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INSTITUTIONUM CALCULI INTEGRALIS

VOLUMEN TERTIUM

IN QUO METHODUS INVENIENDI FUNCTIONES DUARUM ET PLURIUM VARIABILIUM, EX DATA RELATIONE DIFFERENTIALIUM CUJUSVIS GRADUS PERTRACTATUR.

UNA CUM APPENDICE DE CALCULO VARIATIONUM ET SUPPLEMENTO, EVOLUTIONEM CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM CONTINENTE.

Editio tertia.

PETROPOLI,
Impensis Academiae Imperialis Scientiarum
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in Volumine tertio contentorum.

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CAPUT I.

DE

NATURA AEQUATIONUM DIFFERENTIALIUM QUIBUS FUNCTIONES DUARUM VARIABILIUM DETERMINANTUR IN GENERE.

Problema 1.

1.

Si z sit functio quaecunque duarum variabilium x et y, definire indolem aequationis differentialis, qua relatio differentialium ∂x, ∂y et ∂z exprimitur.

Solutio.

Sit $P\partial x + Q\partial y + R\partial z = 0$ aequatio relationem differentialium ∂x, ∂y et ∂z exprimens, in qua P, Q et R sint functiones quaecunque ipsarum x, y et z. Ac primo quidem necesse est, ut haec aequatio nata sit ex differentiatione aequationis cujuspiam finitae, postquam differentiale per quampiam quantitatem fuerit divisum. Dabitur ergo quidam multiplicator puta M, per quem formula $P\partial x + Q\partial y + R\partial z$ multiplicata fiat integrabilis; nisi enim talis multiplicator existeret, aequatio differentialis proposita foret absurda, nihilque omnino declararet. Totum ergo negotium huc redit, ut character assignetur, cujus ope hujusmodi aequationes diffe-

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rentiales absurdae nihilque significantes a realibus dignosci queant. Hunc in finem contemplemur aequationem propositam $P\partial x + Q\partial y + R\partial z = 0$ tanquam realem. Sit M multiplicator eam reddens integrabilem, ita ut haec formula

$$MP\partial x + MQ\partial y + MR\partial z$$

sit verum differentiale cujuspiam functionis trium variabilium x, y et z; quae functio si ponatur $= V$, haec aequatio $V = \text{Const.}$ futura sit integrale completum aequationis propositae. Sive igitur x, sive y, sive z accipiatur constans, singulas has formulas

$$MQ\partial y + MR\partial z,\quad MR\partial z + MP\partial x,\quad MP\partial x + MQ\partial y,$$

seorsim integrabiles esse oportet; unde ex natura differentialium erit

$$\left(\frac{\partial \,.\, MQ}{\partial z}\right) - \left(\frac{\partial \,.\, MR}{\partial y}\right) = 0,\quad \left(\frac{\partial \,.\, MR}{\partial x}\right) - \left(\frac{\partial \,.\, MP}{\partial z}\right) = 0,$$
$$\left(\frac{\partial \,.\, MP}{\partial y}\right) - \left(\frac{\partial \,.\, MQ}{\partial x}\right) = 0,$$

unde per evolutionem hae tres oriuntur aequationes

$$\text{I.}\quad M\left(\frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial M}{\partial z}\right) - M\left(\frac{\partial R}{\partial y}\right) - R\left(\frac{\partial M}{\partial y}\right) = 0$$

$$\text{II.}\quad M\left(\frac{\partial R}{\partial x}\right) + R\left(\frac{\partial M}{\partial x}\right) - M\left(\frac{\partial P}{\partial z}\right) - P\left(\frac{\partial M}{\partial z}\right) = 0$$

$$\text{III.}\quad M\left(\frac{\partial P}{\partial y}\right) + P\left(\frac{\partial M}{\partial y}\right) - M\left(\frac{\partial Q}{\partial x}\right) - Q\left(\frac{\partial M}{\partial x}\right) = 0$$

quarum si prima per P, secunda per Q et tertia per R multiplicetur, in summa omnia differentialia ipsius M se tollent, et reliqua aequatio per M divisa erit

$$P\left(\frac{\partial Q}{\partial z}\right) - P\left(\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}\right) - Q\left(\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y}\right) - R\left(\frac{\partial Q}{\partial x}\right) = 0$$

quae continet characterem, aequationes differentiales reales ab absurdis discernentem, et quoties inter quantitates P, Q et R haec conditio locum habet, toties aequatio differentialis proposita

$$P\partial x + Q\partial y + R\partial z = 0$$

est realis. Caeterum hic meminisse oportet, hujusmodi formulam uncinulis inclusam $\left(\frac{\partial Q}{\partial z}\right)$ significare valorem $\frac{\partial Q}{\partial z}$, si in differentiatione ipsius Q sola quantitas z ut variabilis tractetur; quod idem de caeteris est tenendum, quae ergo semper ad functiones finitas reducuntur.

Corollarium 1.

2. Proposita ergo aequatione differentiali inter tres variabiles

$$P\partial x + Q\partial y + R\partial z = 0,$$

ante omnia dispiciendum est, utrum character inventus locum habeat, nec ne? priori casu aequatio erit realis, posteriori vero absurda et nihil plane significans, neque unquam ad talem aequationem ullius problematis solutio perducere valet.

Corollarium 2.

3. Character inventus etiam hoc modo exprimi potest

$$\left(\frac{P\partial Q - Q\partial P}{\partial z}\right) + \left(\frac{Q\partial R - R\partial Q}{\partial x}\right) + \left(\frac{R\partial P - P\partial R}{\partial y}\right) = 0,$$

quandoquidem uncinulae non quantitates finitas afficiunt, sed solam differentiationem ad certam variabilem restringunt.

Corollarium 3.

4. Simili modo si aequatio haec characterem continens per PQR dividatur, ea hanc formam induet

$$\left(\frac{\partial . l\frac{Q}{P}}{R\partial z}\right) + \left(\frac{\partial . l\frac{R}{Q}}{P\partial x}\right) + \left(\frac{\partial . l\frac{P}{R}}{Q\partial y}\right) = 0$$

quae etiam ita exprimi potest

$$\left(\frac{\frac{\partial Q}{Q} - \frac{\partial P}{P}}{R\partial z}\right) + \left(\frac{\frac{\partial R}{R} - \frac{\partial Q}{Q}}{P\partial x}\right) + \left(\frac{\frac{\partial P}{P} - \frac{\partial R}{R}}{Q\partial y}\right) = 0.$$

Scholion 1.

5. Quemadmodum omnes aequationes differentiales inter binas variabiles semper sunt reales, semperque per eas relatio certa inter ipsas variabiles definitur, ita hinc discimus, rem secus se habere in aequationibus differentialibus, quae tres variabiles involvant, atque hujusmodi aequationes

$$P\partial x + Q\partial y + R\partial z = 0$$

non certam relationem inter ipsas quantitates finitas x, y et z declarare, nisi quantitates P, Q, R ita fuerint comparatae, ut character inventus locum habeat. Ex quo intelligitur infinitas hujusmodi aequationes differentiales inter ternas variabiles proponi posse, quibus nulla prorsus relatio finita conveniat, et quae propterea nihil plane definiant. Pro arbitrio scilicet hujusmodi aequationes formari possunt, nullo scopo proposito ad quem sint accommodatae; statim enim ac certum quoddam problema ad aequationem differentialem inter ternas variabiles perducit, semper necesse est characterem assignatum ei convenire, cum alioquin nihil omnino significaret. Talis aequatio nihil significans est exempli gratia

$$z\partial x + x\partial y + y\partial z = 0,$$

neque pro z ulla quidem functio ipsarum x et y cogitari potest quae isti aequationi satisfaciat; quin etiam character noster pro hoc exemplo dat $-x-y-z$, quae quantitas cum non evanescat, absurditatem illius aequationis declarat.

Scholion 2.

6. Quo character inventus facilius ad quosvis casus oblatos accommodari queat, ex aequatione

$$P\partial x + Q\partial y + R\partial z = 0$$

primo evolvantur sequentes valores

$$\left(\frac{\partial Q}{\partial z}\right) - \left(\frac{\partial R}{\partial y}\right) = L, \quad \left(\frac{\partial R}{\partial x}\right) - \left(\frac{\partial P}{\partial z}\right) = M, \quad \left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial x}\right) = N,$$

et character noster hac continebitur expressione

$$LP + MQ + NR,$$

quae si evanescat, aequatio proposita erit realis, et aequationem quandam finitam agnoscet; sin autem ea ad nihilum non redigatur, aequatio proposita erit absurda, atque de ejus integratione ne cogitandum quidem erit. Ita in exemplo supra posito erit

$$P = z, \quad Q = x, \quad R = y,$$

hinc

$$L = -1, \quad M = -1, \quad N = -1,$$

unde character $-x-y-z$ absurditatem indicat. Proferamus vero etiam exemplum aequationis realis

$$\partial x\,(yy + nyz + zz) - x\,(y + nz)\,\partial y - xz\partial z = 0,$$

in qua ob

$$P = yy + nyz + zz, \quad Q = -xy - nxz \quad \text{et} \quad R = -xz,$$

erit

$$L = -nx, \quad M = -3z - ny \quad \text{et} \quad N = 3y + 2nz,$$

unde

$LP + MQ + NR$

$$= -nx\,(yy + nyz + zz) + x\,(y + nz)\,(3z + ny) - xz\,(3y + 2nz)$$

$$= x\,[-nyy - nnyz - nzz + 3yz + 3nzz + nyy + nnyz - 3yz - 2nzz] = 0,$$

quare cum hic character evanescat, aequatio haec differentialis pro reali est habenda. Simili modo proposita hac aequatione

$$2\partial x\,(y + z) + \partial y\,(x + 3y + 2z) + \partial z\,(x + y) = 0, \quad \text{ob}$$

$$P = 2y + 2z, \quad Q = x + 3y + 2z, \quad R = x + y, \quad \text{fit}$$

$$L = 2 - 1 = 1, \quad M = 1 - 2 = -1, \quad \text{et} \quad N = 2 - 1 = 1,$$

hincque

$$LP + MQ + NR = 2y + 2z - x - 3y - 2z + x + y = 0,$$

unde ista aequatio differentialis erit realis.

Problema 2.

7. Proposita aequatione differentiali inter ternas variabiles x, y, z, quae sit realis, ejus integrale investigare, unde pateat, qualis functio una earum sit binarum reliquarum.

Solutio.

Sit aequatio differentialis proposita

$$P\partial x + Q\partial y + R\partial z = 0,$$

in qua P, Q, R, ejusmodi sint functiones ipsarum x, y et z, ut character realitatis ante inventus satisfaciat. Nisi enim ista aequatio esset realis, ridiculum foret, ejus integrationem tentare. Sumamus ergo hanc aequationem esse realem, atque dabitur relatio inter ipsas quantitates x, y et z, aequationi propositae satisfaciens, ad quam inveniendam perpendatur, si in aequatione integrali una variabilium, puta z, constans spectetur, ex ejus differentiali nihilo aequali posito nasci debere aequationem

$$P\partial x + Q\partial y = 0.$$

Vicissim ergo una variabili puta z ut constante tractata, integratio aequationis differentialis

$$P\partial x + Q\partial y = 0,$$

quae duas tantum variabiles continet, perducet ad aequationem integralem quaesitam, si modo in quantitatem constantem per integrationem ingressam illa quantitas z rite involvatur. Ex quo hanc regulam pro integratione aequationis propositae colligimus. Consideretur una variabilium puta z ut constans, ut habeatur haec aequatio $P\partial x + Q\partial y = 0$, duas tantum variabiles x et y implicans; tum ejus investigetur aequatio integralis completa, quae ergo constantem arbitrariam C complectetur. Deinde haec constans C consideretur ut functio quaecunque ipsius z, atque hac z nunc etiam pro variabili habita, aequatio integralis inventa denuo differentietur; ut omnes tres x, y et z tanquam variabiles tractentur, et aequatio differentialis resultans comparetur cum proposita

$$P\partial x + Q\partial y + R\partial z = 0,$$

vbi quidem functiones P et Q sponte prodibunt, at functio R cum ea quantitate, qua elementum ∂z afficitur, collata determinabit rationem, qua quantitas z in illam litteram C ingreditur, sicque obtinebitur aequatio integralis quaesita, quae simul erit completa, cum semper in illa litterae C pars quaedam constans vere arbitraria relinquatur, cum haec determinatio ex differentiali ipsius C sit petenda.

Corollarium 1.

8. Reducitur ergo integratio hujusmodi aequationum differentialium tres variabiles continentium ad integrationem aequationum differentialium inter duas tantum variabiles, quae ergo quoties licet per methodos in superiori libro traditas, est instituenda.

Corollarium 2.

9. Haec ergo integratio tribus modis institui potest, prout primo vel z, vel y, vel x tanquam constans spectatur. Semper autem necesse est, ut eadem aequatio integralis resultet, siquidem aequatio differentialis fuerit realis.

Corollarium 3.

10. Quodsi haec methodus tentetur in aequatione differentiali impossibili, determinatio illius constantis C non ita succedet,

ut eam variabilem, quae pro constante est habita, solam involvat; atque etiam ex hoc criterium realitatis peti poterit.

Scholion.

11. Quo haec operatio facilius intelligatur, periculum faciamus primo in aequatione impossibili hac

$$z \partial x + x \partial y + y \partial z = 0,$$

hic sumta z pro constante erit

$$z \partial x + x \partial y = 0, \text{ seu } \frac{z \partial x}{x} + \partial y = 0,$$

cujus integrale est $zlx + y = C$, existente C functione ipsius z. Differentietur ergo haec aequatio sumendo etiam z variabile, positoque $\partial C = D \partial z$, ut D sit etiam functio ipsius z tantum, erit

$$\frac{z \partial x}{x} + \partial y + \partial z l x = D \partial z, \text{ seu}$$
$$z \partial x + x \partial y + \partial z (x l x - D x) = 0:$$

deberet ergo esse $xlx - Dx = y$, seu $D = lx - \frac{y}{x}$, quod est absurdum.

Deinde in aequatione reali

$$2 \partial x (y + z) + \partial y (x + 3y + 2z) + \partial z (x + y) = 0$$

operatio exposita ita instituatur. Sumatur y constans, ut sit

$$2\partial x (y + z) + \partial z (x + y) = 0, \text{ seu } \frac{2 \partial x}{x + y} + \frac{\partial z}{y + z} = 0,$$

cujus integrale est

$$2 l (x + y) + l (y + z) = C,$$

ubi C etiam y involvat. Sit ergo $\partial C = D \partial y$, et sumto etiam y variabili, differentiatio praebet

$$\frac{2 \partial x + 2 \partial y}{x + y} + \frac{\partial y + \partial z}{y + z} = D \partial y, \text{ seu}$$

$$2\partial x(y+z)+2\partial y(y+z)+\partial y(x+y)+\partial z(x+y)$$
$$=D\partial y(x+y)(y+z),$$

quae expressio cum forma proposita collata praebet $D=0$, ideoque $\partial C=0$, et C fit constans vera; ita ut integrale sit

$$(x+y)^2(y+z)=\text{Const.}$$

Hujusmodi igitur exempla aliquot evolvamus.

Exemplum 1.

12. *Hujus aequationis realis*

$$\partial x(y+z)+\partial y(x+z)+\partial z(x+y)=0$$

integrale investigare.

Primo quidem patet hanc aequationem esse realem, cum sit

$$P=y+z,\quad L=1-1=0,$$
$$Q=x+z,\quad M=1-1=0,$$
$$R=x+y,\quad N=1-1=0,$$

sumatur igitur z constans, et aequatio prodibit

$$\partial x(y+z)+\partial y(x+z)=0,\ \text{seu}\ \frac{\partial x}{x+z}+\frac{\partial y}{y+z}=0,$$

cujus integrale est

$$l(x+z)+l(y+z)=f:z.$$

Statuatur ergo

$$(x+z)(y+z)=Z,$$

ubi natura functionis Z ex differentiatione debet erui. Fit autem

$$\partial x(y+z)+\partial y(x+z)+\partial z(x+y+2z)=\partial Z,$$

a qua si proposita auferatur, relinquitur $2z\partial z=\partial Z$; hinc $Z=zz+C$, ita ut aequatio integralis completa sit

$$(x+z)(y+z) = zz + C, \text{ seu } xy + xz + yz = C;$$

quae quidem ex ipsa proposita

$$y\partial x + z\partial x + x\partial y + z\,\partial y + x\partial z + y\partial z = 0,$$

facile elicitur, cum bina membra juncta sit integrabilia.

Exemplum 2.

13. *Hujus differentialis aequationis realis*

$$\partial x\,(ay - bz) + \partial y\,(cz - ax) + \partial z\,(bx - cy) = 0$$

aequationem integralem completam invenire.

Realitas hujus aequationis ita ostenditur. Cum sit

$$\begin{aligned} P &= ay - bz, & \text{erit } L &= 2c, \\ Q &= cz - ax, & M &= 2b, \\ R &= bx - cy, & N &= 2a, \end{aligned}$$

hincque manifesto $LP + MQ + NR = 0$. Jam sumatur z constans; ut habeatur

$$\frac{\partial x}{cz - ax} + \frac{\partial y}{ay - bz} = 0, \text{ ergo } \frac{1}{a}\, l \frac{ay - bz}{cz - ax} = f:z,$$

statuatur ergo $\frac{ay - bz}{cz - ax} = Z$, et differentiatio praebet

$$\frac{a\partial x\,(ay - bz) + a\partial y\,(cz - ax) + a\partial z\,(bx - cy)}{(cz - ax)^2} = \partial Z,$$

ex cujus comparatione cum proposita fit $\partial Z = 0$ et $Z = C$, ita ut aequatio integralis completa sit

$$\frac{ay - bz}{cz - ax} = n, \text{ seu } ay + nax = (b + nc)\,z.$$

Quodsi aequatio integralis ponatur

$$Ax + By + Cz = 0,$$

hae constantes ita debent esse comparatae, ut sit

$$Ac + Bb + Ca = 0,$$

sicque constans arbitraria concinnius inducitur.

Corollarium.

14. Haec ergo aequatio integrabilis redditur, si dividatur per $(cz - ax)^2$, atque ob eandem rationem etiam hi divisores

$$(ay - bz)^2 \text{ et } (bx - cy)^2$$

idem praestant. Vi enim integralis hi divisores constantem inter se tenent rationem. Namque si $\frac{ay - bz}{cz - ax} = n$, erit

$$\frac{bx - cy}{cz - ax} = \frac{-b - nc}{a}, \text{ et } \frac{bx - cy}{ay - bz} = \frac{-b - nc}{na}.$$

Exemplum 3.

15. *Hujus aequationis differentialis realis*

$$\partial x(yy + yz + zz) + \partial y(zz + xz + xx) + \partial y(xx + xy + yy) = 0$$

aequationem integralem investigare.

Realitas hujus aequationis inde patet, quod sit

$P = yy + yz + zz$, hincque $L = 2z + x - x - 2y = 2(z - y)$,
$Q = zz + xz + xx$, $M = 2x + y - y - 2z = 2(x - z)$,
$R = xx + xy + yy$, $N = 2y + z - z - 2x = 2(y - x)$,

unde fit

$$LP + MQ + NR = 2(z^3 - y^3) + 2(x^3 - z^3) + 2(y^3 - x^3) = 0.$$

Ad integrale ergo investigandum sumatur z constans, eritque

$$\frac{\partial x}{xx + xz + zz} + \frac{\partial y}{yy + yz + zz} = 0,$$

cujus integrale est

$$\frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{x\sqrt{3}}{2z + x} + \frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{y\sqrt{3}}{2z + y} = f:z,$$

quae per collectionem horum angulorum abit in

$$\frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{(xz + yz + xy)\sqrt{3}}{2zz + xz + yz - xy} = f:z.$$

Statuatur ergo $\frac{xz + yz + xy}{2zz + xz + yz + xy} = Z$, haecque aequatio differentietur sumtis omnibus tribus x, y et z variabilibus, ac prodibit

$$\frac{2z\partial x(yy+yz+zz)+2z\partial y(zz+xz+xx)-2x\partial z(zz+yz+yy)-2y\partial z(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

cum igitur ex aequatione proposita sit

$$\partial x(yy+yz+zz)+\partial y(zz+xz+xx)=-\partial z(xx+xy+yy),$$

erit facta substitutione

$$\frac{-2z\partial z(xx+xy+yy)-2x\partial z(zz+yz+yy)-2y\partial z(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

seu

$$\frac{-2\partial z(xxz+xzz+yyz+yzz+xxy+xyy+3xyz)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

quae in hanc formam reducitur

$$\frac{-2\partial z(x+y+z)(xy+xz+yz)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

At ob $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$, erit

$$\frac{-2ZZ\partial z(x+y+z)}{xy+xz+yz}=\partial Z,\ \text{seu}\ \frac{-\partial Z}{ZZ}=\frac{2\partial z(x+y+z)}{xy+xz+yz}.$$

Necesse ergo est ut etiam $\frac{xy+xz+yz}{x+y+z}$ sit functio ipsius z tantum, quae vocetur Σ, ut sit $-\frac{\partial Z}{ZZ}=\frac{2\partial z}{\Sigma}$. Verum ex sola forma functionis Z negotium confici oportet; quod ita expediri potest. Cum sit

$$Z=\frac{xy+xz+yz}{2zz+xz+yz-xy},\ \text{erit}\ 1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xy},$$

hinc $\frac{1+Z}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$, cujus valoris ope quantitates x et y ex aequatione differentiali eliduntur, fitque

$$-\frac{\partial Z}{ZZ}=\partial z\cdot\frac{2(x+y+z)}{xy+xz+yz}=\partial z\cdot\frac{1+Z}{Zz};\ \text{unde}$$

$$\frac{-\partial Z}{Z(1+Z)}=\frac{\partial z}{z}=\frac{-\partial Z}{Z}+\frac{\partial Z}{1+Z},$$

et integrando $lz=l\frac{1+Z}{Z}+la.$

Ergo $\frac{1+Z}{Z}=\frac{z}{a}$ et $Z=\frac{a}{z-a}$,

ita ut aequatio integralis quaesita sit

$$\frac{a}{z-a} = \frac{xy+xz+yz}{2zz+xz+yz-xy}, \text{ seu } xy+xz+yz = a(x+y+z)$$

quae simplicissima forma statim colligitur ex aequatione

$$\frac{2z(x+y+z)}{xy+xz+yz} = \frac{1+Z}{Z} = \frac{z}{a},$$

Corollarium.

