Quaeratur ergo maxima radix haec modo ante exposito per evolutionem fractionis

$$\frac{1}{1 - 3z \ast + 4z^3},$$

quae dabit hanc seriem recurrentem

$$1,\ 3,\ 9,\ 23,\ 57,\ 135,\ 313,\ 711,\ 1593 \text{ etc.},$$

ubi quidem quivis terminus per praecedentem divisus dat quotum binario maiorem. Cuius ratio ex termino generali facillime patet. Reiectis enim in eo terminis $\mathfrak{C}q^n$ etc. erit terminus potestati z^n respondens

$$= (n+1)\mathfrak{A}p^n + \mathfrak{B}p^n,$$

sequens

$$= (n+2)\mathfrak{A}p^{n+1} + \mathfrak{B}p^{n+1},$$

qui per illum divisus dat

$$\frac{(n+2)\mathfrak{A}+\mathfrak{B}}{(n+1)\mathfrak{A}+\mathfrak{B}}\,p > p,$$

nisi n iam in infinitum excreverit.

EXEMPLUM 2

Sit iam proposita aequatio

$$x^3 - xx - 5x - 3 = 0,$$

cuius maxima radix $= 3$, reliquae duae aequales $= -1$.

Quaeratur maxima radix ope seriei recurrentis, cuius scala relationis est $1, +5, +3$; unde oritur

$$1,\ 1,\ 6,\ 14,\ 47,\ 135,\ 412,\ 1228 \text{ etc.},$$

quae ideo satis cito valorem 3 exhibet, quod potestates minoris radicis -1, etiamsi multiplicentur per $n+1$, tamen mox prae potestatibus ipsius 3 evanescant.

EXEMPLUM 3

Sin autem proponeretur aequatio

$$x^3 + xx - 8x - 12 = 0,$$

cuius radices sunt $3, -2, -2$, multo tardius maxima sese prodet.

Orietur enim haec series

$$1, \ -1, \ 9, \ -5, \ 65, \ 3, \ 457, \ 347, \ 3345, \ 4915 \text{ etc.},$$

quae adhuc longissime continuari deberet, antequam pateret radicem inde oriundam esse $= 3$.

347. Simili modo si tres factores essent aequales, ita ut denominatoris factor unus esset $(1-pz)^3$, reliqui $1-qz$, $1-rz$ etc., seriei recurrentis terminus generalis erit

$$= z^n\left(\frac{(n+1)(n+2)}{1\cdot 2}\mathfrak{A}p^n + (n+1)\mathfrak{B}p^n + \mathfrak{C}p^n + \mathfrak{D}q^n + \mathfrak{E}r^n + \text{etc.}\right).$$

Si ergo p fuerit maxima radix atque n fuerit numerus tantus, ut potestates q^n, r^n etc. prae p^n evanescant, tum ex serie recurrente orietur radix

$$= \frac{\frac{1}{2}(n+2)(n+3)\mathfrak{A} + (n+2)\mathfrak{B} + \mathfrak{C}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A} + (n+1)\mathfrak{B} + \mathfrak{C}}p,$$

quae, nisi sit n numerus maximus et quasi infinitus, verum ipsius p valorem [non] indicabit. Erit autem iste radicis valor

$$= p + \frac{(n+2)\mathfrak{A} + \mathfrak{B}}{\frac{1}{2}(n+1)(n+2)\mathfrak{A} + (n+1)\mathfrak{B} + \mathfrak{C}}p.$$

Quodsi autem p non fuerit radix maxima, tum inventio maximae multo magis adhuc impedietur; unde sequitur aequationes, quae contineant radices aequales, hac methodo per series recurrentes multo difficilius resolvi, quam si omnes radices essent inter se inaequales.

348. Videamus nunc, quomodo series recurrens in infinitum continuata debeat esse comparata, quando denominator fractionis habet factores imaginarios. Sint igitur fractionis

$$\frac{a + bz + cz^2 + dz^3 + \text{etc.}}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \text{etc.}}$$

factores denominatoris reales

$$1 - qz, \quad 1 - rz \quad \text{etc.}$$

insuperque factor trinomialis

$$1 - 2pz \cos.\varphi + ppzz$$

continens duos factores simplices imaginarios. Quodsi ergo series recurrens ex illa fractione orta fuerit

$$A + Bz + Cz^2 + Dz^3 + \cdots + Pz^n + Qz^{n+1} + \text{etc.},$$

erit per ea, quae supra [§ 218] exposuimus, coefficiens P

$$= \frac{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin.n\varphi}{\sin.\varphi} p^n + \mathfrak{C} q^n + \mathfrak{D} r^n + \text{etc.}$$

Si igitur numerus p minor fuerit quam unus ceterorum q, r etc., ita ut maxima radix aequationis

$$x^m - \alpha x^{m-1} - \beta x^{m-2} - \gamma x^{m-3} - \text{etc.} = 0$$

sit realis, tum ea per series recurrentes aeque reperietur, ac si nullae radices inessent imaginariae.

349. Inventio ergo maximae radicis realis per radices imaginarias non perturbabitur, si hae ita fuerint comparatae, ut binarum, quae factorem realem componunt, productum non sit maius quadrato radicis maximae. Sin autem binae eiusmodi insint radices imaginariae, ut earum productum adaequet vel adeo superet quadratum maximae radicis realis, tum investigatio ante exposita nihil declarabit, propterea quod potestas p^n prae simili potestate radicis maximae nunquam evanescit, etiamsi series in infinitum continuetur. Cuius exempla illustrationis causa hic adiicere visum est.

EXEMPLUM 1

Sit proposita aequatio

$$x^3 - 2x - 4 = 0,$$

cuius radicem maximam investigari oporteat.

Resolvitur haec aequatio in duos factores

$$(x-2)(xx+2x+2);$$

unde unam habet radicem realem 2 et duas reliquas imaginarias, quarum productum est 2, minus quam quadratum radicis realis. Quamobrem ea per modum hactenus traditum cognosci poterit. Formetur ergo series recurrens ex scala relationis 0, $+2$, $+4$, quae erit

$$1, \ 0, \ 2, \ 4, \ 4, \ 16, \ 24, \ 48, \ 112, \ 192, \ 416, \ 832 \text{ etc.},$$

unde satis luculenter radix realis 2 cognosci potest.