16. Cum aequationis propositae integrale completum sit

$$xy+xz+yz = (x+y+z) \text{ seu } \frac{xy+xz+yz}{x+y+z} = \text{Const.}$$

ex hujus differentiatione etiam ipsa aequatio proposita resultare deprehenditur. Unde patet aequationem propositam integrabilem reddi, si dividatur per

$$(x+y+z)^2, \text{ vel etiam per } (xy+xz+yz)^2.$$

Scholion.

17. Ex hoc exemplo intelligitur, determinationem functionis per integrationem illatae interdum haud exiguis difficultatibus esse obnoxiam; siquidem hic functionem Z non sine ambagibus elicuimus. Verum et hic ista investigatio multo facilius institui potuisset; statim enim atque invenimus

$$\frac{xy+xz+yz}{2zz+xz+yz-xy} = Z = f:z,$$

hanc ipsam expressionem concinniorem reddere licuisset. Nempe cum sit

$$\frac{1}{Z} = \frac{2zz+xz+yz-xy}{xy+xz+yz}, \text{ erit}$$

$$1+\frac{1}{Z} = \frac{2z(x+y+z)}{xy+xz+yz}, \text{ ideoque}$$

$$\frac{xy+xz+yz}{x+y+z} = \frac{2Zz}{1+Z} = f:z.$$

Relicta ergo functione Z statim ponatur

$$\frac{xy+xz+yz}{x+y+z} = \Sigma = f:z,$$

et sumtis differentialibus per se liquebit, fieri $\partial\Sigma = 0$, ideoque

$\Sigma =$ Const. Adhuc facilius hoc problema resolvitur, si etiam sumto y constante ejus integrale quaeratur, tum enim simili modo pervenitur ad hujusmodi aequationem

$$\frac{xy + xz + yz}{x + y + z} = Y = f : y;$$

quare cum haec expressio aeque esse debeat functio ipsius z atque ipsius y, necesse est, ut ea sit constans; eritque propterea aequatio integralis completa

$$xy + xz + yz = a\,(x + y + z).$$

Exemplum 4.

18. *Hujus aequationis differentialis realis*

$$\partial x(xx - yy + zz) - zz\partial y + z\partial z(y - x) + \frac{x\partial z}{z}(yy - xx) = 0$$

aequationem integralem completam investigare.

Realitas hujus aequationis ita ostenditur.

Ob $P = xx - yy + zz$, erit $L = -3z - \frac{2xy}{z}$

$Q = -zz$, $M = -3z + \frac{yy}{z} - \frac{3xx}{z}$

$R = z\,(y - x) + \frac{x}{z}(yy - xx)$, $N = -2y$;

unde calculo subducto formula $LP + MQ + NR$ evanescit.

Sumamus jam z constans, et habebimus hanc aequationem

$$\partial x\,(xx - yy + zz) - zz\partial y = 0,$$

cujus quidem integratio non constaret, nisi perspiceremus ei satisfacere particulariter $y = x$. Hinc autem ponendo $y = x + \frac{zz}{v}$, integrale completum eruere poterimus; fit enim

$$\partial x\,(zz - \frac{2xzz}{v} - \frac{z^4}{vv}) - zz\partial x + \frac{z^4\partial v}{vv} = 0$$

hincque $\partial v - \frac{2xv\partial x}{zz} = \partial x$, quae per $e^{\frac{-xx}{zz}}$ multiplicata praebet integrale

$$e^{\frac{-xx}{zz}} v = \int e^{\frac{-xx}{zz}} \partial x + f:z;$$

ubi quidem notandum est in integratione formulae $\int e^{\frac{-xx}{zz}} \partial x$ quantitatem z ut constantem tractari, esseque $v = \frac{zz}{y-x}$: ita ut sit

$$\int e^{\frac{-xx}{zz}} \partial x = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z.$$

Quodsi jam hanc aequationem differentiare velimus sumta etiam z variabili, difficultas hic occurrit, quomodo quantitas $\int e^{\frac{-xx}{zz}} \partial x$ differentiale ex variabilitate ipsius z oriundum definiri debeat. Hic ex principiis repeti debet, si fuerit $\partial V = S \partial x + T \partial z$, fore $\left(\frac{\partial T}{\partial x}\right) = \left(\frac{\partial S}{\partial z}\right)$, ideoque si z constans sumatur, $T = \int \partial x \left(\frac{\partial S}{\partial z}\right)$. Jam nostro casu est

$$S = e^{\frac{-xx}{zz}} \text{ et } V = \int e^{\frac{-xx}{zz}} \partial x, \text{ sumta } z \text{ constante};$$

quare cum sit

$$\left(\frac{\partial S}{\partial z}\right) = e^{\frac{-xx}{zz}} \cdot \frac{2xx}{z^3}, \text{ erit } T = \frac{2}{z^3} \int e^{\frac{-xx}{zz}} xx \partial x.$$

Quocirca quantitatis $\int e^{\frac{-xx}{zz}} \partial x$ differentiale plenum ex variabilitate utriusque x et z oriundum est

$$e^{\frac{-xx}{zz}} \partial x + \frac{2 \partial z}{z^3} \int e^{\frac{-xx}{zz}} xx \partial x,$$

cui aequari debet alterius partis $\frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z$ differentiale, quod est

$$e^{\frac{-xx}{zz}} \left(\frac{2z\partial z}{y-x} - \frac{zz\partial y + zz\partial x}{(y-x)^2} + \frac{2xx\partial z - 2xz\partial x}{z(y-x)}\right) + \partial Z.$$

Turbat vero adhuc formula integralis $\int e^{-\frac{xx}{zz}} x x \partial x$, in qua z pro constante habetur: reduci autem potest ad priorem $\int e^{-\frac{xx}{zz}} \partial x$, si ponatur

$$\int e^{-\frac{xx}{zz}} x x \partial x = A e^{-\frac{xx}{zz}} x + B \int e^{-\frac{xx}{zz}} \partial x,$$

prodit enim sola x pro variabili habita, differentiando

$$x x \partial x = A \partial x - \frac{2 A x x \partial x}{zz} + B \partial x; \text{ ergo}$$

$$A = -\tfrac{1}{2} z z, \text{ et } B = -A = \tfrac{1}{2} z z,$$

ita ut sit

$$\int e^{-\frac{xx}{zz}} x x \partial x = -\tfrac{1}{2} e^{-\frac{xx}{zz}} x z z + \tfrac{1}{2} z z \int e^{-\frac{xx}{zz}} \partial x:$$

quare cum sit

$$\int e^{-\frac{xx}{zz}} \partial x = \frac{e^{-\frac{xx}{zz}} z z}{y - x} + Z, \text{ erit}$$

$$\int e^{-\frac{xx}{zz}} x x \partial x = -\tfrac{1}{2} e^{-\frac{xx}{zz}} x z z + \frac{e^{-\frac{xx}{zz}} z^4}{2(y-x)} + \tfrac{1}{2} Z z z.$$

Facta ergo substitutione haec orietur aequatio differentialis

$$e^{-\frac{xx}{zz}} \left(\partial x - \frac{x \partial z}{z} + \frac{z \partial z}{y-x}\right) + \frac{Z \partial z}{z} =$$

$$e^{-\frac{xx}{zz}} \left(\frac{2 z \partial z}{y - x} - \frac{z z \partial y}{(y-x)^2} + \frac{z z \partial x}{(y-x)^2} - \frac{2 x \partial x}{y - x} + \frac{2 x x \partial z}{z (y-x)}\right) + \partial Z,$$

quae transit in hanc formam

$$e^{-\frac{xx}{zz}} \left(\frac{\partial x \, (y+x)}{y - x} - \frac{z z \partial x}{(y-x)^2} + \frac{z z \partial y}{(y-x)^2} - \frac{z \partial z}{y-x} - \frac{x (y+x) \partial z}{z (y-x)}\right) = \frac{z \partial Z - Z \partial z}{z}$$

seu

$$\frac{e^{-\frac{xx}{zz}}}{(y-x)^2} \left[\partial x (yy-xx-zz) + zz \partial y - z \partial z (y-x) - \frac{x \partial z}{z} (yy-xx)\right] = \frac{z \partial Z - Z \partial z}{z}.$$

qua cum proposita collata evidens est, esse debere

$$z \partial Z - Z \partial z = 0 \text{ seu } Z = nz;$$

ita ut aequationis propositae integrale completum sit

$$\int e^{\frac{-xx}{zz}} \partial x = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + nz,$$

siquidem in integrali $\int e^{\frac{-xx}{zz}} \partial x$ quantitas z pro constante habeatur.

Corollarium.

19. Aequatio ergo proposita integrabilis redditur, si multiplicetur per $\frac{1}{(y-x)^2} e^{\frac{-xx}{zz}}$; ac tum integrale est ipsa aequatio, quam invenimus.

Scholion 1.

20. Exemplum hoc imprimis est notatu dignum, quod in ejus solutione quaedam artificia sunt in subsidium vocata, quibus in praecedentibus non erat opus. Per formulam autem $\int e^{\frac{-xx}{zz}} \partial x$ integrale non satis determinatum videtur. Cum enim in ea z constans ponatur, constans per integrationem introducenda per nz non definitur, siquidem lex non praescribitur, secundum quam integrale $\int e^{\frac{-xx}{zz}} \partial x$ capi oporteat, utrum ita ut evanescat facto $x = 0$, an alio quocunque modo? Dubium autem hoc diluetur, si aequationem inventam per z dividamus, ut formula integralis sit $\int e^{\frac{-xx}{zz}} \frac{\partial x}{z}$; ubi cum $\frac{\partial x}{z}$ sit $\partial . \frac{x}{z}$, evidens est ea exprimi functionem quandam ipsius $\frac{x}{z}$; ac si ponatur $\frac{x}{z} = p$, fore aequationem nostram integralem

$$\int e^{-pp} \partial p + \text{Const.} = e^{-pp} \frac{z}{y-x}$$

neque hic amplius conditio illa, qua in formula integrali quantitas z pro constante sit habenda, locum habet, sed integrale perinde determinatur, ac si aequatio duas tantum variabiles contineret. Hanc circumstantiam si perpendissemus, plenum differentiale formulae $\int e^{\frac{-xx}{zz}}\partial x$, ex variabilitate utriusque x et z nullam difficultatem peperisset. Postquam enim pervenimus ad aequationem

$$\int e^{\frac{-xx}{zz}}\partial x = e^{\frac{-xx}{zz}}\cdot\frac{zz}{y-z}+f:z,$$

eam ita repraesentemus

$$\int e^{\frac{-xx}{zz}}\cdot\frac{\partial x}{z}=\int e^{\frac{-xx}{zz}}\partial\cdot\frac{x}{z}=e^{\frac{-xx}{zz}}\cdot\frac{z}{y-z}+Z,$$

ubi cum in formulam integralem etiam variabilitas ipsius z sit inducta, si ea differentietur sumtis omnibus x, y et z variabilibus orietur

$$e^{\frac{-xx}{zz}}\left(\frac{\partial x}{z}-\frac{x\partial z}{zz}\right)=e^{\frac{-xx}{zz}}\left(\frac{\partial z}{y-z}+\frac{z\partial x-z\partial y}{(y-x)^2}-\frac{2x\partial x}{z(y-x)}+\frac{2xx\partial z}{zz(y-x)}\right)+\partial Z$$

seu

$$e^{\frac{-xx}{zz}}\left(\frac{\partial x(y+x)}{z(y-x)}-\frac{z\partial x}{(y-x)^2}+\frac{z\partial y}{(y-x)^2}-\frac{x\partial z(y+x)}{zz(y-x)}-\frac{\partial z}{y-x}\right)=\partial Z$$

quae reducitur ad hanc formam.

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2}\left[\partial x(yy-xx-zz)+zz\partial y-z\partial z(y-x)-\frac{x\partial z}{z}(yy-xx)\right]=\partial Z;$$

unde patet esse debere $\partial Z=0$ et $Z=$ Const. sicque elicitur aequatio integralis ante inventa.

Scholion 2.

21. Idem integrale prodiisset, si loco z altera reliquarum x vel y pro constante fuisset assumta; ubi in genere notari convenit, si hujusmodi aequationem

$$P\partial x + Q\partial y + R\partial z = 0$$

sumta z constante tractare licuerit, etiam resolutionem, quaecunque trium variabilium pro constante assumatur, succedere debere, etiamsi id quandoque minus perspiciatur. Ita in aequatione proposita si y pro constante habeatur, resolvenda erit haec aequatio

$$\partial x\,(xx + zz - yy) - z\partial z\,(x-y) - \frac{x\partial z}{z}(xx - yy) = 0,$$

quae per z multiplicata cum in hanc formam abeat

$$(z\partial x - x\partial z)(xx + zz - yy) + yzz\partial z = 0,$$

facile patet, eam simpliciorem reddi ponendo $x = pz$, tum enim ob

$$z\partial x - x\partial z = zz\partial p$$

prodibit

$$\partial p\,(ppzz + zz - yy) + y\partial z = 0;$$

sit porro $z = qy$, fietque

$$\partial p\,(ppqq + qq - 1) + \partial q = 0,$$

cui cum satisfaciat $q = \frac{1}{p}$, statuatur $q = \frac{1}{p} + \frac{1}{r}$, habebiturque

$$\partial p\left(\frac{2p}{r} + \frac{pp}{rr} + \frac{1}{pp} + \frac{2}{pr} + \frac{1}{rr}\right) - \frac{\partial p}{pp} - \frac{\partial r}{rr} = 0, \quad \text{seu}$$

$$\partial p\,(2ppr + p^3 + 2r + p) - p\partial r = 0, \quad \text{vel}$$

$$\partial r - \frac{2r\partial p\,(pp+1)}{p} = \partial p\,(pp + 1),$$

quae multiplicata per $\frac{1}{pp}e^{-pp}$ et integrata dat

$$e^{-pp}\frac{r}{pp} = \int e^{-pp}\,.\,\frac{\partial p\,(1+pp)}{pp}.$$

$$\text{At } \int e^{-pp}\frac{\partial p}{pp} = -e^{-pp}\frac{1}{p} - 2\int e^{-pp}\partial p,$$

$$\text{unde } e^{-pp}\left(\frac{r}{pp} + \frac{1}{p}\right) = -\int e^{-pp}\partial p.$$

Cum nunc sit

$$p = \frac{x}{z} \text{ et } \frac{1}{r} = \frac{z}{y} - \frac{z}{x} = \frac{z(x-y)}{xy}, \text{ erit}$$

$$r = \frac{xy}{z(x-y)}, \quad \frac{r}{pp} = \frac{yz}{x(x-y)} \text{ et } \frac{r}{pp} + \frac{1}{p} = \frac{z}{x-y}.$$

Unde aequatio nostra integralis erit

$$\int e^{\frac{-xx}{zz}} \partial \,.\, \frac{x}{z} = e^{\frac{-xx}{zz}} \,.\, \frac{z}{y-x} + f:y,$$

cujus differentiale, si etiam y pro variabili habeatur, cum aequatione proposita comparatum, dabit ut ante $f:y = \text{Const.}$

Caeterum cum in his exemplis variabiles x, y, z ubique eundem dimensionem numerum impleant, methodum generalem hujusmodi aequationes tractandi exponam.

Problema 3.

22. Si in aequatione differentiali

$$P\partial x + Q\partial y + R\partial z = 0$$

functiones P, Q, R fuerint homogeneae ipsarum x, y et z ejusdem numeri dimensionum; ejus integrationem, si quidem fuerit realis, investigare.

Solutio.

Sit n numerus dimensionum, quas ternae variabiles x, y et z in functionibus P, Q, R constituunt; ac posito $x = pz$ et $y = qz$, fiet

$$P = z^n S, \quad Q = z^n T \text{ et } R = z^n V,$$

ita ut jam S, T, V, futurae sint functiones binarum tantum variabilium p et q. Cum jam sit

$$\partial x = p\partial z + z\partial p \text{ et } \partial y = q\partial z + z\partial q,$$

aequatio nostra hanc induet formam

$$\partial z(pS + qT + V) + Sz\partial p + Tz\partial q = 0, \text{ seu}$$

$$\frac{\partial z}{z} + \frac{S\partial p + T\partial q}{pS + qT + V} = 0$$

quae aequatio realis esse nequit, nisi formula differentialis binas variabiles p et q involvens $\frac{S\partial p + T\partial q}{pS + qT + V}$ per se fuerit integrabilis; quod eveniet si fuerit

$$(qT + V)\left(\frac{\partial S}{\partial q}\right) + pT\left(\frac{\partial S}{\partial p}\right) - (pS + V)\left(\frac{\partial T}{\partial p}\right) - qS\left(\frac{\partial T}{\partial q}\right)$$
$$- S\left(\frac{\partial V}{\partial q}\right) + T\left(\frac{\partial V}{\partial p}\right) = 0.$$

Quoties ergo hic character locum habet, nostra aequatio erit realis, ejusque integrale erit

$$lz + \int \frac{S\partial p + T\partial q}{pS + qT + V} = \text{Const.}$$

ubi tantum opus est, ut loco litterarum p et q valores assumti $\frac{x}{z}$ et $\frac{y}{z}$ restituantur.

Corollarium 1.

23. Ita in nostro primo exemplo (§. 12.) cum sit
$P = y + z$, $Q = x + z$, $R = x + y$, erit
$S = q + 1$, $T = p + 1$, $V = p + q$ et
$$\frac{\partial z}{z} + \frac{(q+1)\partial p + (p+1)\partial q}{2pq + 2p + 2q} = 0;$$
cujus integrale est
$$lz + \tfrac{1}{2}l(pq + p + q) = \tfrac{1}{2}l(xy + xz + yz) = C, \text{ seu}$$
$$xy + xz + yz = C.$$

Corollarium 2.

24. In secundo exemplo (§. 13.) est
$P = ay - bz$, $Q = cz - ax$, $R = bx - cy$, hinc
$S = aq - b$, $T = c - ap$, $V = bp - cq$.
Ergo $$\frac{\partial z}{z} + \frac{(aq - b)\partial p + (c - ap)\partial q}{0} = 0;$$
hincque
$$(aq - b)\partial p + (c - ap)\partial q = 0.$$

et integrando

$$l\,\frac{aq - b}{c - ap} = l\,\frac{ay - bz}{cz - ax} = C.$$

Corollarium 3.

25. In tertio exemplo (§. 14.) fit

$$S = qq + q + 1,\ T = pp + p + 1,\ \text{et}\ V = pp + pq + qq,$$

hincque

$$\frac{\partial z}{z} + \frac{\partial p\,(qq + q + 1) + \partial q\,(pp + p + 1)}{ppq + pqq + pp + 3pq + qq + p + q} = 0,$$

qui denominator est $= (p + q + 1)(pq + p + q)$, unde haec fractio resolvitur in has duas

$$\frac{-\partial p - \partial q}{p + q + 1} + \frac{\partial p\,(q + 1) + \partial q\,(p + 1)}{pq + p + q}:$$

ex quo integrale a logarithmis ad numeros perductum oritur

$$\frac{z\,(pq + p + q)}{p + q + 1} = \frac{xy + xz + yz}{x + y + z} = C.$$

Corollarium 4.

26. In exemplo quarto (§. 18.) fit

$$S = pp - qq + 1,\ T = -1,\ V = q - p + p\,(qq - pp),$$

hincque

$$\frac{\partial z}{z} + \frac{\partial p\,(pp - qq + 1) - \partial q}{0} = 0,$$

ideoque

$$\partial q = \partial p\,(pp - qq + 1).$$

Cum ergo satisfaciat $q = p$, ponatur $q = p + \frac{1}{r}$, fiet

$$\partial r - 2pr\partial p = \partial p;\ \text{et integrando}$$

$$e^{-pp}\,r = \int e^{-pp}\,\partial p = e^{-pp}\,.\,\frac{1}{q - p},$$

ita ut integrale sit

$$e^{\frac{-xx}{zz}}\,.\,\frac{z}{y - x} = \int e^{\frac{-xx}{zz}}\,\partial\,.\,\frac{x}{z} + \text{Const.}$$

Scholion.

27. Cum igitur aequationes differentiales tres variabiles involventes nullam habeant difficultatem sibi propriam, quoniam earum resolutio, siquidem fuerint reales, semper ad aequationes differentiales duarum variabilium reduci potest; hoc argumentum fusius non prosequor. Quod enim ad ejusmodi aequationes differentiales trium variabilium attinet, in quibus ipsa differentialia ad plures dimensiones ascendunt, veluti est

$$P\partial x^2 + Q\partial y^2 + R\partial z^2 + 2S\partial x\partial y + 2T\partial x\partial z + 2V\partial y\partial z = 0,$$

de iis generatim tenendum est, nisi per radicis extractionem ad formam

$$P\partial x + Q\partial y + R\partial z = 0,$$

reduci queant, eas semper esse absurdas. Quomodocunque enim aequatio integralis esset comparata, ex ea valor ipsius z ita definiri posset, ut z aequetur functioni binarum variabilium x et y, unde foret $\partial z = p\partial x + q\partial y$; neque hae variabiles x et y ullo modo a se penderent. Hic ergo valor $p\partial x + q\partial y$ loco ∂z in aequatione differentiali substitutus, ita satisfacere deberet, ut omnes termini se mutuo destruerent, quod autem fieri non posset, si ex aequationis resolutione ∂z ita definiretur, ut differentialia ∂x et ∂y signis radicalibus essent involuta. Hinc aequatio illa exempli loco allata, cum per resolutionem det

$$\partial z = \frac{-T\partial x - V\partial y \pm \sqrt{[(TT - PR)\partial x^2 + (TV - RS)\partial x\partial y + (VV - QR)\partial y^2]}}{R},$$

realis esse nequit, nisi radix extrahi queat, hoc est nisi ipsa aequatio in factores formae

$$P\partial x + Q\partial y + R\partial z,$$

resolvi possit. Atque etiamsi hoc eveniat, et hi factores nihilo aequales statuantur, tamen aequatio non erit realis, nisi criterium supra traditum locum habeat. Ex his perspicuum est, ne ejusmodi quidem aequationes, quae quatuor pluresve variabiles involvant, plus difficultatis habere.

Problema 4.

28. Si V sit functio quaecunque binarum variabilium x et y, in formula autem integrali $\int V \partial x$ quantitas y pro constante sit habita, definire hujus formae $\int V \partial x$ differentiale, si praeter x etiam y variabilis assumatur.

Solutio.

Ponatur ista formula integralis $\int V \partial x = Z$, eritque Z utique functio ambarum variabilium x et y, etiamsi in ipsa integratione y pro constante habeatur. Evidens autem est, si vicissim in differentiatione y constans sumatur, fore $\partial Z = V \partial x$. Quare si etiam y variabilis statuatur, differentiale ipsius $Z = \int V \partial x$ hujusmodi habebit formam

$$\partial Z = V \partial x + Q \partial y,$$

et quaestio huc redit, ut ista quantitas Q determinetur. Quia autem forma $V \partial x + Q \partial y$ est verum differentiale, necesse est sit $\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$; hincque $\partial x \left(\frac{\partial Q}{\partial x}\right) = \partial x \left(\frac{\partial V}{\partial y}\right)$. At $\partial x \left(\frac{\partial Q}{\partial x}\right)$ est differentiale ipsius Q, si y pro constante habeatur; unde Q reperietur si formula $\partial x \left(\frac{\partial V}{\partial y}\right)$ ita integretur, ut y tanquam constans tractetur, seu erit $Q = \int \partial x \left(\frac{\partial V}{\partial y}\right)$. Quocirca formulae $Z = \int V \partial x$ differentiale ex variabilitate utriusque x et y oriundum erit

$$\partial Z = V \partial x + \partial y \,.\, \int \partial x \left(\frac{\partial V}{\partial y}\right).$$

Corollarium 1.

29. Quoniam V est functio ipsarum x et y, si ponatur $\partial V = R \partial x + S \partial y$, erit $S = \left(\frac{\partial V}{\partial y}\right)$; unde fit

$$\partial Z = \partial \,.\, \int V \partial x = V \partial x + \partial y \int S \partial x,$$

scilicet in formulae $\int S\partial x$ integratione, perinde ac formulae $\int V\partial x$ sola quantitas x pro variabili est habenda.

Corollarium 2.

30. Si V fuerit functio homogenea ipsarum x et y existente numero dimensionum $= n$, posito $\partial V = R\partial x + S\partial y$, erit $Rx + Sy = nV$, ideoque $S = \frac{nV}{y} - \frac{Rx}{y}$, hinc

$$\int S\partial x = \frac{n}{y}\int V\partial x - \frac{1}{y}\int Rx\partial x.$$

At ob y constans est $R\partial x = \partial V$, hinc

$$\int Rx\partial x = \int x\partial V = Vx - \int V\partial x, \text{ ideoque}$$

$$\int S\partial x = \frac{n+1}{y}\int V\partial x - \frac{Vx}{y}, \text{ et}$$

$$\partial Z = \partial . \int V\partial x = V\partial x - \frac{Vx\partial y}{y} + \frac{(n+1)\partial y}{y}\int V\partial x.$$

Corollarium 3.

31. Idem facilius invenitur ex consideratione quod functio $Z = \int V\partial x$ futura sit homogenea $n+1$ dimensionum, quare posito $\partial Z = V\partial x + Q\partial y$, erit $Vx + Qy = (n+1)Z$; ideoque $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$, ut ante.

Scholion.

32. Problemate jam ante, et in praecedente quidem libro, sum usus, neque tamen abs re fore putavi, si id data opera hic tractarem, quandoquidem hic liber in functionibus binarum pluriumve variabilium occupatur. Praecipuum autem negotium non in ejusmodi aequationibus differentialibus, quales in hoc capite integrare docui, versatur, quod quidem brevi esset absolum, sed cum differentiatio functionis binarum variabilium x et y duplices formulas $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ suppeditet, existente V hujusmodi functione, hoc loco ejusmodi quaestiones potissimum contemplabimur, quibus talis functio

V ex data quacunque relatione harum duarum formularum $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ est difinienda. Relatio autem haec per aequationem inter istas formulas et binas variabiles x et y, quam etiam ipsa functio quaesita V ingredi potest, exprimitur, ex cujus aequationis indole divisio tractationis erit petenda. Problema scilicet generale, in quo solvendo ista sectio est occupata, ita se habet, ut ea binarum variabilium x et y functio V inveniatur, quae satisfaciat aequationi cuicunque inter quantitates x, y, V, $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ propositae. Quodsi in hanc aequationem altera tantum binarum formularum differentialium $\left(\frac{\partial V}{\partial x}\right)$ vel $\left(\frac{\partial V}{\partial y}\right)$ ingrediatur, resolutio non est difficilis, atque ad casum aequationum differentialium duas tantum variabiles involventium reducitur; quando autem ambae istae formulae in aequatione proposita insunt, quaestio multo magis est ardua ac saepenumero ne resolvi quidem potest, etiamsi resolutio aequationum differentialium duas tantum variabiles complectentium admittatur: in hoc enim negotio, quoties resolutionem ad integrationem aequationum differentialium inter duas variabiles reducere licet, problema pro resoluto erit habendum. Cum igitur ex aequatione proposita formula $\left(\frac{\partial V}{\partial y}\right)$ aequetur functioni utcunque ex quantitatibus x, y, V et $\left(\frac{\partial V}{\partial x}\right)$ conflatae, ex indole hujus functionis, prout fuerit simplicior, et vel solam formulam $\left(\frac{\partial V}{\partial x}\right)$, vel praeter eam unicam ex reliquis, vel etiam binas, vel adeo omnes comprehendat, tractationem sequentem distribuemus. Hoc enim ordine servato facillime apparebit, quantum adhuc praestare liceat, et quantum adhuc desideretur. Praeterea vero nonnulla subsidia circa transformationem binarum formularum differentialium ad alias variabiles exponenda occurrent.

Divisio hujus Sectionis.

Quo partes, quas in hac sectione pertractari conveniet, clarius conspectui exponantur, quoniam hae quaestiones circa functiones binarum variabilium versantur, sint x et y binae variabiles, et z earum functio et data quadam differentialium relatione definienda, ita ut aequatio finita inter x, y et z requiratur. Ponamus autem $\partial z = p\partial x + q\partial y$, ita ut sit recepto signandi modo $p = (\frac{\partial z}{\partial x})$ et $q = (\frac{\partial z}{\partial y})$, atque ideo p et q sint formulae differentiales, quae in relationem propositam ingrediantur. In genere ergo relatio ista erit aequatio quaecunque inter quantitates p, q, x, y et z proposita, atque haec sectio perfecte absolveretur, si methodus constaret, ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem unicae variabilis praestari possit, multo minus hic est expectandum, ex quo eos casus tantum evolvi conveniet, qui resolutionem admittant. Primo autem resolutio succedit, si in aequatione proposita altera formularum differentialium p vel q plane desit, ita ut aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur. Deinde aequationes, quae solas binas formulas differentiales p et q continent, ita ut altera debeat esse functio quaecunque alterius, commode resolvere licet. Tum igitur sequentur aequationes, quae praeter p et q unicam quantitatem finitarum x vel y vel z complectantur, ex quo genere cujusmodi casus resolvi queant videamus. Ordo porro postulat, ut ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum, vel x et y, vel x et z, vel y et z, involvunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium, agemus, subsidia transformationis deinceps exposituri.

CAPUT II.

DE RESOLUTIONE AEQUATIONUM QUIBUS ALTERA FORMULA DIFFERENTIALIS PER QUANTITATES FINITAS UTCUNQUE DATUR.

Problema 1.

33.

Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right) = p$ sit quantitas constans $= a$.

Solutio.

Posito ergo $\partial z = p\partial x + q\partial y$, ea functionis z indoles quaeritur, ut sit $p = a$, seu $\partial z = a\partial x + q\partial y$: ad quam inveniendam sumatur y pro constante, erit $\partial z = a\partial x$, et integrando $z = ax +$ Const. ubi notari oportet hanc constantem utcunque involvere posse quantitatem y. Quare ut solutionem generalem exhibeamus, erit $z = ax + f:y$, denotante $f:y$ functionem quamcunque ipsius y, quae per se nullo modo determinatur, sed penitus ab arbitrio nostro pendet. Quod etiam differentiatio vicissim declarat; si enim hujus functionis $f:y$ differentiale per $\partial y f':y$ indicemus, erit utique

$$\partial z = a\partial x + \partial y f':y;$$

ideoque $\left(\frac{\partial z}{\partial x}\right) = a$, prorsus uti quaestio postulat; unde patet hoc casu alteram formulam differentialem $q = \left(\frac{\partial z}{\partial y}\right)$, functioni solius y aequari, cum sit $q = \left(\frac{\partial z}{\partial y}\right)$.

Corollarium 1.

34. Si ergo ejusmodi quaeratur functio z binarum variabilium x et y, ut sit $\left(\frac{\partial z}{\partial x}\right) = a$, erit $z = ax + f{:}y$, et altera formula differentialis $\left(\frac{\partial z}{\partial x}\right)$ necessaria aequatur functioni ipsius y tantum.

Corollarium 2.

35. Si talis requiratur functio, ut sit $\left(\frac{\partial z}{\partial x}\right) = 0$, ea necessario erit functio ipsius y tantum, seu quantitatem x plane non involvet; cum enim a variatione ipsius x nullam mutationem pati debeat, haec quantitas x quoque in ejus determinationem plane non ingredietur.

Corollarium 3.

36. Hinc etiam patet aequationem differentialem

$$\partial z = a\partial x + q\partial y$$

realem esse non posse, nisi q sit functio ipsius y tantum; quod etiam character supra expositus declarat, aequatione enim ad hanc formam $a\partial x + q\partial y - \partial z = 0$ reducta, ob $P = a$, $Q = q$, et $R = -1$, erit $L = \left(\frac{\partial q}{\partial z}\right)$, $M = 0$, et $N = -\left(\frac{\partial q}{\partial x}\right)$, ideoque realitas postulat, ut sit

$$a\left(\frac{\partial q}{\partial z}\right) + \left(\frac{\partial q}{\partial x}\right) = 0.$$

At per hypothesin q non pendet a z, unde ob $\left(\frac{\partial q}{\partial z}\right) = 0$, erit $\left(\frac{\partial q}{\partial x}\right) = 0$, ideoque etiam q ab x non pendet.

Scholion 1.

37. Ex allatis satis patet hanc operationem, qua functionem z determinavimus, veram esse integrationem, qua uti in vulgaribus

integrationibus aliquid indeterminati introducitur. Hic scilicet ingressa est functio quaecunque ipsius y, cujus indoles per se nullo modo determinatur; eam quoque ita concipere licet, ut descripta curva quacunque, si ejus abscissae per y indicentur, applicatae exhibeant ejusmodi functionem ipsius y. Neque vero opus est, ut haec curva sit regularis et aequatione quapiam contenta; sed curva quaecunque libero manus ductu descripta eundem praestat effectum, etiamsi sit maxime irregularis, et ex pluribus partibus diversarum curvarum conflata. Hujusmodi functiones irregulares appellare licet discontinuas seu nexu continuitatis destitutas; unde hoc imprimis notatu dignum occurrit, quod cum prioris generis integrationes alias functiones praeter continuas non admittant, hic etiam functiones discontinuae calculo subjiciantur, quod pluribus insignibus Geometris adeo calculi principiis adversari est visum. Verum integrationum in hoc secundo libro tradendarum vis praecipua in eo consistit, quod etiam functionum discontinuarum sint capaces; ex quo per hunc quasi novum calculum fines Analyseos maxime proferri sunt censendi.

Scholion 2.

33. Quemadmodum deinde in vulgaribus integrationibus constans arbitraria ingressa, semper ex indole problematis, cujus solutio eo perduxerat, determinatur, ita etiam hic natura problematis, cujus solutio hujusmodi integratione absolvitur, semper indolem functionis arbitrariae per integrationem ingressae determinabit. Ita si cordae tensae figura quaecunque inducatur, eaque subito dimittatur, ut oscillationes peragat, ope principiorum mechanicorum ad quodvis tempus figura, quam corda tum sit habitura, definiri potest, hocque fit ejusmodi integratione, qua functio quaedam arbitraria introducitur; quam autem deinceps ita determinari convenit, ut pro ipso motus initio ipsa illa figura cordae inducta prodeat; et cum solutio debeat esse generalis, ut satisfaciat figurae cuicunque initiali,

necesse est ut etiam ad eos casus pateat, quibus cordae initio figura irregularis nullo continuitatis nexu praedita inducatur, quod fieri non posset, nisi per integrationem ejusmodi functio arbitrio nostro relicta ingrederetur, quam etiam ad figuras irregulares adaptare liceret. Hujusmodi functiones arbitrarias, prouti hic feci, ejusmodi signandi modo $f:y$ indicabo, unde cavendum erit ne littera f pro quantitate habeatur, quocirca ipsi *colon* suffigere visum est. Simili modo in sequentibus haec scriptio $f:(x+y)$ denotabit functionem arbitrariam quantitatis $x+y$; ac ubi plures tales functiones in calculum ingrediuntur, praeter litteram f etiam his characteribus Φ, ψ, θ, etc. cum simili significatione utar.

Problema 5.

39. Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right)=p$ aequalis fiat functioni datae ipsius x, quae sit X, ita ut sit $p=X$.

Solutio.

Posito $\partial z=p\partial x+q\partial y$, ob $p=X$ erit $\partial z=X\partial x+q\partial y$; quia jam hujus differentialis pars $X\partial x$ est data, ad integrale inveniendum accipiatur y constans, et cum sit $\partial z=X\partial x$, erit integrando $z=\int X\partial x+$ Const. quae constans cum etiam quantitatem y utcunque implicare possit, pro ea assumere licebit functionem quamcunque arbitrariam ipsius y, eritque ergo integrale quaesitum $z=\int X\partial x+f:y$, quae per differentiationem praebet

$$\partial z=X\partial x+\partial y f':y,$$

ita ut sit $q=f':y$, atque $\left(\frac{\partial z}{\partial x}\right)=X$, plane ut requirebatur.

Corollarium 1.

40. Aequationis ergo $\left(\frac{\partial z}{\partial x}\right)=X$, existente z functione duarum variabilium x et y, integrale est $z=\int X\partial x+f:y$, ubi ob

X datum, formula integralis $\int X\partial x$ datam functionem ipsius x denotat; quandoquidem constans hac integratione ingressa in functione arbitraria $f:y$ comprehendi potest.

Corollarium 2.

41. Hinc sequitur aequationem differentialem

$$\partial z = X\partial x + q\partial y$$

realem esse non posse, nisi q sit functio ipsius y; quod quidem cum hac limitatione est intelligendum, nisi q etiam involvat quantitatem z; quem casum autem hinc removemus.

Scholion.

42. Si enim q etiam a z pendere queat, aequatio $\partial z = X\partial x + q\partial y$ realis erit, si q fuerit functio quaecunque binarum quantitatum $z - \int X\partial x$ et y; id quod hinc facillime patet, si ponatur $z - \int X\partial x = u$, ita ut jam q futura sit functio binarum quantitatum u et y. Tum enim aequatio differentialis, quae fit $\partial u = p\partial y$, duas tantum continet variabiles u et y, ideoque certo est realis; et quomodocunque ejus integrale se habeat, inde semper u aequabitur certae functioni ipsius y, unde fit $u = z - \int X\partial x = f:y$, prorsus ut ante. Quoties ergo esse debet $\left(\frac{\partial z}{\partial x}\right) = X$, etiam ne hoc quidem casu excepto, quo forte q ipsam quantitatem z implicat, integrale erit

$$z = \int X\partial x + f:y,$$

neque unquam alia solutio locum habere potest. Erit ergo hoc integrale completum, propterea quod functionem arbitrariam involvit, id quod pro certissimo criterio integralis completi est habendum. Hic igitur ad integrale completum requiritur, ut non tam constans quaedam arbitraria, sed functio adeo variabilis arbitraria ingrediatur; ita si quis pro casu $\left(\frac{\partial z}{\partial x}\right) = axx$ exhibeat hoc integrale

$$z = \tfrac{1}{3}ax^3 + A + By + Cy^2 + \text{etc.}$$

id tantum erit particulare, etiamsi plures constantes arbitrarias A, B, C, etc. ac fortasse infinitas compleotatur; verum enim integrale completum

$$z = \tfrac{1}{3}ax^3 + f.:y$$

infinite latius patet; id quod ad sequentia recte intelligenda probe notari oportet. Occurrent autem utique casus, quibus ob defectum methodi integrale completum investigandi, integralibus particularibus contenti esse debemus, quae etiamsi adeo infinitas constantes arbitratias comprehendant, tamen pro solutionibus particularibus tantum sunt habenda. Hanc observationem in sequentibus perpetuo meminisse oportet, ne circa integralia particularia et completa unquam decipiamur.

Problema 6.

43. Si z debeat esse ejusmodi functio binarum variabilium x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right) = p$ aequetur functioni cuipiam datae ipsarum x et y, definire in genere indolem functionis quaesitae z.

Solutio.

Sit V functio ista data ipsarum x et y, cui formula differenrentialis $\left(\frac{\partial z}{\partial x}\right) = p$ aequalis esse debet, ac posito

$$\partial z = p\,\partial x + q\,\partial y$$

requiritur ut sit $p = V$. Jam ad formam functionis z inveniendam consideretur quantitas y tanquam constans, eritque $\partial z = V\partial x$. Integretur igitur formula $\int V\partial x$ spectata sola x ut variabili, quia y pro constante sumitur, ita ut in hac formula unica insit variabilis x, ideoque ejus integratio nulli obnoxia sit difficultati, id tantum est tenendum, constantem integratione ingressam utcumque involvere posse alteram quantitatem y, sicque pro functione quaesita z haec habebitur expressio

$$z = \int V \partial x + f : y$$

integrali $\int V \partial x$ ita sumto, quasi quantitas y esset constans solaque x variabilis; at $f : y$ denotat functionem quamcunque arbitrariam ipsius y, ne exclusis quidem formis discontinuis, quae nullis expressionibus analyticis exhiberi queant; atque ob hanc ipsam functionem arbitrariam integratio pro completa est habenda.

Corollarium 1.

44. Cum V sit functio data ipsarum x et y, formula integralis $\int V \partial x$ erit etiam functio cognita et determinata earundem quantitatem x et y, quod enim per integrationem arbitrarii ingreditur, in altera parte $f : y$ comprehenditur.

Corollarium 2.

45. Hinc etiam differentialis ∂z altera pars $q \partial y$ ex variabilitate ipsius y oriunda definitur. Nam per §. 28. est formae $\int V \partial x$ differentiale ex utraque variabili x et y ortum

$$V \partial x + \partial y \int \partial x \left(\frac{\partial V}{\partial y}\right);$$

ac si functionis $f : y$ differentiale indicetur per $\partial y f' : y$, erit

$$\partial z = V \partial x + \partial y \int \partial x \left(\frac{\partial V}{\partial y}\right) + \partial y f' : y.$$

Corollarium 3.

46. Cum ergo posuerimus $\partial z = p \partial x + q \partial y$, sitque $p = V$, erit

$$q = \int \partial x \left(\frac{\partial V}{\partial y}\right) + f' : y,$$

ubi ob V functionem datam ipsarum x et y, etiam $\left(\frac{\partial V}{\partial y}\right)$ erit functio data, et in integratione $\int \partial x \left(\frac{\partial V}{\partial y}\right)$ sola x pro variabili habetur.

Exemplum 1.

47. *Quaeratur ejusmodi functio z ipsarum x et y, ut sit*

$$\left(\frac{\partial z}{\partial x}\right) = \frac{x}{\sqrt{(xx+yy)}}.$$

Ob $V = \frac{x}{\sqrt{(xx+yy)}}$, erit $\int V \partial x = \sqrt{(xx+yy)}$, ideoque habemus

$$z = \sqrt{(xx+yy)} + f : y,$$

unde fit

$$\left(\frac{\partial z}{\partial y}\right) = q = \frac{y}{\sqrt{(xx+yy)}} + f' : y,$$

id quod etiam per regulam datam prodit. Erit enim

$$\left(\frac{\partial V}{\partial y}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}},$$

hinc sumta y constante

$$\int \partial x \left(\frac{\partial V}{\partial x}\right) = -y \int \frac{x \partial x}{(xx+yy)^{\frac{3}{2}}} = \frac{y}{\sqrt{(xx+yy)}}.$$

Exemplum 2.

48. *Quaeratur ejusmodi functio z ipsarum x et y, ut sit*

$$\left(\frac{\partial z}{\partial x}\right) = \frac{y}{\sqrt{(yy-xx)}}.$$

Cum sit $V = \frac{y}{\sqrt{(yy-xx)}}$, erit

$$\int V \partial x = y \text{ Ang. sin. } \frac{x}{y},$$

hincque

$$z = y \text{ Ang. sin. } \frac{x}{y} + f : y$$

cujus differentiale ex ipsius y variabilitate oriundum, si desideremus, ob

$$\left(\frac{\partial V}{\partial y}\right) = \frac{-xx}{(yy-xx)^{\frac{3}{2}}}, \text{ erit}$$

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = -\int \frac{xx\partial x}{(yy-xx)^{\frac{3}{2}}} = \int \frac{\partial x}{\sqrt{(yy-xx)}} - yy \int \frac{\partial x}{(yy-xx)^{\frac{3}{2}}},$$

ideoque

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = \text{Ang. sin. } \frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}}, \text{ et}$$

$$q = \text{Ang. sin. } \frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}} + f' : y.$$

Idem reperitur ex differentiatione expressionis pro z inventae

$$\partial z = \partial y \text{ Ang. sin. } \frac{x}{y} + \frac{y\partial x - x\partial y}{\sqrt{(yy-xx)}} + \partial y f' : y,$$

unde pro $q = \left(\frac{\partial z}{\partial y}\right)$ idem valor prodit.