EXEMPLUM 2

Proposita sit aequatio

$$x^3-4xx+8x-8=0,$$

cuius radix una realis est 2, binarum imaginariarum productum vero $=4$ ideoque aequale quadrato radicis realis 2.

Quaeramus ergo radicem per seriem recurrentem; quod quo facilius fieri queat, ponamus $x=2y$, ut habeatur

$$y^3-2yy+2y-1=0,$$

unde formetur series recurrens

$$1, \ 2, \ 2, \ 1, \ 0, \ 0, \ 1, \ 2, \ 2, \ 1, \ 0, \ 0, \ 1, \ 2, \ 2, \ 1 \text{ etc.};$$

in qua cum iidem termini perpetuo revertantur, nihil inde aliud colligi potest, nisi radicem maximam vel non esse realem vel dari imaginarias, quarum productum aequale sit aut superet quadratum radicis realis.

EXEMPLUM 3

Sit iam proposita aequatio

$$x^3-3xx+4x-2=0,$$

cuius radix realis est 1, imaginariarum vero productum $=2$.

Formetur ergo ex scala relationis 3, -4, $+2$ series

$$1,\ 3,\ 5,\ 5,\ 1,\ -7,\ -15,\ -15,\ +1,\ 33,\ 65,\ 65,\ 1 \text{ etc.};$$

in qua cum termini modo fiant affirmativi modo negativi, radix realis 1 inde nullo modo cognosci poterit. Huiusmodi vero revolutiones semper ostendunt radicem, quam series praebere debebat, esse imaginariam; hic enim radices imaginariae potestate sunt maiores quam realis 1.

350. Sit igitur in fractione generali productum binarum radicum imaginariarum pp maius quam ullius radicis realis quadratum, ita ut prae p^n reliquae potestates q^n, r^n etc. evanescant, si n sit numerus infinitus. Hoc ergo casu fiet

$$P = \frac{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi}{\sin. \varphi} p^n$$

et

$$Q = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin.(n+1)\varphi}{\sin. \varphi} p^{n+1}$$

ideoque

$$\frac{Q}{P} = \frac{\mathfrak{A} \sin.(n+2)\varphi + \mathfrak{B} \sin.(n+1)\varphi}{\mathfrak{A} \sin.(n+1)\varphi + \mathfrak{B} \sin. n\varphi} p.$$

Quae expressio nunquam valorem constantem induet, etiamsi n sit numerus infinitus. Sinus enim angulorum perpetuo maxime manent mutabiles, ita ut mox sint affirmativi mox negativi.

351. Interim tamen si fractiones sequentes $\frac{R}{Q}$, $\frac{S}{R}$ simili modo sumantur indeque litterae $\mathfrak{A}$ et $\mathfrak{B}$ eliminentur, simul numerus n ex calculo egredietur; reperietur enim[1])

$$Ppp + R = 2\,Qp \cos. \varphi,$$

unde fit

$$\cos. \varphi = \frac{Ppp + R}{2\,Qp};$$

similiter vero erit

$$\cos. \varphi = \frac{Qpp + S}{2\,Rp},$$

1) Vide paragraphum sequentem. F. R.

ex quorum duorum valorum comparatione fit

$$p = \sqrt{\frac{RR - QS}{QQ - PR}}$$

atque

$$\cos.\varphi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(R^2 - QS)}}.$$

Quamobrem si series recurrens iam eousque fuerit continuata, ut prae p^n reliquarum radicum potestates evanescant, tum hoc modo factor trinomialis $1 - 2pz\cos.\varphi + ppzz$ poterit inveniri.

352. Quoniam iste calculus non satis exercitatis molestiam creare posset, eum totum hic apponam. Ex valore ipsius $\frac{Q}{P}$ invento oritur

$$\mathfrak{A}Pp\sin.(n+2)\varphi + \mathfrak{B}Pp\sin.(n+1)\varphi = \mathfrak{A}Q\sin.(n+1)\varphi + \mathfrak{B}Q\sin.n\varphi,$$

unde fit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{Q\sin.n\varphi - Pp\sin.(n+1)\varphi}{Pp\sin.(n+2)\varphi - Q\sin.(n+1)\varphi}.$$

Pari ratione erit

$$\frac{\mathfrak{A}}{\mathfrak{B}} = \frac{R\sin.(n+1)\varphi - Qp\sin.(n+2)\varphi}{Qp\sin.(n+3)\varphi - R\sin.(n+2)\varphi}.$$

Aequatis his duobus valoribus fiet

$$\begin{aligned} 0 = {} & QQp\sin.n\varphi\sin.(n+3)\varphi - QR\sin.n\varphi\sin.(n+2)\varphi \\ & - PQpp\sin.(n+1)\varphi\sin.(n+3)\varphi - QQp\sin.(n+1)\varphi\sin.(n+2)\varphi \\ & + QR\sin.(n+1)\varphi\sin.(n+1)\varphi + PQpp\sin.(n+2)\varphi\sin.(n+2)\varphi. \end{aligned}$$

Cum autem sit

$$\sin.a\sin.b = \tfrac{1}{2}\cos.(a-b) - \tfrac{1}{2}\cos.(a+b),$$

fiet

$$0 = \tfrac{1}{2}QQp(\cos.3\varphi - \cos.\varphi) + \tfrac{1}{2}QR(1 - \cos.2\varphi) + \tfrac{1}{2}PQpp(1 - \cos.2\varphi),$$

quae per $\frac{1}{2}Q$ divisa dat

$$(Ppp + R)(1 - \cos. 2\varphi) = Qp(\cos. \varphi - \cos. 3\varphi).$$

At est

$$\cos. \varphi = \cos. 2\varphi \cos. \varphi + \sin. 2\varphi \sin. \varphi$$

et

$$\cos. 3\varphi = \cos. 2\varphi \cos. \varphi - \sin. 2\varphi \sin. \varphi,$$

unde

$$\cos. \varphi - \cos. 3\varphi = 2 \sin. 2\varphi \sin. \varphi = 4 \sin. \varphi^2 \cos. \varphi,$$