Exemplum 3.

49. *Quaeratur ejusmodi functio z ipsarum x et y, ut sit*

$$\left(\frac{\partial z}{\partial x}\right) = \frac{a}{\sqrt{(aa-yy-xx)}}.$$

Ob $V = \frac{a}{\sqrt{(aa-yy-xx)}}$, erit

$$\int V \partial x = a \text{ Ang. sin. } \frac{x}{\sqrt{(aa-yy)}},$$

unde functionis z forma quaesita est

$$z = a \text{ Ang. sin. } \frac{x}{\sqrt{(aa-yy)}} + f : y.$$

Deinde quia

$$\left(\frac{\partial V}{\partial y}\right) = \frac{ay}{(aa-yy-xx)^{\frac{3}{2}}}, \text{ erit}$$

$$\int \partial x \left(\frac{\partial V}{\partial y}\right) = ay \int \frac{\partial x}{(aa-yy-xx)^{\frac{3}{2}}} = \frac{ay}{aa-yy} \cdot \frac{x}{\sqrt{(aa-yy-xx)}}.$$

ideoque

$$\left(\frac{\partial x}{\partial y}\right) = q = \frac{axy}{(aa-yy)\sqrt{(aa-yy-xx)}} + f' : y$$

quae eadem expressio etiam ex ipsa differentiatione ipsius z eruitur.

Scholion 1.

50. In hoc calculo tamen adhuc quaedam incertitudo relinquitur, qua valor quantitatis q afficitur. Cum enim valor ipsius $z = \int V\partial x + f:y$ sit determinatus, quandoquidem integrale $\int V\partial x$ respectu ipsius x ita fuerit determinatum, ut pro dato ipsius x valore etiam datum valorem obtineat; adeoque in ejus differentiali pleno nulla incertitudo inesse potest, sed necesse est, ut valor ipsius p aeque prodeat determinatus atque ipsius p: interim tamen formula integralis $\int\partial x\left(\frac{\partial V}{\partial y}\right)$ non determinatur, sed novam functionem arbitrariam a priori non pendentem introducere videtur. Ut igitur talis significatus vagus evitetur, spectari oportet conditionem, qua integrale $\int V\partial x$ determinatur, eademque conditio in formulae $\int\partial x\left(\frac{\partial V}{\partial y}\right)$ integratione adhiberi debet. Nam ponamus integrale $\int V\partial x$ ita capi ut evanescat posito $x = a$, sitque ejus valor determinatus $\int V\partial x = S$; isque igitur potentia saltem habebit factorem $a - x$ seu $a^n - x^n$; qui cum non contineat y, etiam $\left(\frac{\partial S}{\partial y}\right)$ eundem factorem continebit, ideoque $\left(\frac{\partial S}{\partial y}\right)$ evanescet posito $x = a$.

Est vero $\left(\frac{\partial S}{\partial y}\right) = \int\partial x\left(\frac{\partial V}{\partial y}\right)$,

ex quo perspicitur, si integrale $\int V\partial x$ ita capiatur ut evanescat posito $x = a$, etiam alterum integrale $\int\partial x\left(\frac{\partial V}{\partial y}\right)$ ita capi debere, ut evanescat posito $x = a$. In allatis binis postremis exemplis, utraque integratio ita est instituta, ut evanescat posito $x = 0$, in primo autem nulla hujusmodi regula est observata; sin autem eandem legem adhibeamus, habebimus

$$\int V\partial x = \sqrt{(xx+yy)} - y \text{ et } \int \partial x \left(\frac{\partial V}{\partial y}\right) = \frac{y}{\sqrt{(xx+yy)}} - 1;$$

unde quidem eadem solutio emergit; quia ibi $-y$ continetur in $f:y$, et hic -1 in $f':y$. Perinde autem est quacunque lege prior integratio determinetur, dummodo eadem lege et in posteriori utamur.

Scholion 2.

51. Principium hujus determinationis isto innititur Theoremate aeque elegante ac notatu digno:

Si S sit ejusmodi functio binarum variabilium x et y, quae evanescat posito $x = a$, fueritque $\partial S = P\partial x + Q\partial y$, tum etiam quantitas Q evanescet posito $x = a$.

Unde simul colligitur, si S evanescat posito $y = b$, tum etiam fieri $P = 0$ si ponatur $y = b$. Hic autem probe observandum est, quae de simili determinatione binarum formularum integralium $\int V\partial x$ et $\int \partial x \left(\frac{\partial V}{\partial y}\right)$ sunt praecepta, tantum valere si valor a ipsi x tribuendus fuerit constans; neque etiam superius Theorema locum habet, si verbi gratia functio S evanescat posito $x = y$, inde enim neutiquam sequitur, eodem casu quantitatem Q esse evanituram. Etiamsi enim functio S factorem habeat $x - y$ vel $x^n - y^n$, minime sequitur, formulam $\left(\frac{\partial S}{\partial y}\right)$ seu Q eundem factorem esse habituram, quemadmodum usu venit, si factor fuerit $x - a$ seu $x^n - a^n$. Dixi autem non opus esse, ut talis factor revera adsit, dum modo quasi potentia in functione S contineatur. Veluti si fuerit

$$S = a - x + y - \sqrt{(aa - xx + yy)},$$

quae functio posito $x = a$ utique evanescit, etiamsi neque factorem $x - a$ neque $x^n - a^n$ contineat; simul vero etiam

$$\left(\frac{\partial S}{\partial y}\right) = 1 - \frac{y}{\sqrt{(aa - xx + yy)}}$$

posito $x = a$ evanescit. In hujusmodi ergo calculo, quo in his problematibus utimur, ubi integrale formulae $\int V \partial x$ exhiberi debet, id semper ex duabus partibus compositum spectamus, altera indeterminata per functionem $f:y$ indicata, altera autem, quam proprie per $\int V \partial x$ exprimimus, determinata, quae scilicet posito $x = a$, evanescat; hicque semper perinde est qualis constans pro a assumatur, dum discrimen perpetuo alteri parti indeterminatae involvitur.

Problema 7.

52. Si z debeat ita determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right) = p$ aequatur datae cuipiam functioni ipsarum y et z, quae sit V, definire in genere indolem functionis z per x et y.

Solutio.

Cum posito $\partial z = p \partial x + q \partial y$, sit $p = V$, si quantitatem y pro constante capiamus, erit $\partial z = V \partial x$, ubi cum V sit functio data ipsarum y et z, et y pro constante habeatur, aequatio $\frac{\partial z}{V} = \partial x$ erit integrabilis, ex cujus integratione completa oritur

$$\int \frac{\partial z}{V} = x + f:y,$$

qua aequatione relatio inter ternas variabiles x, y et z ita in genere exprimitur, ut ex ea z per x et y definiri, indolesque functionis z assignari possit.

Quodsi hinc alteram quoque differentialis partem $q \partial y$ seu functionem $q = \left(\frac{\partial z}{\partial y}\right)$ indagare velimus, ponamus integrale $\int \frac{\partial z}{V}$, ubi y ut constans spectatur, ita capi ut evanescat posito $z = c$, eritque quantitatem $\int \frac{\partial z}{V}$ denuo differentiando ut etiam y variabilis assumatur

$$\partial \cdot \int \frac{\partial z}{V} = \frac{\partial z}{V} + \partial y \int \partial z \left(\frac{\partial (1:V)}{\partial y}\right), \text{ seu}$$

$$\partial \cdot \int \frac{\partial z}{V} = \frac{\partial z}{V} - \partial y \int \frac{\partial z}{VV} \left(\frac{\partial V}{\partial y}\right),$$

ubi in integrali $\int \frac{\partial z}{VV} \left(\frac{\partial V}{\partial y}\right)$ quantitas y iterum pro constante habetur, hocque integrale ita capi debet, ut posito $z = c$ evanescat. Quo facto cum aequationis inventae differentiale sit

$$\frac{\partial z}{V} - \partial y \int \frac{\partial z}{VV} \left(\frac{\partial V}{\partial z}\right) = \partial x + \partial y f' : y,$$

pro forma proposita habebimus

$$\partial z = V \partial x + \partial y \left(V \int \frac{\partial z}{VV} \left(\frac{\partial V}{\partial y}\right) + V f' : y\right),$$

unde quantitas q innotescit.

Corollarium 1.

53. In hoc problemate facillime definitur, qualis functio quantitas x futura sit binarum reliquarum y et z, cum sit

$$x = \int \frac{\partial z}{V} - f : y,$$

siquidem V per y et z detur. Perinde autem est sive z per x et y, sive x per y et z determinetur.

Corollarium 2.

54. Cum relatio inter ternas variabiles x, y et z ita sit determinata, ut fiat $\left(\frac{\partial z}{\partial x}\right) = V$ functioni datae ipsarum y et z, ob $\partial x = \frac{\partial z}{V}$ sumto y constante erit x ejusmodi functio ipsarum y et z, ut sit $\left(\frac{\partial x}{\partial z}\right) = \frac{1}{V}$, ideoque $\left(\frac{\partial z}{\partial x}\right) \cdot \left(\frac{\partial x}{\partial z}\right) = 1$.

Scholion.

55. In genere autem quaecunque relatio inter ternas variabiles x, y et z proponatur, unde unaquaeque per binas reliquas determinari et tanquam earundem functio spectari possit, semper erit $\left(\frac{\partial z}{\partial x}\right) \cdot \left(\frac{\partial x}{\partial z}\right) = 1$. Ponamus enim aequatione illam relationem exprimente differentiata prodire

$$P\partial x + Q\partial y + R\partial z = 0,$$

ac manifestum est sumta y pro constante fore

$$P\partial x + R\partial z = 0,$$

ideoque tam $\left(\frac{\partial z}{\partial x}\right) = \frac{-P}{R}$ quam $\left(\frac{\partial x}{\partial z}\right) = \frac{-R}{P}$; simile autem modo erit

$$\left(\frac{\partial x}{\partial y}\right) = \frac{-Q}{P}; \left(\frac{\partial y}{\partial x}\right) = \frac{-P}{Q}, \left(\frac{\partial z}{\partial y}\right) = \frac{-Q}{R}, \left(\frac{\partial y}{\partial z}\right) = \frac{-R}{Q},$$

unde propositum patet, etiamsi relatio inter plures tribus variabiles locum habeat. Caeterum hic casus a praecedentibus differt, quod hic natura functionis z, quatenus ex binis reliquis x et y formatur, non explicite exhibeatur, sed per resolutionem demum aequationis inventae definiri debet, cujus rei aliquot exempla evoluisse juvabit.

Exemplum 1.

56. *Quaeratur ejusmodi functio z ipsarum x et y, ut sit* $\left(\frac{\partial z}{\partial x}\right) = \frac{y}{z}$.

Cum ergo sit $\partial z = \frac{y\partial x}{z}$, erit y pro constante sumendo

$$z\partial z = y\partial x \quad \text{et} \quad \tfrac{1}{2}zz = xy + f:y.$$

Pro q inveniendo differentietur haec aequatio generaliter

$$z\partial z = y\partial x + x\partial y + \partial y f':y,$$

eritque

$$q = \frac{x}{z} + \frac{1}{z}f':y;$$

quod idem per regulam datam reperitur. Nam ob $V = \frac{y}{z}$, erit $\int\frac{\partial z}{V} = \frac{zz}{2y}$, integrali ita sumto ut evanescat posito $z = 0$; tum vero ob $\left(\frac{\partial V}{\partial y}\right) = \frac{1}{z}$, erit

$$\int\frac{\partial z}{VV}\left(\frac{\partial V}{\partial y}\right) = \int\frac{z\partial z}{yy} = \frac{zz}{2yy};$$

eadem integrationis lege observata. Hinc fit

$$\partial z = \frac{y\partial x}{z} + \frac{y\partial y}{z}\left(\frac{zz}{2yy} + f' : y\right) \text{ et } q = \frac{z}{2y} + \frac{y}{z} f' : y,$$

quae expressio cum praecedente convenit; ex comparatione enim fit

$$x + f' : y = \frac{zz}{2y} + y f' : y,$$

unde x aequatur ut ante quantitati $\frac{zz}{2y}$ una cum functione ipsius y. Hoc tantum notetur, quod ad consensum perfectum hic pro $f : y$ scribere debuissemus $y f : y$.

Exemplum 2.

67. *Quaeratur ejusmodi functio z binarum variabilium x et y, ut sit $\left(\frac{\partial z}{\partial x}\right) = \frac{\sqrt{(yy - zz)}}{z}$.*

Cum ergo sit

$$\partial z = \frac{\partial x \sqrt{(yy - zz)}}{z} + q\partial y,$$

sumta y constante fit

$$\partial x = \frac{z\partial z}{\sqrt{(yy - zz)}}, \text{ et integrando}$$

$$x = y - \sqrt{(yy - zz)} - f : y,$$

unde vicissim differentiando oritur

$$\partial x = \partial y - \frac{y\partial y + z\partial z}{\sqrt{(yy - zz)}} - \partial y f' : y, \text{ seu}$$

$$\partial z = \frac{\partial x \sqrt{(yy - zz)}}{z} + \partial y \left[\frac{y}{z} - \frac{\sqrt{(yy - zz)}}{z}(1 - f' : z)\right].$$

Per regulam autem datam ob $V = \frac{\sqrt{(yy - zz)}}{z}$, est

$$\int \frac{\partial z}{V} = y - \sqrt{(yy - zz)},$$

integrali ita sumto ut evanescat posito $z = 0$. Jam vero est

$$\left(\frac{\partial V}{\partial y}\right) = \frac{y}{z\sqrt{(yy - zz)}} \text{ et } \frac{1}{VV}\left(\frac{\partial V}{\partial y}\right) = \frac{yz}{(yy - zz)^{\frac{3}{2}}},$$

hinc

$$\int \frac{\partial z}{VV}\left(\frac{\partial V}{\partial y}\right) = \frac{y}{\sqrt{(yy - zz)}} - 1,$$

integrali eadem lege sumto. Quocirca colligitur

$$q = \frac{\sqrt{(yy-zz)}}{z}\left(\frac{y}{\sqrt{(yy-zz)}} - 1 + f':y\right) = \frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1 - f':y),$$

prorsus ut ante.

Problema 8.

58. Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right) = p$ aequetur functioni cuipiam datae ipsarum x et z, quae sit $= V$, definire in genere indolem functionis z per x et y.

Solutio.

Ponatur $\partial z = p\partial x + q\partial y$, et cum sit $p = V$, sumatur quantitas y constans, eritque $\partial z - V\partial x = 0$, quae aequatio duas tantum quantitates variabiles x et z continens, integrabilis reddetur ope cujusdam multiplicatoris, qui sit $= M$, ita ut $M\partial z - MV\partial x$ sit differentiale verum cujuspiam functionis ipsarum x et z, quae functio sit $= S$, quantitatem y non involvens. Ex quo aequatio nostra integralis erit $S = f:y$, unde indoles functionis z quemadmodum per x et y determinatur, innotescit. Differentiemus hanc aequationem sumto praeter x et z etiam y variabili, eritque

$$\partial S = M\partial z - MV\partial x = \partial y f':y, \quad \text{seu}$$
$$\partial z = V\partial x + \frac{\partial y}{M} f':y, \quad \text{ita ut sit } q = \frac{1}{M} f':y.$$

Corollarium 1.

59. Multiplicator etiam M formulam $\partial z - V\partial x$ integrabilem reddens, quantitatem y non continebit, quia in functione data V non inest y. Statim autem hoc multiplicatore invento, valor ipsius $q = \frac{1}{M} f':y$ colligitur.

Corollarium 2.

60. Si formulae differentialis $M\partial z - MV\partial x$ integrale fuerit S functio ipsarum x et z, pro solutione problematis habebimus $S = f:y$, unde patet constantem, quam quis forte ad S adjicere voluerit, jam in functione arbitraria $f:y$ contineri.

Exemplum 1.

61. *Quaeratur ejusmodi functio z ipsarum x et y, ut sit* $\left(\frac{\partial z}{\partial x}\right) = \frac{nz}{x}$.

Posito $\partial z = \frac{nz\partial x}{x} + q\partial y$, sumto y constante erit $\partial z - \frac{nz\partial x}{x} = 0$, quae aequatio per $\frac{1}{z}$ multiplicata fit integrabilis, ita ut sit multiplicator $M = \frac{1}{z}$, hincque integrale $S = lz - lx^n$: ergo aequatio nostra integralis quaesita erit $l\frac{z}{x^n} = f:y$; unde etiam $\frac{z}{x^n}$ aequabitur functioni cuicunque ipsius y, ita ut sit $z = x^n f:y$.

Exemplum 2.

62. *Quaeratur ejusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{\partial z}{\partial x}\right) = nx - z$.

Posito $\partial z = (nx - z)\partial x + q\partial y$, sumto y constante erit $\partial z + z\partial x - nx\partial x = 0$, quae ope multiplicatoris $M = e^x$ dat

$$S = e^x z - n\int e^x x\partial x = e^x z - ne^x x + ne^x;$$

unde aequatio quaesitam relationem inter x, y et z exprimens est

$$e^x z - ne^x x + ne^x = f:y, \text{ sive}$$

$$z = n(x-1) + e^{-x} f:y,$$

tum vero erit

$$q = \left(\frac{\partial z}{\partial y}\right) = e^{-x}\, f' : y.$$

Exemplum 3.

63. *Quaeratur ejusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{\partial z}{\partial x}\right) = \frac{xz}{xx+zz}$.

Ponatur ergo $\partial z = \frac{xz\partial x}{xx+zz} + q\partial y$, et posito y constante quaeratur integrale hujus aequationis differentialis

$$\partial z - \frac{xz\partial x}{xx+zz} = 0,$$

quae integrabilis redditur ope cujusdam divisoris, qui ob homogeneitatem reperitur scribendo x et z loco differentialium ∂x et ∂z, ita ut hic divisor sit

$$z - \frac{xxz}{xx+zz} = \frac{z^3}{xx+zz},$$

hincque multiplicator $M = \frac{xx+zz}{z^3}$. Quare erit

$$\partial S = \frac{(xx+zz)\,\partial z}{z^3} - \frac{x\partial x}{zz}, \text{ ideoque}$$

$$S = \frac{-xx}{2zz} + lz;$$

unde aequatio nostra quaesita erit

$$lz - \frac{xx}{2zz} = f : y \quad \text{et} \quad q = \frac{z^3}{xx+zz}\, f' : y,$$

ex qua cum posito $lz - \frac{xx}{2zz} = u$ sit $u = f : y$, etiam vicissim concludi potest fore $y = f : u$.

Problema 9.

64. Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{\partial z}{\partial x}\right)$ aequetur functioni cuipiam datae omnes tres variabiles x, y et z implicanti, quae sit $= V$, definire in genere indolem functionis z per x et y.

Solutio.

Cum sit $\partial z = V\partial x + q\partial y$, si sumamus y constans, erit $\partial z = V\partial x$, quae ergo aequatio duas tantum continet variabiles x et z, litteram autem y functione V involvens. Dabitur ergo multiplicator M hanc aequationem integrabilem reddens, ita ut sit

$$M\partial z - MV\partial x = \partial S,$$

unde aequatio integralis relationem inter x, y et z exprimens erit

$$S = f:y:$$

uci S erit functio certa ipsarum x, y et z, fierique potest ut etiam M omnes has tres litteras comprehendat. Convenit autem functioni S per integrationem inventae valorem determinatum tribui, quoniam pars indeterminata in functione arbitraria $f:y$ includitur. Ponamus ergo S ita capi, ut evanescat si ponatur $x = a$ et $z = c$.

Quod si hinc aequationis differentialis propositae alteram partem $q\partial y$ invenire velimus, differentiemus functionem S sumto etiam y variabili, sitque

$$\partial S = M\partial z - MV\partial x + Q\partial y = \partial y f':y,$$

ubi cum sit

$$\left(\frac{\partial Q}{\partial x}\right) = \left(\frac{\partial M}{\partial y}\right) \text{ vel } \left(\frac{\partial Q}{\partial x}\right) = -\left(\frac{\partial . MV}{\partial y}\right),$$

erit sumto iterum y constante

$$\partial Q = \partial z\left(\frac{\partial Q}{\partial z}\right) + \partial x\left(\frac{\partial Q}{\partial x}\right) = \partial z\left(\frac{\partial M}{\partial y}\right) - \partial x\left(\frac{\partial . MV}{\partial y}\right),$$

quae formula certo erit integrabilis. Capi autem Q eadem lege debet, qua S sumsimus, ita ut evanescat posito $x = a$ et $z = c$, atque inventa hac quantitate Q, cum habeamus

$$\partial z = V \partial x - \frac{Q \partial y}{M} + \frac{\partial y}{M} f' : y, \text{ erit}$$

$$q = \left(\frac{\partial z}{\partial y}\right) = \frac{-Q + f' : y}{M}.$$

Haec determinatio isto nititur fundamento, quod si S fuerit ejusmodi functio ipsarum x, y et z, quae posito $x = a$ et $z = c$ evanescat, etiam formula differentialis $\left(\frac{\partial S}{\partial y}\right)$ eodem casu evanescat.

Corollarium 1.

65. Reducitur ergo resolutio hujus problematis ad integrationem aequationis differentialis

$$\partial z - V \partial x = 0,$$

in qua quantitas y ut constans spectatur, etiamsi V contineat omnes tres litteras x, y et z. Dabitur ergo utique multiplicatur M, qui hanc aequationem integrabilem reddat, ut sit

$$M \partial z - M V \partial x = \partial S,$$

existente S certa quadam functione ipsarum x, y et z.