et

$$1 - \cos. 2\varphi = 2 \sin. \varphi^2,$$

ex quo erit

$$Ppp + R = 2Qp \cos. \varphi$$

et

$$\cos. \varphi = \frac{Ppp + R}{2Qp}$$

atque

$$\cos. \varphi = \frac{Qpp + S}{2Rp},$$

unde superiores valores prodeunt, scilicet

$$p = \sqrt{\frac{RR - QS}{QQ - PR}}$$

et

$$\cos. \varphi = \frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}.$$

353. Si denominator fractionis, ex qua series recurrens formatur, plures habeat factores trinomiales inter se aequales, tum spectata forma termini generalis supra [§ 219 et sq.] data patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quaecunque radix realis iam proxime fuerit detecta, tum aequationis transformatione semper valor eiusdem radicis multo propior eruetur. Ponatur enim x aequalis valori illi iam detecto $+ y$ atque novae aequationis quaeratur minima radix pro y, quae addita ad illum valorem praebebit verum ipsius x valorem.

EXEMPLUM

Sit proposita ista aequatio

$$x^3 - 3xx + 5x - 4 = 0;$$

cuius unam radicem fere esse $= 1$ inde constat, quod posita $x = 1$ prodit

$$x^3 - 3xx + 5x - 4 = -1.$$

Ponatur ergo $x = 1 + y$ fietque

$$1 - 2y - y^3 = 0,$$

unde pro radice minima invenienda formetur series recurrens, cuius scala relationis 2, 0, $+1$, quae erit

1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296 etc.,

unde radix minima ipsius y erit proxime

$$\frac{1041}{2296} = 0{,}453397,$$

ita ut sit

$$x = 1{,}453397,$$

qui valor tam prope vix alia methodo aeque facile obtineri poterit.

354. Quodsi autem series quaecunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cuiusnam aequationis radix sit futura quotus, qui ex divisione unius termini per praecedentem oritur. Sint

$$P,\ Q,\ R,\ S,\ T \text{ etc.}$$

termini seriei recurrentis a principio iam longissime remoti, ita ut cum progressione geometrica confundantur, sitque

$$T = \alpha S + \beta R + \gamma Q + \delta P$$

seu scala relationis α, $+\beta$, $+\gamma$, $+\delta$. Ponatur valor fractionis $\frac{Q}{P} = x$; erit

$$\frac{R}{P} = xx, \quad \frac{S}{P} = x^3 \quad \text{et} \quad \frac{T}{P} = x^4,$$

qui in superiori aequatione substituti dabunt

$$x^4 = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

unde patet quotum $\frac{Q}{P}$ tandem praebere radicem unam aequationis inventae. Hoc vero et praecedens methodus indicat, praeterea vero docet fractionem $\frac{Q}{P}$ dare maximam aequationis radicem.

355. Potest quoque haec methodus investigandarum radicum saepenumero utiliter adhiberi, si aequatio sit infinita. Ad quod ostendendum proposita sit aequatio

$$\frac{1}{2} = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} + \text{etc.},$$

cuius radix minima z exhibet arcum 30^0 seu semiperipheriae circuli sextantem. Perducatur ergo aequatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \text{etc.} = 0.$$

Hinc ergo formetur series recurrens, cuius scala relationis est infinita, scilicet

$$2,\ 0,\ -\frac{1}{3},\ 0,\ +\frac{1}{60},\ 0,\ -\frac{1}{2520},\ 0 \ \text{etc.},$$

eritque series recurrens

$$1,\ 2,\ 4,\ \frac{23}{3},\ \frac{44}{3},\ \frac{1681}{60},\ \frac{2408}{45}\ \text{etc.};$$

erit ergo proxime

$$z = \frac{1681 \cdot 45}{2408 \cdot 60} = \frac{1681 \cdot 3}{2408 \cdot 4} = \frac{5043}{9632} = 0{,}52356.$$

At ex proportione peripheriae ad diametrum cognita debebat esse $z = 0{,}523598$, ita ut radix inventa tantum parte $\frac{3}{100\,000}$ a vero discrepet.[1]) Hoc autem in hac aequatione commode usu venit, quod eius omnes radices sint reales atque a minima reliquae satis notabiliter discrepent. Quae conditio cum rarissime in aequationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

1) Valores accuratiores sunt $z = 0{,}523\,567$ et $z = 0{,}523\,599$, quorum tamen differentia itidem est $\frac{3}{100\,000}$. F. R.

CAPUT XVIII

DE FRACTIONIBUS CONTINUIS

356. Quoniam in praecedentibus capitibus plura cum de seriebus infinitis tum de productis ex infinitis factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim iam aliquoties[1]) eiusmodi specimina, quibus haec expectatio non parum probabilis redditur. Imprimis vero ad ipsam arithmeticam et algebram communem non contemnenda subsidia affert ista speculatio, quae hoc capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco eiusmodi fractionem, cuius denominator constat ex numero integro cum fractione, cuius denominator denuo est aggregatum ex integro et fractione, quae porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Huiusmodi ergo fractio continua erit sequens expressio

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}} \qquad \text{vel} \qquad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

1) Vide L. EULERI Commentationes 71 et 123 (indicis ENESTROEMIANI): *De fractionibus continuis*, Comment. acad. sc. Petrop. **9** (1737), 1744, p. 98, et *De fractionibus continuis observationes*, Comment. acad. sc. Petrop. **11** (1739), 1750, p. 32; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. F. R.

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Exposita ergo fractionum harum continuarum forma primum videndum est, quemadmodum earum significatio consueto more expressa inveniri queat. Quae ut facilius inveniri possit, progrediamur per gradus abrumpendo illas fractiones primo in prima, tum in secunda, post in tertia et ita porro fractione; quo facto patebit fore

$$a = a,$$

$$a + \frac{1}{b} = \frac{ab+1}{b},$$

$$a + \cfrac{1}{b + \cfrac{1}{c}} = \frac{abc+a+c}{bc+1},$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d},$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

etc.