Corollarium 2.

66. Invento autem hoc multiplicatore M indeque integrali S, quantitas z ita per binas variabiles x et y definietur, ut sit $S = f : y$, ubi $f : y$ denotat functionem quamcunque ipsius y sive continuam sive etiam discontinuam, ob cujus naturam integratio pro completa est habenda.

Corollarium 3.

67. Cum hoc modo relatio inter z, x, y, fuerit definita, erit ea ita differentiata, ut omnes tres litterae x, y et z variabiles sumantur

$$\partial z = V \partial x + \left(\frac{f' : y - Q}{M}\right) \partial y,$$

ubi quantitas Q ex suo differentiali

$$\partial Q = \partial z \left(\frac{\partial M}{\partial y}\right) - \left(\frac{\partial . MV}{\partial y}\right)$$

definiri debet, sumta y constante, integrationem ita temperando, ut si S evanescat casu $x = a$ et $z = c$, etiam Q eodem casu evanescat.

Scholion:

68. Hic ergo ad insigne hoc Theorema deducimur:

Quod si fuerit S ejusmodi functio ipsarum x, y et z, quae evanescat ponendo $x = a$ et $z = c$, tum etiam pro eadem positione formulam $\left(\frac{\partial S}{\partial y}\right)$ esse evanituram.

Veluti si fuerit

$$S = Axx + Bxyz + Czz - Aaa - Bacy - Ccc,$$

erit $\left(\frac{\partial S}{\partial y}\right) = Bxz - Bac$,

quarum utraque expressio casu $x = a$ et $z = c$ evanescit. Pluribus autem hujusmodi exemplis evolutis veritas Theorematis ita patet, ut demonstratio solennis non desideretur. Interim hujusmodi functio semper, quantitates solam y continentes a reliquis separando, ita evolvi potest, ut in talem formam transmutetur

$$S = PY + QY' + RY'' + \text{etc.}$$

ubi per hypothesin P, Q, R, etc. sunt functiones ipsarum x et z tantum, et tales quidem quae ponendo $x = a$ et $z = c$ singulae evanescant. Hinc jam perspicuum est fore

$$\left(\frac{\partial S}{\partial y}\right) = P \cdot \frac{\partial Y}{\partial y} + Q \cdot \frac{\partial Y'}{\partial y} + R \cdot \frac{\partial Y''}{\partial y} + \text{etc.}$$

quae forma manifesto sub iisdem conditionibus evanescit. Quomodocunque autem functio S hac indole praedita fuerit complicata, tam formulis irrationalibus quam transcendentibus, eam semper in ejusmodi formam evolvere licet, quae etsi in infinitum progrediatur, haec demonstratio tamen vim suam retinet.

Exemplum 1.

69. *Quaeratur ejusmodi functio z duarum variabilium x et y, ut sit formula differentialis* $\left(\frac{\partial z}{\partial x}\right) = \frac{xz}{ay}$.

Ponamus ergo $\partial z = \frac{xz\partial x}{ay} + q\partial y$, et sumta y constante habebitur aequatio $\partial z - \frac{xz\partial x}{ay} = 0$, ut sit $V = \frac{xz}{ay}$, et multiplicator erit $M = \frac{1}{z}$; unde fit

$$S = l\frac{z}{c} - \frac{xx + aa}{2ay},$$

et aequatio integralis completa functionem z determinans erit

$$l\frac{z}{c} + \frac{aa - xx}{2ay} = f : y.$$

Porro ad quantitatem q inveniendam, ob $M = \frac{1}{z}$ et $MV = \frac{x}{ay}$, erit $\partial Q = 0$ et $Q = 0$; unde fit $q = zf' : y$. Hic idem autem valor ex differentiatione aequationis inventae eruitur, quae praebet

$$\frac{\partial z}{z} - \frac{x\partial x}{ay} = \partial y f' : y, \text{ ideoque}$$

$$\partial z = \frac{xz\partial x}{ay} + z\partial y f' : y, \text{ ita ut sit } q = zf' : y.$$

Exemplum 2.

70. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial z}{\partial x}\right) = \frac{y}{x + z}$.

Cum sit $V = \frac{y}{x + z}$, habebitur sumto y constante haec aequatio

$$\partial z - \frac{y\partial x}{x + z} = 0,$$

ad cujus multiplicatorem inveniendum, multiplicetur primo per $x + z$, ut prodeat

$$x\partial z + z\partial z - y\partial x = 0, \text{ seu } \partial x - \frac{x\partial z}{y} = \frac{z\partial z}{y},$$

quae multiplicata per $e^{-\frac{z}{y}}$ integrabilis evadit, proditque

$$e^{-\frac{z}{y}}x = \int e^{-\frac{z}{y}}\frac{z\partial z}{y} = -e^{-\frac{z}{y}}z + \int e^{-\frac{z}{y}}\partial z;$$

hincque

$$e^{-\frac{z}{y}}x = -e^{-\frac{z}{y}}z - ye^{-\frac{z}{y}} + C.$$

Quocirca erit multiplicator

$$M = (x+z).-\frac{1}{y}.e^{-\frac{z}{y}} = -\frac{(x+z)}{y}e^{-\frac{z}{y}}, \text{ et}$$

$$S = e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y),$$

ex quo aequatio integralis completa erit

$$e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y) = f:y.$$

Nunc porro cum sit $MV = -e^{-\frac{z}{y}}$, erit

$$\left(\frac{\partial M}{\partial y}\right) = e^{-\frac{z}{y}}\left(\frac{x+z}{yy} - \frac{z(x+z)}{y^3}\right) = e^{-\frac{z}{y}}(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right), \text{ et}$$

$$\left(\frac{\partial . MV}{\partial y}\right) = -e^{-\frac{z}{y}}.\frac{z}{yy}, \text{ hincque}$$

$$\partial Q = e^{-\frac{z}{y}}\left[\partial z(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right) + \frac{z\partial x}{yy}\right].$$

sumto y constante, unde integrando obtinebitur

$$Q = e^{-\frac{z}{y}}\left(\frac{xz}{yy} + 1 + \frac{z}{y} + \frac{zz}{yy}\right) - e^{-\frac{c}{y}}\left(\frac{ac}{yy} + 1 + \frac{c}{y} + \frac{cc}{yy}\right),$$

hinc

$$q = \frac{z}{y} + \frac{y+z}{x+z} - e^{\frac{z-c}{y}}\left(\frac{ac+cc+cy+yy}{y(x+z)}\right) - \frac{y}{x+z}e^{\frac{z}{y}}f':y,$$

ita ut sit

$$\partial z = \frac{y\partial x}{x+z} + q\partial y.$$

Aequatio autem inventa si differentietur dat

$$-e^{-\frac{z}{y}}\frac{(x+z)\,\partial z}{y}+e^{-\frac{z}{y}}\partial x+e^{-\frac{z}{y}}\partial y\left(1+\frac{z}{y}+\frac{xz}{yy}+\frac{zz}{yy}\right)$$

$$-e^{-\frac{c}{y}}\partial y\left(1+\frac{c}{y}+c\frac{(a+c)}{yy}\right)=\partial y\, f':y,$$

unde idem prorsus valor pro q concluditur.

Exemplum 3.

71. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial z}{\partial x}\right)=\frac{yy+zz}{yy+xx}$.

Posito $\partial z=\frac{yy+zz}{yy+xx}\partial x+q\partial y$, sumatur quantitas y constans, et cum sit $\partial z-\frac{(yy+zz)\,\partial x}{yy+xx}=0$, evidens est multiplicatorem idoneum esse $M=\frac{y}{yy+zz}$, unde cum sit

$$\frac{y\partial z}{yy+zz}-\frac{y\partial x}{yy+xx}=0,$$

erit per integrationem

$$S=\text{A tang.}\frac{z}{y}-\text{A tang.}\frac{x}{y}+C=\text{A tang.}\frac{yz-xy}{yy+xz}-\text{A tang.}\frac{(c-a)y}{ac+yy};$$

et functio quaesita z hac aequatione definietur

$$\text{A tang.}\frac{y(z-x)}{yy+xz}-\text{A tang.}\frac{(c-a)y}{ac+yy}=f:y.$$

Cum porro sit $MV=\frac{x}{yy+xx}$, erit

$$\left(\frac{\partial M}{\partial y}\right)=\frac{zz-yy}{(yy+zz)^2}\text{ et }\left(\frac{\partial .MV}{\partial y}\right)=\frac{xx-yy}{(yy+xx)^2},$$

hincque

$$\partial Q=\frac{(zz-yy)\,\partial z}{(yy+zz)^2}-\frac{(xx-yy)\,\partial x}{(yy+xx)^2},$$

sumto y constante. Ergo

$$Q=\frac{-z}{yy+zz}+\frac{x}{yy+xx}+\frac{c}{yy+cc}-\frac{a}{yy+aa};$$

et $q=\frac{-Q+f':y}{M}$, qui idem valor etiam ex differentiatione prodit.

Caeterum cum constantes a et c pro lubitu accipi queant, sumtis iis nihilo aequalibus, seu saltem $c = a$, erit aequatio integralis

$$\text{A tang.}\ \frac{y(z-x)}{yy+xz} = \text{f} : y.,$$

unde erit etiam

$$\frac{y(z-x)}{yy+xz} = \text{f} : y \quad \text{et} \quad \frac{yy+xz}{z-x} = \text{f} : y.,$$

quae functio si dicatur Y habebitur

$$z = \frac{yy+x\text{Y}}{\text{Y}-x}.$$

Scholion.

72. Vix opus est notari, saepe fieri posse, ut solutio hujusmodi quaestionum superet vires analyseos, quando scilicet aequatio dtfferentialis resolvenda artificiis adhuc cognitis integrari nequit. Veluti si proponatur casus $\left(\frac{\partial z}{\partial x}\right) = \frac{yy}{xx+zz}$, unde sumto y constante fieri debet $yy\partial x = xx\partial z + zz\partial z$, cujus integrationem nondum expedire licet. Interim quia integrale per seriem exhiberi potest, modo id fiat complete, etiam solutio per seriem obtinebitur. Posito scilicet $x = \frac{-yy\partial u}{u\partial z}$, et sumto elemento ∂z constante, oritur haec aequatio differentio-differentialis

$$y^4 \partial\partial u + uzz\partial z^2 = 0,$$

unde per series integrando reperitur

$$u = \text{A}\left(1 - \frac{z^4}{3.4y^4} + \frac{z^8}{3.4.7.8y^8} - \text{etc.}\right) + \text{B}z\left(1 - \frac{z^4}{4.5y^4} + \frac{z^8}{4.5.8.9y^8} - \text{etc.}\right),$$

ubi pro A et B functiones quaecunque ipsius y accipi possunt. Quare si ponatur $\frac{\text{A}}{\text{B}} = \text{f} : y$, erit

$$x = \frac{yy\,\text{f} : y.\left(\frac{z^3}{3y^4} - \frac{z^7}{3.4.7y^8} + \text{etc.}\right) - yy\left(1 - \frac{z^4}{4y^4} + \frac{z^8}{4.5.8y^8} - \text{etc.}\right)}{\text{f} : y.\left(1 - \frac{z^4}{3.4y^4} + \frac{z^8}{3.4.7.8y^8} - \text{etc.}\right) + z\left(1 - \frac{z^4}{4.5y^4} + \frac{z^8}{4.5.8.9y^8} - \text{etc.}\right)},$$

qua aequatione functio quaesita z per binas variabiles x et y generalissime exprimitur. Quoniam ergo methodos aperuimus aequationes differentiales quaecunque per approximationes integrandi, idque complete; his methodis in subsidium vocandis, omnia problemata huc pertinentia saltem per approximationem sesolvi poterunt. Caeterum in hac parte Analyseos sublimiori resolutionem aequationum differentialium ad priorem partem Analysis pertinentium pro concessa assumere possumus, omnino uti, quo longius in Analysi progredimur, ea semper quae ad partes praecedentes pertinent, etiamsi non penitus sunt evoluta, tanquam confecta spectare solemus.

CAPUT III.

DE RESOLUTIONE AEQUATIONUM QUIBUS BINARUM FORMULARUM DIFFERENTIALIUM ALTERA PER ALTERAM UTCUNQUE DATUR.

Problema 10.

73.

Si z ejusmodi esse debeat functio binarum variabilium x et y, ut formulae differentiales $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ inter se fiant aequales, indolem istius functionis in genere determinare.

Solutio.

Ponatur $\left(\frac{\partial z}{\partial x}\right) = p$ et $\left(\frac{\partial z}{\partial y}\right) = q$, ut sit $\partial z = p\partial x + q\partial y$, haecque formula $p\partial x + q\partial y$ integrationem sponte admittat. Quoniam igitur requiritur ut sit $q = p$, erit $\partial z = p(\partial x + \partial y)$, et posito $x + y = u$, fiet $\partial z = p\partial u$, quae formula cum debeat esse per se integrabilis, necesse est ut p sit functio quantitatis variabilis u, nullam praeterea aliam variabilem involuens; hincque integrando ipsa quantitas $z = \int p\partial u$ aequabitur functioni ipsius u, seu prodibit $z = \mathrm{f}:u$, quae functio omnino arbitrio nostro relinquitur, ita ut pro z functio quaecunque ipsius u sive continua sive etiam discontinua assumta problemati satisfaciat. Quare cum sit $u = x + y$, erit pro solutione nostri problematis $z = \mathrm{f}:(x + y)$. Quae forma, quo facilius appareat, quomodo conditioni praescriptae satisfaciat, fit $\partial.\mathrm{f}:u = \partial u \mathrm{f}':u$, ideoque ob $u = x + y$ erit

$$\partial z = (\partial x + \partial y)\, f' : (x+y) = \partial x\, f' : (x+y) + \partial y\, f' : (x+y),$$

ideoque et

$$\left(\frac{\partial z}{\partial x}\right) = p = f' : (x+y) \quad \text{et} \quad \left(\frac{\partial z}{\partial y}\right) = q = f' : (x+y),$$

ac propterea $\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)$, seu $q = p$, omnino uti problema postulat.

Corollarium 1.

74. Quaecunque ergo functio quantitatis $x+y$ formetur, ea pro z assumta praescriptam habebit proprietatem, ut sit $\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)$. Talem autem functionem indicamus signo $f : (x+y)$, ita ut sit $z = f : (x+y)$.

Corollarium 2.

75. Geometrice haec solutio ita referri potest. Descripta super axe linea curva quacunque sive regulari sive irregulari, si abscissa exprimatur per $x+y$, applicata semper idoneum valorem pro functione z exhibebit.

Corollarium 3.

76. Universalitas hujus solutionis per integrationem erutae in hoc consistit, quod quantitatis $x+y$ functionem qualemcunque sive continuam sive etiam discontinuam pro z invenerimus; quippe quae conditioni problematis semper satisfacit.

Scholion 1.

77. Fundamentum solutionis hoc nititur principio, quod formula differentialis $p\,\partial u$ integrabilis esse nequeat, nisi quantitas p sit functio ipsius u, vel vicissim u functio ipsius p, ita ut nulla alia variabilis in computum ingrediatur. Quin etiam qualiscunque

fuerit p functio ipsius u, integrale nisi actu exhiberi, semper tamen concipi potest; si enim u denotet abscissam, et p applicatam curvae cujuscunque sive regularis, sive irregularis, qua ratione utique functio quaecunque ipsius z in sensu latissimo repraesentari potest, ejus curvae area $\int p \partial u$ praebet valorem formulae integralis $\int p \partial u$; quae iterum ut functio ipsius u spectari potest; ex quo vicissim functio quaecunque ipsius u naturam formulae integralis $\int p \partial u$ exhaurit. Quod autem functio quaecunque quantitatis $x+y$ pro z assumta satisfaciat conditioni, ut in differentiali $\partial z = p \partial x + q \partial y$ fiat $p = q$, seu $\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)$, ita per se est perspicuum, ut illustratione per exempla non egeat. Si enim verbi gratia ponatur

$$z = aa + b(x+y) + (x+y)^2 = aa + bx + by + xx + 2xy + yy,$$

erit differentiando

$$\left(\frac{\partial z}{\partial x}\right) = b + 2x + 2y \text{ et } \left(\frac{\partial z}{\partial y}\right) = b + 2x + 2y,$$

qui valores inter se utique sunt aequales.

Scholion 2.

78. Cum z sit functio binarum variabilium x et y, ac ponatur $\partial z = p \partial x + q \partial y$, ut sit

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = q,$$

in hoc capite ejusmodi quaestiones evoluere est propositum, in quibus aequatio quaecunque inter p et q praescribitur, in quam reliquarum variabilium x, y et z nulla ingrediatur. Proposita ergo aequatione quacunque inter binas formulas p et q et constantes, quaeri oportet indolem functionis z binarum variabilium x et y, ut formulis inde per differentiationem natis $p = \left(\frac{\partial z}{\partial x}\right)$ et $q = \left(\frac{\partial z}{\partial y}\right)$ praescripta illa conditio conveniat. Quam tractationem quidem exorsi sumus ab exemplo simplicissimo $p = q$, cujus solutio etiam ope principii modo expositi confici potest. At vero idem principium sufficit problemati sequenti latius patenti resolvendo.

Problema 11.

79. Si z eiusmodi esse debeat functio binarum variabilium x et y, ut fiat $\alpha\left(\frac{\partial z}{\partial x}\right)+\beta\left(\frac{\partial z}{\partial y}\right)=\gamma$, indolem istius functionis z in genere definire.

Solutio.

Posito $\partial z=p\,\partial x+q\,\partial y$, requiritur ut sit $\alpha p+\beta q=\gamma$. Hinc cum sit $q=\frac{\gamma-\alpha p}{\beta}$, erit

$$\partial z=p\,\partial x+\frac{(\gamma-\alpha p)}{\beta}\partial y, \text{ seu}$$

$$\partial z=\frac{\gamma}{\beta}\partial y+\frac{p}{\beta}(\beta\,\partial x-\alpha\,\partial y),$$

quam formulam integrabilem esse oportet. Cum autem pars $\frac{\gamma}{\beta}\partial y$ per se sit integrabilis, altera pars etiam integrabilis sit, necesse est; unde posito $\beta x-\alpha y=u$, ut altera pars fiat $\frac{p}{\beta}\partial u$, evidens est, p functionem esse debere ipsius u, indeque etiam integrale proditurum esse functionem ipsius $u=\beta x-\alpha y$. Quare ponamus

$$\int p(\beta\,\partial x-\alpha\,\partial y)=\mathrm{f}:(\beta x-\alpha y),$$

eritque

$$z=\frac{\gamma}{\beta}y+\frac{1}{\beta}\mathrm{f}:(\beta x-\alpha y),$$

seu aequatio quaesita indolem functionis z determinans erit

$$\beta z=\gamma y+\mathrm{f}:(\beta x-\alpha y),$$

denotante signo f: functionem quamcunque sive continuam sive discontinuam formulae suffixae $\beta x-\alpha y$. Atque indicando formulae $\mathrm{f}:u$ differentiale per $\partial u\,\mathrm{f}':u$, erit

$$p = f':(\beta x - \alpha y) \text{ et } q = \frac{\gamma}{\beta} - \frac{\alpha}{\beta} f':(\beta x - \alpha y);$$

unde manifesto resultat $\alpha p + \beta q = \gamma$.

Corollarium 1.

80. Eodem solutio redit, si pro p ejus valorem $p = \frac{\gamma - \beta q}{\alpha}$ substituamus, unde fit

$$\partial z = \frac{\gamma}{\alpha} \partial x + \frac{q}{\alpha}(\alpha \partial y - \beta \partial x),$$

hincque eodem modo

$$z = \frac{\gamma x}{\alpha} + \frac{1}{\alpha} f:(\alpha y - \beta x).$$

Etsi enim haec forma a praecedente differre videtur, tamen facile eo reducitur, ponendo ibi

$$f:(\beta x - \alpha y) = \frac{\gamma(\beta x - \alpha y)}{\alpha} + \frac{\beta}{\alpha} \Phi:(\alpha y - \beta x),$$

quae forma utique est functio ipsius $\beta x - \alpha y$.

Corollarium 2.

81. Si ergo in forma $\partial z = p \partial x + q \partial y$ debeat esse $p + q = 1$, ob $\alpha = 1$, $\beta = 1$ et $\gamma = 1$, solutio huc redit, ut fiat

$$z = y + f:(x - y).$$

Constructa ergo curva quaecunque, si abscissae $x - y$ respondeat applicata v, erit $z = y + v$.

Scholion.

82. Si alia proponatur relatio inter p et q, eadem methodo solutionem obtinere non licet; sed alio principio uti convenit, cujus quidem veritas ex primis calculi integralis elementis est manifesta. Notari scilicet oportet esse

$$\int p \partial x = p x - \int x \partial p,$$

similique modo

$$\int q \partial y = q y - \int y \partial q,$$

ita ut cum sit

$$z = \int (p \partial x + q \partial y),$$

futurum sit

$$z = p x + q y - \int (x \partial p + y \partial q).$$

Quomodo autem hoc principium ad solutionem hujusmodi quaestionum, quae ad hoc caput sint referendae, applicandum sit, in sequentibus problematibus docebitur.

Problema 12.

83. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z = p \partial x + q \partial y$, fiat $p q = 1$, indolem istius functionis z in genere definire.

Solutio.

Ex principio ante stabilito notemus fore

$$z = p x + q y - \int (x \partial p + y \partial q).$$

Cum jam ob $p q = 1$ sit $q = \frac{1}{p}$, erit

$$z = p x + \frac{y}{p} - \int \left(x \partial p - \frac{y \partial p}{p p}\right).$$

Integrabilis ergo esse debet haec forma $\int \left(x - \frac{y}{p p}\right) \partial p$, at in genere formula $\int u \partial p$ integrationem non admittit, nisi sit u functio ipsius p. Quare in nostro casu necesse est sit quantitas $x - \frac{y}{p p}$ functio ipsius p tantum, unde etiam integrale $\int \partial p \left(x - \frac{y}{p p}\right)$ erit functio ipsius p tantum, quae si indicetur per $f:p$ ejusque differentiale per $\partial p f':p$, erit

$$z = p x + \frac{y}{p} - f:p, \quad \text{et} \quad x - \frac{y}{p p} = f':p.$$

Quare ad problema nostrum solvendum, nova variabilis p introduci debet, ex qua cum altera y conjuncta binae reliquae x et z determinentur. Sumta scilicet variabili p ejusque functione quacunque $f:p$, indeque per differentiationem derivata $f':p$, capiatur primo

$$x = \frac{y}{pp} + f':p, \text{ indeque erit}$$
$$z = \frac{2y}{p} + p\, f':p - f:p,$$

quae est solutio problematis quaesita generalis.

Corollarium 1.

84. Hic igitur functio quaesita z per ipsas variabiles x et y explicite evolui nequit; propterea quod quantitatem p ex aequatione $x - \frac{y}{pp} = f':p$ in genere per x et y definire non licet.

Corollarium 2.

85. Nihilo vero minus solutio pro idonea et completa est habenda, quoniam introducendo novam variabilem p, ex binis y et p a se invicem non pendentibus ambae reliquae x et z definiuntur.

Corollarium 3.

86. Si sumamus $f':p = \alpha + \frac{\beta}{pp}$, erit

$$f:p = \alpha p - \frac{\beta}{p} \text{ et } (x - \alpha) = \frac{\beta + y}{pp},$$

hinc $p = \sqrt{\frac{\beta + y}{x - \alpha}}$; unde functio quaesita z ita se habebit

$$z = \frac{2y\sqrt{(x-\alpha)}}{\sqrt{(\beta+y)}} + \frac{\alpha y + \beta x}{\sqrt{(x-\alpha)(\beta+y)}} - \frac{\alpha y + \beta x - 2\alpha\beta}{\sqrt{(x-\alpha)(\beta+y)}},$$

seu $z = 2\sqrt{(x-\alpha)(y+\beta)}$, quae est solutio particularis, et simplicissima est $z = 2\sqrt{xy}$.

Scholion 1.

87. Quemadmodum solutio hujus problematis ex alio principio est deducta, ita etiam forma solutionis a praecedentibus discrepat, quod hic aequationem inter x, y et z explicitam exhibere non liceat, sed nova variabilis p introducatur. Cum igitur ante una aequatio inter ternas variabiles x, y et z solutionem continuisset, nunc accedente nova variabili p, solutio geminam aequationem inter has quatuor variabiles postulat, sicque pro nostro casu invenimus

$$z = px + \frac{y}{p} - f:p \text{ et } x - \frac{y}{pp} = f':p,$$

existente

$$\partial . f:p = \partial p f':p,$$

ubi functionis signum indefinitum f: quod etiam functiones discontinuas admittit, universalitatem solutionis praestat. Quod si hinc litteram p eliminare liceret, aequatio evoluta inter x, y et z obtineretur; haec autem eliminatio succedit, quoties pro $f:p$ functio algebraica ipsius p assumitur, in genere autem nullo modo sperari potest. Nihilo vero minus ope curvae pro lubitu assumtae problema construi potest: sumta enim curva quacunque sive regulari sive irregulari, ponatur abscissa $= p$, sitque applicata $f':p = r$, erit $f:p = \int r \partial p$ area ejus curvae, quae si dicatur $= s$, aequationes binae

$$x - \frac{y}{pp} = r \text{ et } z = px + \frac{y}{p} - s,$$

solutionem completam problematis praebebunt. Scilicet sumto pro x valore quocunque, erit $y = pp(x - r)$, hincque fit

$$z = 2px - pr - s,$$

in qua solutione nihil ad praxin spectans desiderari potest. Hinc patet etiam fortasse fieri posse, ut duae novae variabiles sint introducendae, ac tum solutio tribus aequationibus contineatur; neque etiam tum quicquam deerit ad usum practicum.

Scholion 2.

88. Cum pro formula $\partial z = p\partial x + q\partial y$ requiratur ut sit $pq = 1$, introducendo angulum indefinitum Φ alia solutio concinnior elici potest. Posito enim $p = \text{tang.}\Phi$ erit $q = \text{cot.}\Phi$, et ob $\partial z = \partial x\,\text{tang.}\Phi + \partial y\,\text{cot.}\Phi$, fiet per reductionem supra indicatam

$$z = x\,\text{tang.}\Phi + y\,\text{cot.}\Phi - \int\partial\Phi\left(\frac{x}{\text{cos.}\Phi^2} - \frac{y}{\text{sin.}\Phi^2}\right),$$

unde patet formulam $\frac{x}{\text{cos.}\Phi^2} - \frac{y}{\text{sin.}\Phi^2}$ esse debere functionem ipsius Φ, quae si ponatur $f':\Phi$, et formula integralis

$$\int\partial\Phi\,.\,f':\Phi = f:\Phi,$$

binae aequationes solutionem continentes erunt

$$\frac{x}{\text{cos.}\Phi^2} - \frac{y}{\text{sin.}\Phi^2} = f':\Phi \text{ et } z = x\,\text{tang.}\Phi + y\,\text{cot.}\Phi - f:\Phi,$$

unde jam pro lubitu x vel y eliminare licet. Quin etiam utramque eliminare possumus, ac per binas variabiles z et Φ binae reliquae x et y ita exprimentur

$$x = \tfrac{1}{2}z\,\text{cot.}\Phi + \tfrac{1}{2}\,\text{cot.}\Phi\,.\,f:\Phi + \tfrac{1}{2}\,\text{cos.}\Phi^2\,.\,f':\Phi,$$
$$y = \tfrac{1}{2}z\,\text{tang.}\Phi + \tfrac{1}{2}\,\text{tang.}\Phi\,.\,f:\Phi - \tfrac{1}{2}\,\text{sin.}\Phi^2\,.\,f':\Phi.$$

Quodsi igitur hinc differentialia capiantur, ac ponatur $\partial y = 0$, ex posteriori dabitur relatio inter ∂z et $\partial\Phi$, unde si ipsius $\partial\Phi$ valor in priori substituatur, necesse est prodeat

$$\partial z = \partial x\ \text{tang.}\Phi;$$

simili autem modo si ponatur $\partial x = 0$, ex altera orietur

$$\partial z = \partial y\ \text{cot.}\Phi.$$

Problema 13.

89. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z = p\,\partial x + q\,\partial y$ fiat $pp + qq = 1$, indolem istius functionis z in genere investigare.

Solutio.

Cum per reductionem fiat

$$z = px + qy - \int (x\,\partial p + y\,\partial q),$$

ut irrationalia evitemus, ponamus

$$p = \frac{1 - rr}{1 + rr} \text{ et } q = \frac{2r}{1 + rr},$$

siquidem hinc fit $pp + qq = 1$. Erit autem

$$\partial p = \frac{-4r\,\partial r}{(1 + rr)^2}, \text{ et } \partial q = \frac{2\,\partial r\,(1 - rr)}{(1 + rr)^2},$$

hincque fit

$$z = \frac{(1 - rr)x + 2ry}{1 + rr} + 2\int \frac{2xr\,\partial r - y\,\partial r\,(1 - rr)}{(1 + rr)^2},$$

quae forma integralis cum sit functio ipsius r, statuatur ea $= f:r$, ejusque differentiale $= \partial r\, f':r$, ex quo obtinebimus

$$\frac{2xr - y(1 - rr)}{(1 + rr)^2} = f':r \text{ et}$$

$$z = \frac{(1 - rr)x + 2ry}{1 + rr} + 2\, f:r.$$

Unde si eliciamus

$$x = \frac{(1 - rr)y}{2r} + \frac{(1 + rr)^2}{2r} f':r, \text{ erit}$$

$$z = \frac{(1 + rr)y}{2r} + \frac{1 - r^4}{2r} f':r + 2\, f:r.$$

Corollarium 1.

90. Si irrationalitatem non pertimescamus ob

$$q = \sqrt{(1 - pp)} \text{ et } \partial q = \frac{-p\,\partial p}{\sqrt{(1 - pp)}}, \text{ erit}$$

$$z = px + y\sqrt{(1 - pp)} - \int \partial p \left(x - \frac{py}{\sqrt{(1 - pp)}}\right).$$

Posito ergo $z = px + y\sqrt{(1 - pp)} - f:p$, erit

$$x - \frac{py}{\sqrt{(1 - pp)}} = f':p.$$

Corollarium 2.

91. Solutio simplicissima sine dubio prodit sumendo $f:p = 0$, unde cum sit $x = \frac{py}{\sqrt{(1 - pp)}}$, erit

$$p = \frac{x}{\sqrt{(xx + yy)}} \text{ et } \sqrt{(1 - pp)} = \frac{y}{\sqrt{(xx + yy)}},$$

hincque

$$z = \frac{xx + yy}{\sqrt{(xx + yy)}} = \sqrt{(xx + yy)}.$$

Ex quo valore fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{x}{\sqrt{(xx + yy)}} = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = \frac{y}{\sqrt{(xx + yy)}} = q,$$

ideoque $pp + qq = 1$.

Corollarium 3.

92. Si ponamus $p = \sin.\Phi$, erit $q = \cos.\Phi$, hinc

$$z = x\sin.\Phi + y\cos.\Phi - \int \partial\Phi\,(x\cos.\Phi - y\sin.\Phi),$$

eritque hoc integrale $= f:\Phi$, ejusque differentiale $\partial\Phi\, f':\Phi$. Ex quo habebimus

$$z = x\sin.\Phi + y\cos.\Phi - f:\Phi \text{ et } x\cos.\Phi - y\sin.\Phi = f':\Phi.$$

Problema 14.

93. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z = p\,\partial x + q\,\partial y$, quantitas q aequetur functioni datae ipsius p, indolem hujus functionis z in genere investigare.

Solutio.

Cum q sit functio data ipsius p, ponatur $\partial q = r\,\partial p$, erit etiam r functio data ipsius p. Aequatio ergo nostra generalis solutionem suppeditans induet hanc formam

$$z = px + qy - \int \partial p\,(x + ry),$$

unde patet integrale $\int \partial p\,(x + ry)$ fore functionem ipsius p, quae si generatim per $f : p$ exponatur, ejusque differentiale per $\partial p\, f' : p$, habebimus

$$z = px + qy - f : p \quad \text{et} \quad x + ry = f' : p,$$

quae duae aequationes solutionem problematis universalissime complectuntur, siquidem $f : p$ functionem quamcunque ipsius p sive continuam sive discontinuam denotare potest.

Corollarium 1.

94. Cum sit q functio data ipsius p, indeque $r = \frac{\partial q}{\partial p}$, si functio indefinita ipsius p ponatur $f : p = P$, ob $f' p = \frac{\partial P}{\partial p}$, solutio his aequationibus continebitur

$$z = px + qy - P \quad \text{et} \quad x\partial p + y\partial q = \partial P.$$

Corollarium 2.

95. Si ad constructionem utamur curva quacunque, in qua si abscissa capiatur $= p$, applicata sit $= f' : p$, area ejus curvae dabit valorem ipsius $f : p$. Sin autem applicata indicetur per $f : p$, tum $f' : p$ exprimet tangentem anguli, quem tangens curvae faciet cum axe.

Scholion.

96. Duplici ergo modo curva quaecunque ad libitum descripta, sive sit continua seu aequatione quapiam analytica contenta, sive libero manus ductu utcunque delineata, ad constructionem problematis adhiberi potest. Vel enim abscissa per p indicata, applicata sumi potest ad $f : p$ vel ad $f' : p$ exprimendum, nec facile dici potest, utrum ad praxin commodius sit futurum? Ubi autem hujusmodi

problemata realia occurrunt, reliquae circumstantiae solutionem determinare solent, unde pro quovis casu constructio maxime idonea facile colligetur. Problemata autem mechanica hanc calculi integralis partem postulantia semper ad formulas differentiales secundi altiorumque ordinum deducunt, quarum resolutio ne suscipi quidem posse ante videtur, quam methodus pro formulis differentialibus primi gradus fuerit patefacta. Hactenus quidem problemata proposita absolute resolvere licuit; nunc autem quando conditio praescripta relationem formularum $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ per reliquas variabiles x, y et z definit, negotium in genere non amplius succedit, nisi relatio praescripta unicam tantum variabilem cum binis formulis differentialibus conjungat.

CAPUT IV.

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES ET UNICAM TRIUM QUANTITATUM VARIABILIUM PROPONITUR.

Problema 15.

97.

Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z = p\,\partial x + q\,\partial y$ sit $q = \frac{px}{a}$, indolem hujus functionis in genere investigare.

Solutio.

Cum sit

$$\partial z = p\,\partial x + \frac{px\,\partial y}{a} = px\left(\frac{\partial x}{x} + \frac{\partial y}{a}\right),$$

haecque formula esse debeat integrabilis, necesse est ut px ac proinde etiam z, sit functio quantitatis $lx + \frac{y}{a}$. Quare solutio nostri problematis in genere ita se habebit, ut sit

$$z = f:(lx + \tfrac{y}{a}) \text{ et } px = f':(lx + \tfrac{y}{a}),$$

sumendo scilicet perpetuo $\partial.f:u = \partial u f':u$. Hinc autem erit

$$p = \tfrac{1}{x} f':(lx + \tfrac{y}{a}) \text{ et } q = \tfrac{1}{a} f':(lx + \tfrac{y}{a}),$$

sicque $q = \frac{px}{a}$ omnino uti requiritur.

Corollarium 1.

98. Cum sit

$$z = px - \int x\,\partial p + \int \frac{px\,\partial y}{a} = px + \int px\left(\frac{\partial y}{a} - \frac{\partial p}{p}\right),$$

hinc alia solutio deduci potest. Si enim ponamus

$$\int px\left(\frac{\partial y}{a} - \frac{\partial p}{p}\right) = f:\left(\frac{y}{a} - lp\right), \text{ erit } px = f':\left(\frac{y}{a} - lp\right),$$

indeque

$$z = f':\left(\frac{y}{a} - lp\right) + f:\left(\frac{y}{a} - lp\right).$$

Corollarium 2.

99. Hac ergo solutione nova introducitur variabilis p, ex qua cum y conjuncta definitur primo

$$x = \frac{1}{p} f':\left(\frac{y}{a} - lp\right),$$

tum vero ipsa functio quaesita

$$z = px + f:\left(\frac{y}{a} - lp\right).$$

Huic autem solutioni praecedens sine dubio antecellit, cum illa quantitatem z immediate per x et y exprimat.

Scholion.

100. Quo has duas solutiones inter se comparare queamus, quoniam functio arbitraria in utraque diversae est indolis, etiam charactere diverso utamur. Cum igitur prima praebeat

$$z = f:\left(\frac{y}{a} + lx\right) \text{ et } px = f':\left(\frac{y}{a} + lx\right),$$

altera vero

$$z = F:\left(\frac{y}{a} - lp\right) + F':\left(\frac{y}{a} - lp\right) \text{ et } px = F':\left(\frac{y}{a} - lp\right),$$

patet fore

$$f':\left(\frac{y}{a} + lx\right) = F':\left(\frac{y}{a} - lp\right) \text{ et}$$

$$f':\left(\tfrac{y}{a}+lx\right)=F:\left(\tfrac{y}{a}-lp\right)+F':\left(\tfrac{y}{a}-lp\right),$$

unde non solum relatio inter utriusque functionis f et F indolem definitur, sed etiam inde sequi debet, fore

$$px=f':\left(\tfrac{y}{a}+lx\right);$$

id quod non parum videtur absconditum. Verum ob hoc ipsum istud problema eo magis est notatu dignum, quod solutio altera, qua nova variabilis p introducitur, congruit cum priore, ubi z per x et y immediate definitur, neque tamen consensus harum solutionum perspicue monstrari potest. Quamobrem quando ad ejusmodi solutiones pervenimus, uti in problematibus posterioribus capitis praecedentis usu venit, in quibus nova variabilis introducitur, non omnem statim spem ejus eliminandae abjicere debemus, cum isto casu altera solutio ad priorem certe sit reductibilis, etiamsi methodus reducendi non perspiciatur, quam tamen infra §. 119. exhibebimus.

Problema 16.

101. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z=p\,\partial x+q\,\partial y$, sit $q=pX+T$, existentibus X et T functionibus quibuscunque ipsius x, indolem istius functionis z in genere investigare.

Solutio.

Cum ergo sit $\partial z=p\,\partial x+pX\,\partial y+T\,\partial y$, statuatur $p=r-\frac{T}{X}$ ut prodeat

$$\partial z=r\,\partial x-\frac{T\,\partial x}{X}+rX\,\partial y=\frac{-T\,\partial x}{X}+rX\left(\frac{\partial x}{X}+\partial y\right);$$

qua reductione facta perspicuum est, tam rX quam $\int rX\left(\frac{\partial x}{X}+\partial y\right)$ fore functionem quantitatis $y+\int\frac{\partial x}{X}$. Quare si ponamus

$$\int r\,X\left(\frac{\partial x}{X}+\partial y\right)=f:\left(y+\int\frac{\partial x}{X}\right);\text{ erit}$$

$$rX=f':\left(y+\int\frac{\partial x}{X}\right),$$

ac tum functio quaesita erit

$$z=-\int\frac{T\partial x}{X}+f:\left(y+\int\frac{\partial x}{X}\right);$$

quae ob functionem indefinitam f: est completa. Tum vero erit

$$p=\frac{-T}{X}+\frac{1}{X}f':\left(y+\int\frac{\partial x}{X}\right)\text{ et}$$

$$q=f':\left(y+\int\frac{\partial x}{X}\right):$$

unde patet fore utique $q=pX+T$. Quoniam vero X et T sunt functiones datae ipsius x, formulae integrales $\int\frac{\partial x}{X}$ et $\int\frac{T\partial x}{X}$ solutionem non turbant.

Corollarium 1.

102. Solutio aliquanto facilior redditur sumendo ex conditione praescripta $p=\frac{q}{X}-\frac{T}{X}$, unde fit

$$\partial z=-\frac{T\partial x}{X}+\frac{q\partial x}{X}+q\partial y\text{ et}$$

$$z=-\int\frac{T\partial x}{X}+\int q\left(\partial y+\frac{\partial x}{X}\right).$$

Jam manifesto est

$$\int q\left(\partial y+\frac{\partial x}{X}\right)=f:\left(y+\int\frac{\partial x}{X}\right),$$

sicque ipsa solutio praecedens resultat.

Corollarium 2.

103. Eodem modo resolvitur problema, si proponatur conditio $q=pY+V$, existentibus Y et V functionibus datis ipsius y. Tum enim erit

$$\partial z=p\partial x+pY\partial y+V\partial y,\quad\text{et}\quad z=\int V\partial y+\int p(\partial x+Y\partial y).$$

Hic ergo fit

$$\int p(\partial x + Y\partial y) = f:(x + \int Y\partial y),$$

et solutio erit

$$z = \int V\partial y + f:(x + \int Y\partial y);$$

unde fit

$$p = f':(x + \int Y\partial y) \text{ et } q = V + Y f':(x + \int Y\partial y).$$

Scholion.

104. Ex forma solutionis hic inventae discere poterimus, quomodo problema comparatum esse debeat, ut ejus solutio hac ratione perfici, et functio z per binas variabiles x et y exhiberi queat. Sint enim K et V functiones quaecunque ipsarum x et y, indeque differentiando

$$\partial K = L\partial x + M\partial y \text{ et } \partial V = P\partial x + Q\partial y.$$

Jam a solutione incipiamus, ponamusque

$$z = K + f:V,$$

eritque differentiando

$$\partial z = L\partial x + M\partial y + (P\partial x + Q\partial y) f':V.$$

Cum jam hanc formam cum assumta

$$\partial z = p\partial x + q\partial y$$

comparando, fit

$$p = L + P f':V \text{ et } q = M + Q f':V, \text{ erit}$$

$$Qp - Pq = LQ - MP.$$

Quare si hoc problema proponatur, ut posito

$$\partial z = p\partial x + q\partial y,$$

fieri debeat

$$q = \frac{Q}{P} p + M - \frac{LQ}{P},$$

solutio erit $z = K + f:V$; dummodo M et L itemque P et Q ita sint comparatae, ut sit

$$L\partial x + M\partial y = \partial K \text{ et } P\partial x + Q\partial y = \partial V,$$

verum hi casus ad sequens caput sunt referendi.

Problema 17.

105. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut posito $\partial z = p\partial x + q\partial y$, sit $q = Px + \Pi$ existentibus P et Π functionibus datis ipsius p; indolem istius functionis z in genere investigare.

Solutio.

Cum igitur sit

$$\partial z = p\partial x + Px\partial y + \Pi\partial y, \text{ erit}$$
$$z = px + \int (Px\partial y + \Pi\partial y - x\partial p).$$

Sumatur $Px + \Pi = v$, ut sit $x = \frac{v - \Pi}{P}$, fietque

$$z = px + \int \left(v\partial y - \frac{v\partial p}{P} + \frac{\Pi\partial p}{P}\right).$$

Quare cum P et Π sint functiones ipsius p, ideoque formula $\int \frac{\Pi\partial p}{P}$ data, habebitur

$$z = px + \int \frac{\Pi\partial p}{P} + \int v \left(\partial y - \frac{\partial p}{P}\right),$$

unde patet tam v quam $\int v \left(\partial y - \frac{\partial p}{P}\right)$ functionem esse debere formulae $y - \int \frac{\partial p}{P}$. Ponamus ergo

$$\int v \left(\partial y - \frac{\partial p}{P}\right) = f:\left(y - \int \frac{\partial p}{P}\right), \text{ eritque}$$
$$v = Px + \Pi = f':\left(y - \int \frac{\partial p}{P}\right).$$

et hinc

$$x = \frac{-\Pi}{P} + \frac{1}{P} f':\left(y - \int \frac{\partial p}{P}\right),$$

tum vero

$$z = \int \frac{\Pi\partial p}{P} - \frac{\Pi p}{P} + \frac{p}{P} f':\left(y - \int \frac{\partial p}{P}\right) + f:\left(y - \int \frac{\partial p}{P}\right).$$

Corollarium 1.