359. Etsi in his fractionibus ordinariis non facile lex, secundum quam numerator ac denominator ex litteris a, b, c, d etc. componantur, perspicitur, tamen attendenti statim patebit, quemadmodum quaelibet fractio ex praecedentibus formari queat. Quilibet enim numerator est aggregatum ex numeratore ultimo per novam litteram multiplicato et ex numeratore penultimo simplici; eademque lex in denominatoribus observatur. Scriptis ergo ordine litteris a, b, c, d etc. ex iis fractiones inventae facile formabuntur hoc modo

$$\begin{array}{cccccc} & a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+a+c}{bc+1}, & \frac{abcd+ab+ad+cd+1}{bcd+b+d} & \text{etc.,} \end{array}$$

ubi quilibet numerator invenitur, si praecedentium ultimus per indicem supra scriptum multiplicetur atque ad productum antepenultimus addatur; quae eadem lex pro denominatoribus valet. Quo autem hac lege ab ipso initio uti liceat, praefixi fractionem $\frac{1}{0}$, quae, etiamsi e fractione continua non oriatur, tamen progressionis legem clariorem efficit. Quaelibet autem fractio exhibet valorem fractionis continuae usque ad eam litteram, quae antecedenti imminet, inclusive continuatae.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

dabit, prout aliis aliisque locis abrumpitur, sequentes valores

$$\begin{aligned}
&a &&= a,\\
&a+\frac{\alpha}{b} &&= \frac{ab+\alpha}{b},\\
&a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} &&= \frac{abc+\beta a+\alpha c}{bc+\beta},\\
&a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} &&= \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}
\end{aligned}$$

etc.,

quarum fractionum quaeque ex binis praecedentibus sequentem in modum invenietur:

$$\begin{array}{cccccc}
a & b & c & d & e & \\
\frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\beta a+\alpha c}{bc+\beta}, & \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.}\\
\alpha & \beta & \gamma & \delta & \varepsilon &
\end{array}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d etc., infra autem subscribantur indices $\alpha, \beta, \gamma, \delta$ etc. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$. Tum sequentium quaevis formabitur, si anteceden-

tium ultimae numerator per indicem supra scriptum, penultimae vero numerator per indicem infra scriptum multiplicetur et ambo producta addantur; aggregatum erit numerator fractionis sequentis. Simili modo eius denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum et ex penultimo denominatore per indicem infra scriptum multiplicatis. Quaelibet vero fractio hoc modo inventa praebebit valorem fractionis continuae ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatae inclusive.

362. Quodsi ergo hae fractiones eousque continuentur, quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuae. Praecedentes fractiones vero continuo propius ad hunc valorem accedent ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuae

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.}}}}}$$

esse $=x$ atque manifestum est fractionem primam $\frac{1}{0}$ esse maiorem quam x, secunda vero $\frac{a}{1}$ minor erit quam x, tertia $a+\frac{\alpha}{b}$ iterum vero valore erit maior, quarta denuo minor, atque ita porro hae fractiones alternatim erunt maiores et minores quam x. Porro autem perspicuum est quamlibet fractionem propius accedere ad verum valorem x quam ulla praecedentium[1]), unde hoc pacto citissime et commodissime valor ipsius x proxime obtinetur, etiamsi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ etc. non nimis crescant; sin autem omnes isti numeratores fuerint unitates, tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio huius appropinquationis ad verum fractionis continuae valorem melius percipiatur, consideremus fractionum inventarum differentias.

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1) Hoc autem veritati non semper consentaneum est, nisi sit $\alpha=\beta=\gamma=\cdots=1$. Sic si sit $x=1+\frac{1}{1+\frac{3}{2}}$, fractionum $\frac{a}{1}=1$, $\frac{ab+\alpha}{b}=2$, $\frac{abc+\beta a+\alpha c}{bc+\beta}=\frac{7}{5}$ prima propius accedit ad verum valorem $\frac{7}{5}$ quam secunda. F. R.

Ac prima quidem $\frac{1}{0}$ praetermissa differentia inter secundam ac tertiam est

$$=\frac{\alpha}{b},$$

quarta a tertia subtracta relinquit

$$\frac{\alpha\beta}{b(bc+\beta)},$$

quarta a quinta subtracta relinquit

$$\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}$$

etc. Hinc exprimetur valor fractionis continuae per seriem terminorum consuetam hoc modo, ut sit

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\text{etc.},$$

quae series toties abrumpitur, quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in seriem terminorum, quorum signa alternantur, convertendi, siquidem prima littera a evanescat. Si enim fuerit

$$x=\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

erit per ea, quae modo invenimus,

$$x=\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}$$
$$-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)}+\text{etc.}$$

Unde, si α, β, γ, δ etc. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d etc. numeri integri quicunque affirmativi, valor fractionis continuae exprimetur per seriem terminorum maxime convergentem.

365. His probe consideratis poterit vicissim series quaecunque terminorum alternantium in fractionem continuam converti seu fractio continua inveniri, cuius valor aequalis sit summae seriei propositae. Sit enim proposita haec series

$$x = A - B + C - D + E - F + \text{etc.};$$

erit singulis terminis cum serie ex fractione continua orta comparandis

$A = \frac{\alpha}{b}$,	unde fit $\alpha = Ab$,
$\frac{B}{A} = \frac{\beta}{bc+\beta}$,	$\beta = \frac{Bbc}{A-B}$,
$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}$,	$\gamma = \frac{Cd(bc+\beta)}{b(B-C)}$,
$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}$	$\delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$
etc.,	etc.

At cum sit $\beta = \frac{Bbc}{A-B}$, erit

$$bc + \beta = \frac{Abc}{A-B},$$

unde

$$\gamma = \frac{ACcd}{(A-B)(B-C)}.$$

Porro fit

$$bcd + \beta d + \gamma b = (bc + \beta)d + \gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)},$$

unde erit

$$\frac{bcd+\beta d+\gamma b}{bc+\beta} = \frac{Bd}{B-C}$$

et

$$\delta = \frac{BDde}{(B-C)(C-D)}.$$

Simili modo reperietur

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

et ita porro.