106. In solutione hujus problematis iterum nova variabilis p introducitur, ex qua cum y conjunctim primo variabilis x, tum vero ipsa functio quaesita z determinatur.

Corollarium 2.

107. Neque vero hinc istam novam variabilem p ex calculo elidere licet, uti ante usu venit; propterea quod hic P et Π functiones ipsius p denotant, quarum indoles jam in ipsum problema ingreditur.

Corollarium 3.

108. Simili modo problema resolvetur, si permutandis x et y, quantitas p ita per y et q detur, ut sit $p = \mathfrak{Q}y + \mathfrak{Z}$, denotantibus $\mathfrak{Q}$ et $\mathfrak{Z}$ functiones datas ipsius q.

Scholion.

109. In hoc capite constituimus ejusmodi problemata tractare, quorum conditio aequatione inter binas formulas differentiales $\left(\frac{\partial z}{\partial x}\right) = p$; $\left(\frac{\partial z}{\partial y}\right) = q$ et unam ex tribus variabilibus x, y et z utcunque exprimitur. Problemata autem bina evoluta ex hoc genere certos casus complectuntur, quorum solutio peculiari methodo expediri potest, simulque ad formulas simpliciores perducitur. In posteriori quidem relationem inter p, q et x ita assumsimus, ut sit $q = \mathrm{P}x + \Pi$, seu ut in valore ipsius q, per p et x expresso, quantitas x unam dimensionem non excedat; in priori vero ita ut sit $q = p\mathrm{X} + \mathrm{T}$, seu ut in valore ipsius q, per p et x expresso, quantitas p unicam obtineat dimensionem. In genere autem notasse juvabit, tam quantitates p et x quam q et y inter se esse permutabiles. Cum enim sit

$$\int p\,\partial x = px - \int x\,\partial p,$$

loco

$$z = \int (p\,\partial x + q\,\partial y),$$

erit

$$z = px + \int (q\,\partial y - x\,\partial p).$$

Simili modo est

$$z = qy + \int (p\,\partial x - y\,\partial q),$$

tum vero etiam

$$z = px + qy - \int (x\,\partial p + y\,\partial q).$$

Quibus ergo casibus una harum quatuor formularum integralium redditur integrabilis, iisdem ternae reliquae etiam integrationem admittent. Cum igitur in superiori capite primam formulam resolverimus, si p vel q quomodocunque detur per x et y; ita eodem modo resolvetur formula secunda, si q per p et y, tertia autem si p per x et q, at quarta si vel x per p et q vel y per p et q utcunque datur; quae quaestiones cum generaliter expediri queant, eas in sequenti problemate evolvamus.

Problema 18.

110. Posito $\partial z = p\,\partial x + q\,\partial y$, si relatio inter p, q et x aequatione quacunque definiatur, indolem functionis z, quemadmodum ex binis variabilibus x et y determinetur, in genere investigare.

Solutio.

Ex aequatione inter p, q et x proposita quaeratur valor ipsius x qui functioni cuipiam ipsarum p et q aequabitur. Cum jam sit

$$z = px + qy - \int (x\,\partial p + y\,\partial q),$$

quoniam x est functio data ipsarum p et q, formula $x\partial p$ integretur sumta quantitate q constante, sitque

$$\int x\,\partial p = \mathrm{V} + \mathrm{f}:q,$$

et erit V functio cognita ipsarum p et q, quae differentiata prodeat

$$\partial \mathrm{V} = x\,\partial p + \mathrm{S}\,\partial q,$$

ubi S quoque erit functio data ipsarum p et q. Quia jam forma $\int(x\partial p + y\partial q)$ integrationem admittere debet, aequabitur formae $\mathrm{V} + \mathrm{f}:q$, unde differentiando concluditur

$$x\partial p + y\partial q = x\partial p + \mathrm{S}\partial q + \partial q\,\mathrm{f}':q,$$

ideoque

$$y = \mathrm{S} + \mathrm{f}':q \text{ et } z = px + qy - \mathrm{V} - \mathrm{f}:q, \text{ seu}$$
$$z = px + \mathrm{S}q + q\mathrm{f}':q - \mathrm{f}:q - \mathrm{V}.$$

Solutio ergo ita se habet: primo ex conditione praescripta datur x per p et q, tum sumta q constante sit $\mathrm{V} = \int x\,\partial p$, et vicissim $\partial \mathrm{V} = x\,\partial p + \mathrm{S}\,\partial q$; inventis autem V et S per p et q, reliquae quantitates y et z ita per easdem exprimentur ut sit

$$y = \mathrm{S} + \mathrm{f}':q \text{ et } z = px + \mathrm{S}q + q\mathrm{f}':q - \mathrm{f}:q - \mathrm{V},$$

quae solutio, quia $\mathrm{f}:q$ functionem quamcunque ipsius q sive continuam sive discontinuam denotat, utique pro completa latissimeque patente est habenda.

Aliter.

111. Vel ex aequatione inter p, q et x data, quaeratur valor ipsius p per x et q expressus, ita ut p aequetur functioni cuipiam datae binarum variabilium x et q, per quas etiam reliquas quantitates y et z definire conemur. Ad hoc utamur formula

$$z = qy + \int(p\,\partial x - y\,\partial q),$$

et quia p est functio ipsarum x et q, dabitur earundem ejusmodi functio V ut sit

$$\partial V = p\partial x + R\partial q.$$

Statuatur ergo

$$\int (p\partial x - y\partial q) = V + f : q,$$

eritque

$$y = - R - f' : q \quad \text{et} \quad z = qy + V + f : q.$$

Corollarium 1.

112. Utraque solutio aeque commode adhiberi potest, si ex relatione inter p, q et x proposita, tam quantitatem x quam p aeque commode definire liceat. Sin autem earum altera commodius definiri queat, ea solutione, quae ad istum casum est accommodata, erit utendum.

Corollarium 2.

113. Sin autem neque p neque x commode elici queat, tum nihilo minus hic resolutio aequationum cujusque ordinis, quin etiam transcendentium tanquam concessa assumitur. Caeterum etiamsi q facile per p et x definiatur, hinc calculus nihil juvatur.

Corollarium 3.

114. Ex hoc problemate utpote latissime patente etiam bina praecedentia resolvi possunt; solutio autem hinc inventa a praecedente discrepabit, cum illa ex methodo particulari sit deducta: operae autem pretium erit, has duplices solutiones inter se comparare.

Exemplum 1.

115. *Si fuerit* $q = pX + T$, *existentibus* X *et* T *functionibus ipsius* x, *indolem functionis* z *investigare.*

Hic solutione utendum est posteriori, pro qua est $p = \frac{q-T}{X}$; nunc posita q constante prodit

$$V = \int p\partial x = q\int\frac{\partial x}{X} - \int\frac{T\partial x}{X},$$

hincque

$$R = \left(\frac{\partial V}{\partial q}\right) = \int\frac{\partial x}{X};$$

unde solutio his formulis continetur

$$q = pX + T,\ y = -\int\frac{\partial x}{X} - f' : q,\ z = -\int\frac{T\partial x}{X} - qf' : q + f : q,$$

solutio autem superior ita se habebat

$$q = pX + T,\ q = f' : \left(y + \int\frac{\partial x}{X}\right) \text{ et } z = -\int\frac{T\partial x}{X} + f : \left(y + \int\frac{\partial x}{X}\right).$$

Scholion.

116. Consensus harum duarum solutionum ita ostendi potest, ut ex ea, quam hic invenimus, antecedens per legitimam consequentiam formetur. Cum enim sit

$$f' : q = -y - \int\frac{\partial x}{X},$$

statuatur brevitatis gratia $y + \int\frac{\partial x}{X} = v$, ut sit $f' : q = -v$, erit ergo vicissim q aequalis functioni cuidam ipsius v, quae ponatur $q = F' : v$, unde fit $\partial q = \partial v F'' : v$, ergo

$$\partial q f' : q = -v\partial v F'' : v = -v\partial . F' : v,$$

ergo integrando

$$f : q = -\int v\partial . F' : v = -vF' : v + \int\partial v . F' : v = -vF' : v + F : v.$$

Quare cum sit

$$z = -\int\frac{T\partial x}{X} - qf' : q + f' : q, \text{ erit}$$

$$z = -\int\frac{T\partial x}{X} + vF' : v - vF' : v + F : v, \text{ seu}$$

$$z = -\int\frac{T\partial x}{X} + F : \left(y + \int\frac{\partial x}{X}\right),$$

quae est ipsa solutio praecedens.

Exemplum 2.

117. *Si fuerit* $q = Px + \Pi$, *existentibus* P *et* Π *functionibus datis ipsius* p, *indolem functionis* z, *ut sit*
$$\partial z = p\partial x + q\partial y,$$
investigare.

Hic solutione priori utendum, cum sit $x = \frac{q-\Pi}{P}$. Sumto ergo q constante quaeratur
$$V = \int x\partial p = q\int\frac{\partial p}{P} - \int\frac{\Pi\partial p}{P},$$
unde fit
$$S = \left(\frac{\partial V}{\partial q}\right) = \int\frac{\partial p}{P}.$$
Solutio ergo praebet
$$y = \int\frac{\partial p}{P} + f':q \quad \text{et}$$
$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int\frac{\partial p}{P} + qf':q - f:q - q\int\frac{\partial p}{P} + \int\frac{\Pi\partial p}{P},$$
sive
$$z = \frac{p(q-\Pi)}{P} + \int\frac{\Pi\partial p}{P} + qf':q - f:q.$$
Solutio autem ejusdem casus supra (106.) inventa erat
$$x = \frac{-\Pi}{P} + \frac{1}{P}f':\left(y - \int\frac{\partial p}{P}\right) \text{ et } q = Px + \Pi,$$
atque
$$z = \frac{-p\Pi}{P} + \int\frac{\Pi\partial p}{P} + \frac{p}{P}f':\left(y - \int\frac{\partial p}{P}\right) + f:\left(y - \int\frac{\partial p}{P}\right).$$

Scholion 1.

118. Videamus quomodo solutio hic inventa ad superiorem reduci queat. Cum ibi invenerimus
$$y - \int\frac{\partial p}{P} = f':q,$$
vicissim q aequabitur functioni quantitatis $y - \int\frac{\partial p}{P}$, ponatur ergo
$$q = F':\left(y - \int\frac{\partial p}{P}\right),$$

eritque statim

$$x = \frac{-\Pi}{P} + \frac{1}{P} F' : (y - \int \frac{\partial p}{P});$$

sit brevitatis gratia $y - \int \frac{\partial p}{P} = v$, ut fiat

$$q = F' : v \quad \text{et} \quad v = f' : q, \quad \text{erit}$$

$$F : v = \int q \partial v = qv - \int v \partial q = qv - \int \partial q f' : q.$$

Ergo $F : v = qv - f : q$, ita ut sit

$$f : q = q (y - \int \frac{\partial p}{P}) - F : (y - \int \frac{\partial p}{P}), \quad \text{seu}$$

$$f : q = (y - \int \frac{\partial p}{P}) F' : (y - \int \frac{\partial p}{P}) - F : (y - \int \frac{\partial p}{P}).$$

Quibus valoribus substitutis habebimus

$$x = \frac{-\Pi}{P} + \frac{1}{P} F' : (y - \int \frac{\partial p}{P}) \quad \text{et}$$

$$z = \frac{-p\Pi}{P} + \frac{p}{P} F' : (y - \int \frac{\partial p}{P}) + \int \frac{\Pi \partial p}{P} + (y - \int \frac{\partial p}{P}) F' : (y - \int \frac{\partial p}{P})$$
$$- (y - \int \frac{\partial p}{P}) F' : (y - \int \frac{\partial p}{P}) + F : (y - \int \frac{\partial p}{P}), \quad \text{seu}$$

$$z = \frac{-p\Pi}{P} + \frac{p}{P} F' : (y - \int \frac{\partial p}{P}) + \int \frac{\Pi \partial p}{P} + F : (y - \int \frac{\partial p}{P}),$$

quae est ipsa solutio ante inventa.

Scholion 2.

119. Hoc consensu ostenso etiam consensum supra observatum §. 100. demonstrare poterimus, qui multo magis absconditus videtur. Altera autem solutio ibi inventa erat

$$px = F' : (\frac{y}{a} - lp) \quad \text{et} \quad z = px + F : (\frac{y}{a} - lp),$$

ex quarum formula priori patet, fore vicissim $\frac{y}{a} - lp$ functionem ipsius px; hinc etiam $\frac{y}{a} - lp + lpx$ seu $\frac{y}{a} + lx$ aequabitur functioni ipsius px. Denuo ergo vicissim px aequabitur functioni cuipiam ipsius $\frac{y}{a} + lx$.

$$\partial . F : (\frac{y}{a} - lp) = (\frac{\partial y}{a} - \frac{\partial p}{p}) F' : (\frac{y}{a} - lp), \quad \text{erit}$$

$$F:\left(\frac{y}{a}-lp\right)=\int px\left(\frac{\partial y}{a}-\frac{\partial p}{p}\right)=\int px\left(\frac{\partial y}{a}+\frac{\partial x}{x}\right)-\int px\left(\frac{\partial x}{x}+\frac{\partial p}{p}\right)$$
$$=\int px\left(\frac{\partial y}{a}+\frac{\partial x}{x}\right)-px.$$

Jam pro px substituto valore $f':\left(\frac{y}{a}+lx\right)$, obtinebitur

$$F:\left(\frac{y}{a}-lp\right)=-px+\int\left(\frac{\partial y}{a}+\frac{\partial x}{x}\right)f':\left(\frac{y}{a}+lx\right)=-px+f:\left(\frac{y}{a}+lx\right),$$

ita ut hinc fiat $z=f:\left(\frac{y}{a}+lx\right)$, quae est ipsa solutio altera. Hac igitur reductione haud parum luminis accenditur ad alia mysteria hujus generis investiganda. Summa autem hujus ratiocinii huc redit, ut si fuerit $r=f':s$, fore etiam $r=F':(s+R)$ denotante R functionem ipsius r, quod quidem per se est evidens, quia utrinque r per s determinatur. Cum ergo sit

$$f':s=r=F':(s+R)\text{, erit}$$
$$f:s=\int\partial s\,f':s+\int r\partial s=\int r(\partial s+\partial R-\partial R)$$
$$=\int(\partial s+\partial R)F':(s+R)-\int r\partial R,$$

ideoque

$$f:s=F:(s+R)-\int r\partial R;$$

unde loco functionum quantitatis s, functiones quantitatis $s+R$ introduci possunt. Scilicet si fit $r=f':s$ sumti potest

$$r=F':(s+R)$$

existente R functione quaecunque ipsius r, tum vero erit

$$f:s=F:(s+R)-\int R\partial R.$$

Exemplum 3.

120. *Posito $\partial z=p\partial x+q\partial y$, si x aequetur functioni homogeneae n dimensionum ipsarum p et q, indolem functionis z investigare.*

Cum x detur per p et q, utendum erit solutione priori, et ob $x=$ functioni homogeneae n dimensionum ipsarum p et q, ponatur $p=qr$, fietque $x=q^nR$, existente R functione ipsius r

tantum. Sumatur nunc q constans, et quaeratur

$$V = \int x\partial p = \int q^{n+1} R\partial r, \text{ ob } \partial p = q\partial r,$$

eritque

$$V = q^{n+1}\int R\partial r,$$

quod integrale datur. Hinc differentiando erit

$$\partial V = q^{n+1} R\partial r + (n+1) q^n \partial q \int R\partial r,$$

quae ut cum

$$\partial V = x\partial p + S\partial q = q^n R\partial p + S\partial q$$

comparari possit, quia ob $\partial p = q\partial r + r\partial q$ est

$$\partial V = q^{n+1} R\partial r + q^n Rr\partial q = S\partial q, \text{ erit}$$

$$S = -q^n Rr + (n+1) q^n \int R\partial r;$$

unde fit

$$y = -q^n Rr + (n+1) q^n \int R\partial r + f' : q, \text{ et } x = q^n R,$$

atque

$$z = nq^{n+1}\int R\partial r + q f' : q - f : q, \text{ existente } p = qr.$$

Corollarium 1.

121. Sit $x = \frac{p^m}{q^m}$, et posito $p = qr$ erit $x = r^m$, ideoque $n = 0$ et $R = r^m$; unde fit

$$y = -r^{m+1} + \frac{r^{m+1}}{m+1} + f' : q = \frac{-m}{m+1} r^{m+1} + f' : q \text{ et}$$

$$z = q f' : q - f : q.$$

Quare ob $r = x^{\frac{1}{m}}$, erit $y = \frac{-m}{m+1} x^{\frac{m+1}{m}} + f' : q,$

Corollarium 2.

122. Eodem casu ergo quo $x = \frac{p^m}{q^m}$, aequabitur q functioni

quantitatis $y + \frac{m}{m+1} x^{\frac{m+1}{m}}$, quae quantitas si ponatur $= v$ et $q = F' : z$, ut sit $v = F' : q$, erit

$$f : q = \int \partial q f' : q = \int v \partial v F'' : v, \quad \text{ob } \partial q = \partial v F'' : v;$$

unde concluditur

$$f : q = vF' : v - F : v \text{ et } z = F : v = F : \left(y + \frac{m}{m+1} x^{\frac{m+1}{m}}\right).$$

Exemplum 4.

123. *Duarum variabilium x et y ejusmodi functionem z investigare, ut posito*

$$\partial z = p \partial x + q \partial y \textit{ fiat } p^3 + x^3 = 3pqx.$$

Consideretur forma

$$z = qy + \int (p \partial x - y \partial q),$$

ubi jam formulam $p \partial x - y \partial q$ integrabilem reddi oportet. Statuatur $p = ux$, et conditio praescripta dat

$$x(1 + u^3) = 3qu;$$

unde fit

$$x = \frac{3qu}{1+u^3} \text{ et } p = \frac{3quu}{1+u^3},$$

tum vero

$$\partial x = \frac{3q \partial u (1 - 2u^3)}{(1+u^3)^2} + \frac{3u \partial q}{1+u^3},$$

sicque habebitur

$$z = qy + \int \left(\frac{9qquu \partial u (1 - 2u^3)}{(1+u^3)^3} + \frac{9qu^3 \partial q}{(1+u^3)^2} - y \partial q\right), \text{ at}$$

$$\int \frac{9qq \partial u (1 - 2u^3)}{(1+u^3)^3} = \frac{3qq (1+4u^3)}{2(1+u^3)^2} - \int \frac{3q(1+4u^3) \partial q}{(1+u^3)^2}.$$

Ergo

$$z = qy + \frac{3qq(1+4u^3)}{(1+u^3)^2} - \int \partial q \left(y + \frac{3q}{1+u^3}\right).$$

Quare necesse est esse $\frac{3q}{1+u^3}$ functionem ipsius q tantum, quae sit $= - f' : q$, unde fit

$$y = - \frac{3q}{1+u^2} - f' : q \quad \text{et} \quad z = qy + \frac{3qq(1+4u^2)}{2(1+u^2)^2} + f : q,$$

seu $z = \frac{3qq(2u^2-1)}{2(1+u^2)^2} - qf' : q + f : q$, existente $x = \frac{3qu}{1+u^2}$. Ex quibus tribus aequationibus si eliminentur binae quantitates q et u, orietur aequatio inter z et x, y, quae quaeritur.

Corollarium 1.

124. Ex aequatione pro y inventa colligitur $\frac{3}{1+u^2} = \frac{-y-f':q}{q}$; aequatio autem pro z inventa abit in hanc

$$z = \frac{3qq}{1+u^2} - \frac{9qq}{2(1+u^2)^2} - qf' : q + f : q,$$

quae eliso u transmutatur in hanc

$$z = -qy - 2q\, f' : q - \tfrac{1}{2}(y + f' : q)^2 + f : q;$$

tum vero est

$$x = -u(y + f' : q),$$

unde reperitur $u = \frac{-x}{y + f' : q}$, hincque

$$x^3 = 3q(y + f' : q)^2 + (y + f' : q)^3.$$

Corollarium 2.

125. Si sumamus $f' : q = a$, erit $f : q = aq + b$, et postrema aequatio praebet $q = \frac{x^3 - (y+a)^3}{3(y+a)^2}$. Cum deinde pro hoc casu fiat

$$z = -qy - aq - \tfrac{1}{2}(y+a)^2 + b,$$

proveniet loco q valorem inventum substituendo

$$z = \frac{6b(y+a) - (y+a)^3 - 2x^3}{6(y+a)}.$$

Corollarium 3.

126. Cum in genere sit

$$x^3 = (y + f' : q)^2 (y + 3q + f' : q),$$

ponamus $f':q = a - 3q$, ideoque $f:q = b + aq - \frac{3}{2}qq$, ut fiat $(y + a - 3q)^2 = \frac{x^3}{y+a}$, eritque

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \quad \text{et} \quad q = \tfrac{1}{3}(y+a) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hinc ergo prodit

$$f':q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y \quad \text{et}$$

$$f:q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \tfrac{1}{6}(y+a)^2 + \tfrac{1}{3}x\sqrt{x}(y+a) - \frac{x^3}{6(y+a)}$$

$$\text{seu } f:q = b + \frac{aa-yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)},$$

atque

$$z = -\tfrac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \tfrac{3}{2}qq,$$

$$\text{seu } z = b - \tfrac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \tfrac{9}{2}qq,$$

et facta reductione

$$z = b + \tfrac{1}{6}(y+a)^2 - \tfrac{2}{3}x\sqrt{x}(y+a).$$

Corollarium 4.