366. Quo ista lex clarius appareat, ponamus esse

$$P = b,$$

$$Q = bc + \beta,$$

$$R = bcd + \beta d + \gamma b,$$

$$S = bcde + \beta de + \gamma be + \delta bc + \beta\delta,$$

$$T = bcdef + \text{etc.},$$

$$V = bcdefg + \text{etc.}$$

$$\text{etc.};$$

erit ex lege harum expressionum

$$Q = Pc + \beta,$$

$$R = Qd + \gamma P,$$

$$S = Re + \delta Q,$$

$$T = Sf + \varepsilon R,$$

$$V = Tg + \zeta S$$

$$\text{etc.}$$

Cum igitur his adhibendis litteris fit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \text{etc.}$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - F + \text{etc.},$$

erit

$$A = \frac{\alpha}{P}, \qquad \alpha = AP,$$

$$\frac{B}{A} = \frac{\beta}{Q}, \qquad \beta = \frac{BQ}{A},$$

$$\frac{C}{B} = \frac{\gamma P}{R}, \qquad \gamma = \frac{CR}{BP},$$

$$\frac{D}{C} = \frac{\delta Q}{S}, \qquad \delta = \frac{DS}{CQ},$$

$$\frac{E}{D} = \frac{\varepsilon R}{T}, \qquad \varepsilon = \frac{ET}{DR}$$

etc.

Porro vero differentiis sumendis habebitur

$$A - B = \frac{\alpha(Q-\beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q},$$

$$B - C = \frac{\alpha\beta(R-\gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R},$$

$$C - D = \frac{\alpha\beta\gamma(S-\delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S},$$

$$D - E = \frac{\alpha\beta\gamma\delta(T-\varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T}$$

etc.

Si bini igitur in se invicem ducantur, fiet

$$(A-B)(B-C) = ABcd\,\frac{P}{R} \quad \text{et} \quad \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)},$$

$$(B-C)(C-D) = BCde\,\frac{Q}{S} \quad \text{et} \quad \frac{S}{Q} = \frac{BCde}{(B-C)(C-D)},$$

$$(C-D)(D-E) = CDef\,\frac{R}{T} \quad \text{et} \quad \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$

etc.

Unde, cum sit $P = b$, $Q = \frac{\alpha c}{A-B} = \frac{Abc}{A-B}$, erit

$$\alpha = Ab,$$

$$\beta = \frac{Bbc}{A-B},$$

$$\gamma = \frac{ACcd}{(A-B)(B-C)},$$

$$\delta = \frac{BDde}{(B-C)(C-D)},$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

etc.

368. Inventis ergo valoribus numeratorum α, β, γ, δ etc. denominatores b, c, d, e etc. arbitrio nostro relinquuntur; ita autem eos assumi convenit, ut cum ipsi sint numeri integri, tum valores integros pro α, β, γ, δ etc. exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C etc., utrum sint integri an fracti. Ponamus esse numeros integros atque quaesito satisfiet statuendo

$b = 1,$	unde fit	$\alpha = A,$
$c = A - B,$		$\beta = B,$
$d = B - C,$		$\gamma = AC,$
$e = C - D,$		$\delta = BD,$
$f = D - E$		$\varepsilon = CE$
etc.,		etc.

Quocirca si fuerit

$$x = A - B + C - D + E - F + \text{etc.},$$

idem ipsius x valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{etc.}}}}}}$$

369. Sin autem omnes termini seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{etc.},$$

habebuntur pro α, β, γ, δ etc. sequentes valores

$$\alpha = \frac{b}{A},$$

$$\beta = \frac{Abc}{B-A},$$

$$\gamma = \frac{B^2cd}{(B-A)(C-B)},$$

$$\delta = \frac{C^2de}{(C-B)(D-C)},$$

$$\varepsilon = \frac{D^2ef}{(D-C)(E-D)}$$

etc.

Ponatur ergo, ut sequitur,

$$\begin{array}{lcl} b = A, & \text{unde fit} & \alpha = 1, \\ c = B - A, & & \beta = AA, \\ d = C - B, & & \gamma = BB, \\ e = D - C & & \delta = CC \\ \text{etc.,} & & \text{etc.} \end{array}$$

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \text{etc.}}}}}$$

EXEMPLUM 1

Transformetur haec series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}$$

in fractionem continuam.

Erit ergo

$$A=1,\quad B=2,\quad C=3,\quad D=4\quad \text{etc.},$$

atque cum seriei propositae valor sit $=l2$, erit

$$l2=\cfrac{1}{1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\cfrac{25}{1+\text{etc.}}}}}}}$$

EXEMPLUM 2

Transformetur haec series infinita [§ 140]

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.},$$

ubi π denotat peripheriam circuli, cuius diameter $=1$, in fractionem continuam.

Substitutis loco A, B, C, D etc. numeris 1, 3, 5, 7 etc. orietur

$$\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}}$$

hincque invertendo fractionem erit

$$\frac{4}{\pi}=1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}$$

quae est expressio, quam BROUNCKERUS[1]) primum pro quadratura circuli protulit.

1) Hanc celebrem fractionem continuam W. BROUNCKER (1620—1684) epistola cum J. WALLIS sine demonstratione communicaverat. Vide J. WALLIS, *Arithmetica infinitorum,* Oxoniae 1655, p. 182; *Opera mathematica*, t. I, Oxoniae 1695, p. 355, imprimis p. 469. Vide etiam *LEONHARDI EULERI Opera omnia,* series I, vol. 1, p. 507. F. R.

EXEMPLUM 3

Sit proposita ista series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{etc.},$$

quae ob

$$A = m, \quad B = m + n, \quad C = m + 2n \quad \text{etc.}$$

in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{etc.}}}}}}$$

ex qua fit invertendo

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \text{etc.}}}}}$$

EXEMPLUM 4

Quoniam supra (§ 178) invenimus esse

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \text{etc.},$$

erit pro fractione continuanda

$$A = m, \quad B = n - m, \quad C = n + m, \quad D = 2n - m \quad \text{etc.},$$

unde fiet

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n - 2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n - 2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n - 2m + \text{etc.}}}}}}}$$

370. Si series proposita per continuos factores progrediatur, ut sit

$$x=\frac{1}{A}-\frac{1}{AB}+\frac{1}{ABC}-\frac{1}{ABCD}+\frac{1}{ABCDE}-\text{etc.},$$

tum prodibunt sequentes determinationes

$$\alpha=\frac{b}{A},$$

$$\beta=\frac{bc}{B-1},$$

$$\gamma=\frac{Bcd}{(B-1)(C-1)},$$

$$\delta=\frac{Cde}{(C-1)(D-1)},$$

$$\varepsilon=\frac{Def}{(D-1)(E-1)}$$

etc.

Fiat ergo, ut sequitur,

$$\begin{array}{lcl} b=A, & \text{unde fit} & \alpha=1, \\ c=B-1, & & \beta=A, \\ d=C-1, & & \gamma=B, \\ e=D-1, & & \delta=C, \\ f=E-1 & & \varepsilon=D \\ \text{etc.}, & & \text{etc.}, \end{array}$$

unde consequenter fiet

$$x=\cfrac{1}{A+\cfrac{A}{B-1+\cfrac{B}{C-1+\cfrac{C}{D-1+\cfrac{D}{E-1+\text{etc.}}}}}}.$$

EXEMPLUM 1

Quoniam posito e numero, cuius logarithmus est $=1$, supra [§ 123] invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

haec series in fractionem continuam convertetur ponendo

$$A = 1, \quad B = 2, \quad C = 3, \quad D = 4 \quad \text{etc.};$$

quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}}}$$

unde asymmetria initio reiecta erit

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}}$$

EXEMPLUM 2

Invenimus quoque arcus, qui radio aequalis sumitur, cosinum esse [§ 134]

$$= 1 - \frac{1}{2} + \frac{1}{2\cdot 12} - \frac{1}{2\cdot 12\cdot 30} + \frac{1}{2\cdot 12\cdot 30\cdot 56} - \text{etc.}$$

Si ergo fiat

$$A = 1, \quad B = 2, \quad C = 12, \quad D = 30, \quad E = 56 \quad \text{etc.}$$

atque cosinus arcus, qui radio aequatur, ponatur $=x$, erit

$$x=\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55+\text{etc.}}}}}}$$

seu

$$\frac{1}{x}-1=\cfrac{1}{1+\cfrac{2}{11+\cfrac{12}{29+\cfrac{30}{55+\text{etc.}}}}}$$

371. Sit series insuper cum geometrica coniuncta, scilicet

$$x=A-Bz+Cz^2-Dz^3+Ez^4-Fz^5+\text{etc.};$$

erit

$$\alpha=Ab,$$

$$\beta=\frac{Bbcz}{A-Bz},$$

$$\gamma=\frac{ACcdz}{(A-Bz)(B-Cz)},$$

$$\delta=\frac{BDdez}{(B-Cz)(C-Dz)},$$

$$\varepsilon=\frac{CEefz}{(C-Dz)(D-Ez)}$$

etc.

Ponatur nunc

$$\begin{array}{lll} b=1, & \text{erit} & \alpha=A, \\ c=A-Bz, & & \beta=Bz, \\ d=B-Cz, & & \gamma=ACz, \\ e=C-Dz & & \delta=BDz \\ \text{etc.,} & & \text{etc.,} \end{array}$$

unde fiet

$$x=\cfrac{A}{1+\cfrac{Bz}{A-Bz+\cfrac{ACz}{B-Cz+\cfrac{BDz}{C-Dz+\text{etc.}}}}}$$

372. Quo autem hoc negotium generalius absolvamus, ponamus esse

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} - \frac{Dy^3}{Oz^3} + \frac{Ey^4}{Pz^4} - \text{etc.}$$

fietque comparatione instituta

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{BLbcy}{AMz - BLy},$$

$$\gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)},$$

$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}$$

etc.

Statuantur valores b, c, d etc. sequenti modo

$$\begin{array}{ll} b = L, & \text{erit}\quad \alpha = A, \\ c = AMz - BLy, & \beta = BLLy, \\ d = BNz - CMy, & \gamma = ACM^2yz, \\ e = COz - DNy, & \delta = BDN^2yz, \\ f = DPz - EOy & \varepsilon = CEO^2yz \\ \text{etc.}; & \text{etc.}, \end{array}$$

unde series proposita per sequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - Bly + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \text{etc.}}}}}$$

373. Habeat denique series proposita huiusmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \text{etc.}$$

atque sequentes valores prodibunt

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{Bbcy}{Mz - By},$$

$$\gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)},$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)},$$

$$\varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)}$$

etc.

Ad valores ergo integros inveniendos fiat

$$\begin{array}{ll} b = Lz, & \text{erit} \quad \alpha = Az, \\ c = Mz - By, & \beta = BLyz, \\ d = Nz - Cy, & \gamma = CMyz, \\ e = Oz - Dy, & \delta = DNyz, \\ f = Pz - Ey & \varepsilon = EOyz \\ \text{etc.}; & \text{etc.}, \end{array}$$

unde valor seriei propositae ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \text{etc.}}}}}$$

vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \text{etc.}}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuae in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim ex supra traditis infinitae series, quarum summae constent, ad hoc negotium accommodari queant, unaquaeque transformari poterit in fractionem continuam, cuius adeo valor summae illius seriei est aequalis. Exempla, quae iam hic sunt allata, sufficiunt ad hunc usum ostendendum. Verumtamen optandum esset, ut methodus detegeretur, cuius beneficio, si proposita fuerit fractio continua quaecunque, eius valor immediate inveniri posset. Quanquam enim fractio continua transmutari potest in seriem infinitam, cuius summa per methodos cognitas investigari queat, tamen plerumque istae series tantopere fiunt intricatae, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit.

375. Quo autem clarius perspiciatur dari eiusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}$$

cuius omnes denominatores sunt inter se aequales. Si enim hinc modo supra exposito fractiones formemus

$$\begin{array}{ccccccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \text{ etc.}, \end{array}$$

oritur haec series

$$x = 0 + \frac{1}{2} - \frac{1}{2\cdot 5} + \frac{1}{5\cdot 12} - \frac{1}{12\cdot 29} + \frac{1}{29\cdot 70} - \text{etc.}$$

vel, si bini termini coniungantur, erit

$$x = \frac{2}{1\cdot 5} + \frac{2}{5\cdot 29} + \frac{2}{29\cdot 169} + \text{etc.}$$

vel

$$x = \frac{1}{2} - \frac{2}{2\cdot 12} - \frac{2}{12\cdot 70} - \text{etc.}$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2\cdot 2\cdot 5} + \frac{1}{2\cdot 5\cdot 12} - \frac{1}{2\cdot 12\cdot 29} + \text{etc.}$$

$$+ \frac{1}{4} - \frac{1}{2\cdot 2\cdot 5} + \frac{1}{2\cdot 5\cdot 12} - \frac{1}{2\cdot 12\cdot 29} + \text{etc.},$$

erit

$$x = \frac{1}{4} + \frac{1}{1\cdot 5} - \frac{1}{2\cdot 12} + \frac{1}{5\cdot 29} - \frac{1}{12\cdot 70} + \text{etc.};$$

quae series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro huiusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt aequales vel iidem revertuntur, ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit aequalis, facilis habetur modus earum summas explorandi. In exemplo enim proposito cum sit

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{etc.}}}}}$$

erit

$$x = \frac{1}{2 + x}$$

ideoque

$$xx + 2x = 1$$

et

$$x + 1 = \sqrt{2},$$

ita ut valor huius fractionis continuae sit

$$= \sqrt{2} - 1.$$

Fractiones vero ex fractione continua ante erutae continuo propius ad hunc valorem accedunt idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum inveniri queat.

Est enim $\sqrt{2}-1$ tam prope $\frac{29}{70}$, ut error sit insensibilis; namque radicem extrahendo erit

$$\sqrt{2}-1=0{,}414213562\,37$$

atque

$$\frac{29}{70}=0{,}414285714\,28,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuae commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem

$$x=\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}}$$

erit

$$x=\frac{1}{a+x}$$

et

$$xx+ax=1,$$

unde fit

$$x=-\frac{1}{2}a+\sqrt{\left(1+\frac{1}{4}aa\right)}=\frac{\sqrt{(aa+4)}-a}{2}.$$

Haec ergo fractio continua inserviet valori radicis quadratae ex numero $aa+4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4 etc. reperientur $\sqrt{5}$, $\sqrt{2}$, $\sqrt{13}$, $\sqrt{5}$, $\sqrt{29}$, $\sqrt{10}$, $\sqrt{53}$ etc., perductis scilicet his radicibus ad formam simplicissimam. Erit ergo

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{3}{5}, & \frac{5}{8} \end{array} \text{ etc.} = \frac{\sqrt{5}-1}{2},$$

$$\begin{array}{cccccc} 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70} \end{array} \text{ etc.} = \sqrt{2}-1,$$

$$\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \\ \frac{0}{1}, & \frac{1}{3}, & \frac{3}{10}, & \frac{10}{33}, & \frac{33}{109}, & \frac{109}{360} \end{array} \text{ etc.} = \frac{\sqrt{13}-3}{2},$$

$$\begin{array}{cccccc} 4 & 4 & 4 & 4 & 4 & 4 \\ \frac{0}{1}, & \frac{1}{4}, & \frac{4}{17}, & \frac{17}{72}, & \frac{72}{305}, & \frac{305}{1292} \end{array} \text{ etc.} = \sqrt{5}-2$$

etc.

Notandum autem eo promptiorem esse approximationem, quo maior fuerit numerus a. Sic in ultimo exemplo erit

$$\sqrt{5} = 2\,\frac{305}{1292},$$

ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur haec approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \text{etc.}}}}}}}$$

Erit

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}$$

ideoque

$$axx + abx = b$$

et

$$x = -\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}bb + \frac{b}{a}\right)} = \frac{-ab + \sqrt{(aabb + 4ab)}}{2a}.$$

Unde iam omnium numerorum radices inveniri poterunt. Sit verbi gratia $a = 2$, $b = 7$; erit

$$x = \frac{-14 + \sqrt{14 \cdot 18}}{4} = \frac{-7 + 3\sqrt{7}}{2}.$$

At valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{matrix} 2 & 7 & 2 & 7 & 2 & 7 & \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510} & \text{etc.} \end{matrix}$$

Erit ergo proxime

$$\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$$

et

$$\sqrt{7} = \frac{2024}{765} = 2{,}645751\,63;$$

at revera est

$$\sqrt{7} = 2{,}645751\,31;$$

ita ut error minor sit quam $\frac{33}{100000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \text{etc.}}}}}}}}$$

Erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c},$$

unde

$$(ab+1)xx+(abc+a-b+c)x=bc+1$$

atque

$$x=\frac{-abc-a+b-c+\sqrt{((abc+a+b+c)^2+4)}}{2(ab+1)},$$

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum; neque ergo haec forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma iam suffecerat. Simili modo si quatuor litterae a, b, c, d continuo repetitae denominatores fractionis continuae constituant, tum ea plus non inserviet quam secunda, quae duas tantum litteras continebat, et ita porro.

380. Cum igitur fractiones continuae tam utiliter ad extractionem radicis quadratae adhiberi queant, simul inservient aequationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per aequationem quadraticam affectam determinatur. Potest autem vicissim facile cuiusque aequationis quadratae radix per fractionem continuam hoc modo exprimi. Sit proposita ista aequatio

$$xx=ax+b;$$

ex qua cum sit

$$x=a+\frac{b}{x},$$

substituatur in ultimo termino loco x valor idem iam inventus eritque

$$x=a+\cfrac{b}{a+\cfrac{b}{x}}$$

simili ergo modo procedendo erit per fractionem continuam infinitam

$$x=a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\text{etc.}}}}$$

quae autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem continuam converti posse. Sit enim proposita fractio

$$x = \frac{A}{B},$$

in qua sit $A > B$; dividatur A per B sitque quotus $= a$ et residuum C; tum per hoc residuum C dividatur praecedens divisor B prodeatque quotus b et relinquatur residuum D, per quod denuo praecedens divisor C dividatur; sicque haec operatio, quae vulgo ad maximum communem divisorem numerorum A et B investigandum usurpari solet, continuetur, donec ipsa finiatur, sequenti modo:

$$\begin{array}{l} B)\,\underline{A}\,(a \\ \quad C)\,\underline{B}\,(b \\ \qquad D)\,\underline{C}\,(c \\ \qquad\quad E)\,\underline{D}\,(d \\ \qquad\qquad F \text{ etc.} \end{array}$$

Eritque per naturam divisionis

$$\begin{array}{lll} A = aB + C, \quad \text{unde} & \dfrac{A}{B} = a + \dfrac{C}{B}, & \\[2ex] B = bC + D, & \dfrac{B}{C} = b + \dfrac{D}{C}, & \dfrac{C}{B} = \dfrac{1}{b + \frac{D}{C}}, \\[2ex] C = cD + E, & \dfrac{C}{D} = c + \dfrac{E}{D}, & \dfrac{D}{C} = \dfrac{1}{c + \frac{E}{D}}, \\[2ex] D = dE + F & \dfrac{D}{E} = d + \dfrac{F}{E}, & \dfrac{E}{D} = \dfrac{1}{d + \frac{F}{E}} \\[2ex] \quad \text{etc.,} & & \text{etc.} \end{array}$$

Hinc sequentes valores in praecedentibus substituendo erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \cfrac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{E}{D}}}$$

unde tandem x per meros quotos inventos a, b, c, d etc. sequentem in modum exprimetur, ut sit

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \text{etc.}}}}}}$$

EXEMPLUM 1

Sit proposita ista fractio $\frac{1461}{59}$, quae sequenti modo in fractionem continuam transmutabitur, cuius omnes numeratores erunt unitates.

Instituatur scilicet eadem operatio, qua maximus communis divisor numerorum 59 et 1461 quaeri solet:

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

EXEMPLUM 2

Fractiones quoque decimales eodem modo transmutari poterunt.

Sit enim proposita

$$\sqrt{2} = 1{,}414221356 = \frac{141\,421\,356}{100\,000\,000},$$

unde haec operatio instituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418328	1010160	2
86752	209164	

etc.

Ex qua operatione iam patet omnes denominatores esse 2 atque adeo esse

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\text{etc.}}}}}}$$

cuius expressionis ratio iam ex superioribus patet.

EXEMPLUM 3

Imprimis vero etiam hic attentione dignus est numerus e, cuius logarithmus est $= 1$, qui est

$$e = 2{,}718281828459.$$

Unde oritur

$$\frac{e-1}{2} = 0{,}8591409142295,$$

quae fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704[1])	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	

etc.

Si iste calculus exactius adhuc assumpto valore ipsius e ulterius continuetur, tum prodibunt isti quoti

$$1,\ 6,\ 10,\ 14,\ 18,\ 22,\ 26,\ 30,\ 34 \text{ etc.},$$

qui dempto primo progressionem arithmeticam constituunt, unde patet fore

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\text{etc.}}}}}}}}$$

cuius fractionis ratio ex Calculo infinitesimali dari potest. [2])

382. Cum igitur ex huiusmodi expressionibus fractiones erui queant, quae quam citissime ad verum valorem expressionis deducant, haec methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quae ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita, cuius numerator et denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt, quae, etiamsi propositae non sint peni-

1) Revera oritur 1408590857705, quo valore assumpto calculus sequens non mediocriter mutatur. F. R.

2) Fractionem istam EULERUS primum in Commentatione 71 nota p. 362 laudata exposuit. F. R.

tus aequales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO[1]) olim tractatum facile resolvi potest, quo quaeruntur fractiones minoribus numeris expressae, quae tam prope exhauriant valorem fractionis cuiuspiam in numeris maioribus propositae, quantum fieri poterit numeris non maioribus. Fractiones autem nostra hac methodo ortae tam prope ad valorem fractionis continuae, ex qua eliciuntur, accedunt, ut nullae numeris non maioribus constantes dentur, quae propius accedant.

EXEMPLUM 1

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri maiores adhibeantur.

Si fractio decimalis cognita

$$3{,}1415926535 \text{ etc.}$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3, \ 7, \ 15, \ 1, \ 292, \ 1, \ 1 \ \text{etc.},$$

ex quibus sequentes fractiones formabuntur

$$\frac{1}{0}, \ \frac{3}{1}, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102} \ \text{etc.}$$

Secunda fractio iam ostendit esse diametrum ad peripheriam ut $1:3$ neque certe numeris non maioribus accuratius dari poterit. Tertia fractio dat rationem ARCHIMEDEAM[2]) $7:22$, at quinta METIANAM[3]), quae ad verum tam prope accedit, ut error minor sit parte $\frac{1}{113 \cdot 33102}$. Ceterum hae fractiones alternatim vero sunt maiores minoresque.

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1) J. WALLIS, *Opera mathematica* t. II, Oxoniae 1693, cap. 98—99, p. 418—429. Cf. quoque cap. 56—61, p. 232—250. F. R.

2) ARCHIMEDES (287—212 a. Chr. n.), *Opera* (ed. J. L. HEIBERG), vol. I, Lipsiae 1880, p. 257. F. R.

3) ADRIAAN ANTONISZOON, vulgo cognomine METIUS appellatus (1527—1607). Vide exempli gratia filii ADRIANI METII (1571—1635) librum, qui inscribitur *Arithmeticae libri duo et Geometriae libri VI,* Lugd. Batav. 1626, Geometriae pars prior, cap. X. F. R.

EXEMPLUM 2

Exprimatur ratio diei ad annum solarem medium in numeris minimis proxime.

Cum annus iste sit $365^d\, 5^h\, 48'\, 55''$, continebit in fractione annus unus

$$365\,\frac{20935}{86400}$$

dies. Tantum ergo opus est, ut haec fractio evolvatur, quae dabit sequentes quotos

$$4,\ 7,\ 1,\ 6,\ 1,\ 2,\ 2,\ 4,$$

unde istae eliciuntur fractiones

$$\frac{0}{1},\ \frac{1}{4},\ \frac{7}{29},\ \frac{8}{33},\ \frac{55}{227},\ \frac{63}{260},\ \frac{181}{747}\ \text{etc.}$$

Horae ergo cum minutis primis et secundis, quae supra 365 dies adsunt, quatuor annis unum diem circiter faciunt, unde calendarium JULIANUM originem habet. Exactius autem 33 annis 8 dies implentur vel 747 annis 181 dies; unde sequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium JULIANUM inserat 100 dies, GREGORIANUM quaternis seculis tres annos bissextiles in communes convertit.

FINIS TOMI PRIMI.

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