127. Quodsi hic sumatur $a = 0$ et $b = 0$, erit per expressionem satis simplicem

$$z = \tfrac{1}{6}yy - \tfrac{2}{3}x\sqrt{xy};$$

quae quomodo conditioni praescriptae satisfaciat, ita apparet. Per differentiationem colligitur

$$p = \left(\frac{\partial z}{\partial x}\right) = -\sqrt{xy} \quad \text{et} \quad q = \left(\frac{\partial z}{\partial y}\right) = \tfrac{1}{3}y - \frac{x\sqrt{x}}{3\sqrt{y}},$$

hincque

$$p^3 + x^3 = -xy\sqrt{xy} + x^3; \quad \text{at}$$

$$3pq = xx - y\sqrt{xy}, \quad \text{ideoque}$$

$$3pqx = x^3 - xy\sqrt{xy}; \quad \text{ergo}$$

$$p^3 + x^3 = 3pqx.$$

Scholion.

128. Successit ergo solutio, quando aequatio quaecunque inter p, q et x proponitur, etiamsi casibus, quibus inde neque x neque p elici potest, difficultas quaedam restat, quae autem resolutionem aequationum finitarum potissimum afficit, quam hic merito concedi postulamus. Interim ex postremo exemplo perspicitur, quomodo operatio sit instituenda, si ope substitutionis idoneae aequatio proposita ad resolutionem accommodari queat, cui autem negotio hic amplius non immoror. Neque etiam eos casus, quibus inter p, q et y relatio quaedam praescribitur, hic seorsim evolvam, cum ob permutabilitatem ipsarum x et y, qua etiam p et q permutantur, hi casus ad praecedentes sponte revocentur. Superest igitur casus, quo aequatio inter p, q et z proponitur, ubi quidem statim manifestum est, in aequatione $\partial z = p\partial x + q\partial y$ quantitates p et q non uti functiones ipsarum x et y spectari posse, quoniam etiam a z pendent, neque ergo earum indoles inde determinari poterit, ut formula $p\partial x + q\partial y$ integrabilis evadat. Verum sine discrimine conditio ea est definienda, ut aequatio differentialis

$$\partial z - p\partial x - q\partial y = 0$$

fiat possibilis; ad quod ex principiis supra stabilitis §. 6. requiritur, ut posito

$$\left(\frac{\partial q}{\partial z}\right) = L, \quad -\left(\frac{\partial p}{\partial z}\right) = M, \text{ et } \left(\frac{\partial p}{\partial y}\right) - \left(\frac{\partial q}{\partial x}\right) = N, \text{ sit}$$

$$Lp + Mq - N = 0, \text{ seu } p\left(\frac{\partial q}{\partial z}\right) - q\left(\frac{\partial p}{\partial z}\right) + \left(\frac{\partial q}{\partial x}\right) - \left(\frac{\partial p}{\partial y}\right) = 0.$$

Quare proposita aequatione quacunque inter p, q et z, eas conditiones in genere investigare oportet, ut huic requisito satisfiat.

Problema 19.

129. Si posito $\partial z = p\partial x + q\partial y$ debeat esse $p + q = \frac{z}{a}$, relationem functionis z ad variabiles x et y in genere investigare.

Solutio.

Cum sit $q = \frac{z}{a} - p$, aequatio nostra hanc induet formam

$$\partial z = p\partial x - p\partial y + \frac{z\partial y}{a}, \text{ seu}$$

$$p(\partial x - \partial y) = \frac{a\partial z - z\partial y}{a} = z\left(\frac{\partial z}{z} - \frac{\partial y}{a}\right).$$

Quoniam igitur ambae formulae

$$\partial x - \partial y \text{ et } \frac{\partial z}{z} - \frac{\partial y}{a}$$

per se sunt integrabiles, ob

$$\frac{\partial z}{z} - \frac{\partial y}{a} = \frac{p}{z}(\partial x - \partial y),$$

necesse est ut $\frac{p}{z}$ sit functio quantitatis $x - y$, ponatur ergo

$$\frac{p}{z} = f' : (x - y), \text{ ut fiat } lz - \frac{y}{a} = f : (x - y).$$

Definiri ergo potest z per x et y, et cum sit $e^{f:(x-y)}$ etiam functio ipsius $x - y$, si ea ponatur $= F : (x - y)$, erit

$$z = e^{\frac{y}{a}} F : (x - y), \text{ unde fit}$$

$$\left(\frac{\partial z}{\partial y}\right) = p = e^{\frac{y}{a}} F' : (x - y) \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = q = -e^{\frac{y}{a}} F' : (x - y) + \frac{1}{a} e^{\frac{y}{a}} F : (x - y);$$

ideoque

$$p + q = \frac{1}{a} e^{\frac{y}{a}} F : (x - y) = \frac{z}{a}.$$

uti requiritur.

Corollarium 1.

130. Ex hoc exemplo intelligitur, quomodo certa functio ipsarum p et q quantitati z aequari possit, etiamsi p et q sint functiones ipsarum x et y. Simul scilicet ratio integralis formulae

$$\partial z = p\partial x + q\partial y$$

introducitur in calculum.

Corollarium 2.

131. Forma $e^{\frac{y}{a}} F:(x-y)$ pro valore ipsius z inventa per functionem quamvis ipsius $x-y$ multiplicari potest. Si ergo multiplicetur per

$$e^{\frac{x-y}{a}}, \text{ fit } z = e^{\frac{x}{a}} F:(x-y).$$

Sin autem multiplicetur per

$$e^{\frac{x-y}{2a}}, \text{ fit } z = e^{\frac{x+y}{2a}} F:(x-y),$$

quae formae problemati aeque satisfaciunt.

Problema 20.

132. Si posito $\partial z = p\partial x + q\partial y$, quantitas z aequari debeat functioni datae ipsarum p et q, indolem, qua z per x et y definitur, in genere investigare.

Solutio.

Ex formula proposita habemus $\partial y = \frac{\partial z}{q} - \frac{p\partial x}{q}$; statuatur $p = qr$, ut sit z aequalis functioni ipsarum q et r, et ex $\partial y = \frac{\partial z}{q} - r\partial x$ elicitur

$$y = \frac{z}{q} - rx + \int \left(\frac{z\partial q}{qq} + x\partial r\right),$$

quam formulam integrabilem reddi oportet. Cum igitur z sit functio data ipsarum q et r, posito r constante quaeratur integrale formulae $\frac{z\partial q}{qq}$, sitque

$$\int\frac{z\partial q}{qq} = \mathrm{V} + \mathrm{f} : r,$$

unde differentiando prodeat

$$\partial \mathrm{V} = \frac{z\partial q}{qq} + \mathrm{R}\partial r,$$

ac jam patet esse debere $x = \mathrm{R} + \mathrm{f}' : r$, indeque obtineri

$$y = \frac{z}{q} - \mathrm{R}r - r\mathrm{f}'r + \mathrm{V} + \mathrm{f} : r,$$

quibus duabus aequationibus relatio inter quantitates propositas determinatur. Primo igitur posito $p = qr$, datur z per q et r. Deinde sumto r constante integretur formula $\frac{z\partial q}{qq}$, sitque integrale resultans $\mathrm{V} = \int\frac{z\partial q}{qq}$, quod etiam per q et r datur; unde sumto q constante colligitur $\mathrm{R} = \left(\frac{\partial \mathrm{V}}{\partial r}\right)$. Quibus inventis erit

$$x = \mathrm{R} + \mathrm{f}' : r \text{ et } y = \frac{z}{q} - rx + \mathrm{V} + \mathrm{f} : r,$$

sicque omnes quantitates per binas variabiles q et r determinantur.

Corollarium 1.

133. Quia permutatis x et y litterae p et q permutantur, simili modo nostram investigationem incipere potuissemus ab aequatione

$$\partial x = \frac{\partial z}{p} - \frac{q\partial y}{p},$$

similisque solutio prodiisset, quae quidem forma diversa at re congruens esset.

Corollarium 2.

134. Jam scilicet posito $q = ps$, ut sit

$$\partial x = \frac{\partial z}{p} - s\partial y, \text{ erit}$$

$$x = \frac{z}{p} - sy + \int\left(\frac{z\partial p}{pp} + y\partial s\right).$$

Jam sumto s constante ponatur $\int\frac{z\partial p}{pp} = U$, quae quantitas per p et s determinatur, ex ea vero prodeat $\left(\frac{\partial U}{\partial s}\right) = S$, erit

$$y = S + f' : s \text{ et } x = \frac{z}{p} - sy + U + f : s.$$

Exemplum 1.

135. *Si esse debeat $p + q = \frac{z}{a}$, solutionem pro hoc casu exhibere.*

Posito $p = qr$, erit $z = aq(1 + r)$, nunc sumto r constante erit

$$V = \int\frac{z\partial q}{qq} = a(1 + r)\,lq \text{ et } R = \left(\frac{\partial V}{\partial r}\right) = alq.$$

Hinc reperitur

$$x = alq + f' : r, \text{ et } y = \frac{z}{q} - arlq - rf' : r + a(1 + r)\,lq + f : r, \text{ seu}$$

$$y = a(1 + r) + alq - rf' : r + f : r.$$

Si hinc q elidere velimus, ob $q = \frac{z}{a(1 + r)}$ solutio his duabus aequationibus continetur

$$x = al\frac{z}{a(1 + rr)} + f' : r, \text{ et}$$

$$y = al\frac{z}{a(1 + r)} + a(1 + r) - rf' : r + f : r.$$

Unde sequenti modo praecedens solutio elici potest, ex forma priori est

$$\frac{x}{a} - l\frac{z}{a} = -l(1 + r) + \frac{1}{a}f' : r = \text{funct.}\, r,$$

ex ambabus vero

$$y - x = a(1 + r) - (1 + r)\,f' : r + f : r = \text{funct.}\, r.$$

Cum ergo tam $\frac{x}{a} - l\frac{z}{a}$, seu $ze^{-\frac{x}{a}}$, quam $y - x$ sit functio ipsius r, altera forma aequabitur functioni alterius; unde statui potest

$$ze^{-\frac{x}{a}} = F:(y-x), \quad \text{seu} \quad z = e^{\frac{x}{a}} F:(y-x),$$

quae est solutio ante inventa.

Exemplum 2.

136. *Si posito $\partial z = pdx + q\partial y$ debeat esse $z = apq$, relationem inter x, y et z investigare.*

Posito $p = qr$ erit $z = aqqr$, et sumto r constante sit $V = \int \frac{z\partial q}{qq} = aqr$, hincque $R = \left(\frac{\partial V}{\partial r}\right) = aq$. Quocirca habebimus

$$x = aq + f':r \quad \text{et} \quad y = aqr - rf':r + f:r,$$

seu ob $r = \frac{z}{aqq}$, erit

$$x = aq + f':\frac{z}{aqq} \quad \text{et} \quad y = \frac{z}{q} - \frac{z}{aqq} f':\frac{z}{aqq} + f:\frac{z}{aqq}.$$

Hic in genere notemus si sit $f':r = v$, ponamusque $r = F':v$, ob $\partial r = \partial v F'':v$, fore

$$f:r = \int \partial r f':r = \int v\partial v F'':v = vF':v - F:v, \quad \text{seu}$$

$$f:r = vF':v - F:v, \quad \text{hincque}$$

$$f:r - rf':r = -F:v.$$

Quare cum sit $f':r = x - aq$, si ponamus $r = F':(x-aq)$, erit

$$f:r - rf':r = -F:(x-aq) \quad \text{et}$$

$$y = aqF':(x-aq) - F:(x-aq), \quad \text{atque}$$

$$z = aqqF':(x-aq).$$

Scholion.

137. **Hae postremae formulae ita statim ex conditione quaestionis elici possunt. Nam ob $p = \frac{z}{aq}$ erit**

$$\partial z = \frac{z\partial x}{aq} + q\partial y, \text{ et } \partial y = \frac{\partial x}{q} - \frac{z\partial x}{aqq},$$

hincque

$$y = \frac{z}{q} + \int\left(\frac{z\partial q}{qq} - \frac{z\partial x}{aqq}\right) = \frac{z}{q} + \int\frac{z}{qq}\left(\partial q - \frac{\partial x}{a}\right),$$

ubi manifestum est esse $\frac{z}{qq}$ functionem quantitatis $q - \frac{x}{z}$. Quare posito

$$\frac{z}{qq} = \mathrm{F}' : \left(q - \frac{x}{a}\right), \text{ erit}$$

$$y = \frac{z}{q} + \mathrm{F} : \left(q - \frac{x}{a}\right).$$

Quin etiam indidem alia solutio deduci potest ponendo

$$\partial x = \frac{aq}{z}\left(\partial z - q\partial y\right),$$

quae posito $z = qv$ abit in

$$\partial x = \frac{a}{v}\left(v\partial q + q\partial v - q\partial y\right), \text{ unde}$$

$$x = aq + \int\frac{aq}{v}\left(\partial v - \partial y\right).$$

Quare ponatur

$$\frac{aq}{v} = \mathrm{f}' : (v - y), \text{ eritque}$$

$$x = aq + \mathrm{f} : (v - y).$$

Jam restituto valore $v = \frac{z}{q}$ habebitur

$$\frac{aqq}{z} = \mathrm{f}' : \left(\frac{z}{q} - y\right) \text{ et } x - aq = \mathrm{f} : \left(\frac{z}{q} - y\right).$$

Prima autem solutio ad eliminanda q et r est aptissima in exemplis. Si enim ponatur

$$\mathrm{f}' : r = \frac{b}{\sqrt{r}} + c, \text{ erit } \mathrm{f} : r = 2b\sqrt{r} + cr + d;$$

hinc

$$z = aqqr \text{ et } x = aq + \frac{b}{\sqrt{r}} + c,$$

atque

$$y = aqr + b\sqrt{r} + d.$$

Jam ob $r = \frac{z}{aqq}$ fit

$$x = aq + bq\sqrt{\frac{a}{z}} + c \text{ et } y = \frac{z}{q} + \frac{b}{q}\sqrt{\frac{z}{a}} + d,$$

Hinc

$$x - c = q\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right) \text{ et } y - d = \frac{z}{aq}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right),$$

et multiplicando ilidïtur q, fitque

$$(x - c)(y - d) = \frac{z}{a}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right)^2 = (b + \sqrt{az})^2,$$

ita ut sit

$$b + \sqrt{az} = \sqrt{(x - c)(y - d)},$$

et proinde

$$z = \frac{(x - c)(y - d) - 2b\sqrt{(x - c)(y - d)} + bb}{a},$$

quae si $b = c = d = 0$ dat casum simplicissimum $z = \frac{xy}{a}$.

CAPUT V.

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER QUANTITATES $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, ET BINAS TRIUM VARIABILIUM x, y, z QUAECUNQUE DATUR.

Problema 21.

138.

Si posito $\partial z = p\partial x + q\partial y$, debeat esse $px + qy = 0$, functionis z indolem per x et y in genere investigare.

Solutio.

Cum sit $q = -\frac{px}{y}$, erit

$$\partial z = p\partial x - \frac{px\partial y}{y} = px\left(\frac{\partial x}{x} - \frac{\partial y}{y}\right), \text{ seu}$$

$$\partial z = py\left(\frac{\partial x}{y} - \frac{x\partial y}{yy}\right) = py\,\partial \,.\, \frac{x}{y}.$$

Unde patet py esse debere functionem ipsius $\frac{x}{y}$; ac si ponatur

$$py = f' : \frac{x}{y}, \text{ fore } z = f : \frac{x}{y}.$$

Perpetuo scilicet in designandis functionibus hac lege utemur, ut sit

$$\partial \,.\, f : v = \partial v f' : v,$$

sicque porro

$$\partial \,.\, f' : v = \partial v f'' : v \text{ et } \partial \,.\, f'' : v = \partial v f''' : v, \text{ etc.}$$

At $f : \frac{x}{y}$ denotat functionem quamcunque homogeneam ipsarum x et

y nullius dimensionis, ac si z fuerit talis functio quaecunque, et differentiando prodeat $\partial z = p\partial x + q\partial y$, semper erit

$$px + qy = 0.$$

Corollarium 1.

139. Quodsi ergo z fuerit functio homogenea nullius dimensionis ipsarum x et y, ob

$$p = \left(\frac{\partial z}{\partial x}\right) \text{ et } q = \left(\frac{\partial z}{\partial y}\right), \text{ erit}$$

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0,$$

quam veritatem quidem jam supra elicuimus.

Corollarium 2.

140. Tum vero cum sit

$$p = \frac{1}{y}\, f' : \frac{x}{y} \quad \text{et} \quad q = \frac{-x}{yy}\, f' : \frac{x}{y},$$

erit p functio homogenea ipsarum x et y numeri dimensionem $= -1$, et si sit $q = \frac{-px}{y}$, ipsa functio z reperitur ex integratione $z = \int py\partial \,.\, \frac{x}{y}$.

Scholion.

141. Simili modo solvitur problema, si posito

$$\partial z = p\partial x + q\partial y,$$

fieri debeat $mpx + nqy = a$. Tum enim ob $q = \frac{a}{ny} - \frac{mpx}{ny}$, erit

$$\partial z = \frac{a\partial y}{ny} + p\partial x - \frac{mpx\partial y}{ny}, \text{ seu}$$

$$\partial z = \frac{a\partial y}{ny} + \frac{px}{n}\left(\frac{n\partial x}{x} - \frac{m\partial y}{y}\right) = \frac{a\partial y}{ny} + \frac{py^m}{nx^{n-1}}\,\partial \,.\, \frac{x^n}{y^m};$$

unde solutio praebet

$$\frac{py^m}{nx^{n-1}} = f' : \frac{x^n}{y^m} \quad \text{et} \quad z = \frac{a}{n}\,ly + f : \frac{x^n}{y^m}.$$

Quin etiam hoc generalius problema resolvi potest, quo esse debet $pX + qY = A$, existente X functione ipsius x, et Y ipsius y. Cum enim inde fiat $q = \frac{A}{Y} - \frac{pX}{Y}$, erit

$$\partial z = \frac{A\partial y}{Y} + p\partial x - \frac{pX\partial y}{Y} = \frac{A\partial y}{Y} + pX\left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right).$$

Statui ergo debet

$$pX = f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

indeque fit

$$z = A\int\frac{\partial y}{Y} + f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

Problema 22.

142. Si posito $\partial z = p\partial x + q\partial y$, debet esse $\frac{q}{p}$ aequale functioni datae cuicunque ipsarum x et y, indolem functionis z in genere investigare.

Solutio.

Sit V ista functio data ipsarum x et y, ut sit $q = pV$, et habebitur $\partial z = p(\partial x + V\partial y)$. Dabitur jam multiplicator M itidem functio ipsarum x et y, ut $M(\partial x + V\partial y)$ fiat integrabile. Ponatur ergo $M(\partial x + V\partial y) = \partial S$, ac dabitur etiam S functio ipsarum x et y. Cum ergo sit $\partial z = \frac{p\partial S}{M}$, perspicuum est, quantitatem $\frac{p}{M}$ aequari debere functioni ipsius S, quare si ponamus $\frac{p}{M} = f' : S$, fiet $z = f : S$, indeque erit

$$p = Mf' : S \text{ et } q = MVf' : S.$$

Corollarium 1.

143. Hoc ergo casu functio quaesita z statim invenitur per x et y expressa, quoniam S per x et y datur. Fieri autem potest, ut S prodeat quantitas transcendens; quin etiam ut per methodos adhuc cognitas multiplicator M ne inveniri quidem possit.

Corollarium 2.

144. Si V sit functio nullius dimensionis ipsarum x et y, erit $M = \frac{1}{x + Vy}$. Seu posito $x = vy$, fiet V functio ipsius v, et

$$\partial S = M\,(y\partial v + v\partial y + V\partial y).$$

Capiatur $M = \frac{1}{y\,(v + V)}$, eritque

$$\partial S = \frac{\partial y}{y} + \frac{\partial v}{v + V};\ \text{unde reperitur}$$

$$z = f : \left(ly + \int \frac{\partial v}{v + V}\right).$$

Scholion.

145. Ob permutabilitatem ipsarum p et x item q et y, simili modo sequentia problemata resolvi possunt.

I. Si debeat esse $q = xV$, existente V functione quacunque ipsarum p et y, consideretur forma

$$z = px + \int (q\partial y - x\partial p) = px + \int x\,(V\partial y - \partial p).$$

Quaeratur multiplicator M, ut sit

$$M\,(V\partial y - \partial p) = \partial S,$$

erit S functio ipsarum p et y, atque

$$z = px + \int \frac{x\partial S}{M};$$

ex quo colligitur haec solutio

$$\frac{x}{M} = f' : S \quad \text{et} \quad z = pMf' : S + f : S.$$

II. Si debeat esse $y = pV$, existente V functione quacunque ipsarum x et q. Consideretur forma

$$z = qy + \int (p\partial x - y\partial q) = qy + \int p\,(\partial x - V\partial q).$$

Quaeratur multiplicator M, ut sit

$$M\,(\partial x - V\partial q) = \partial S,$$

erit S functio ipsarum x et q, et

$$z = qy + \int \frac{p\partial S}{M}.$$

Quare fit

$$\frac{p}{M} = f' : S \quad \text{et} \quad z = qy + f : S,$$

seu ob $p = \frac{y}{V}$, erit

$$y = MVf' : S \quad \text{et} \quad z = qMVf' : S + f : S.$$

III. Si debeat esse $y = xV$, existente V functione quaecunque ipsarum p et q, consideretur haec forma

$$z = px + qy - \int (x\partial p + xV\partial q).$$

Quaeratur multiplicator M, ut fiat

$$M(\partial p + V\partial q) = \partial S,$$

erit S functio ipsarum p et q, et

$$z = px + qy - \int \frac{x\partial S}{M};$$

unde haec solutio nascitur

$$\frac{x}{M} = f' : S \quad \text{et} \quad z = px + qy - f : S.$$

Omnes hi casus huc redeunt, ut quaterharum quantitatum p, x, q, y, vel $\frac{q}{p}$, vel $\frac{q}{x}$, vel $\frac{y}{p}$, vel $\frac{y}{x}$, aequetur functioni cuicunque binarum reliquarum.

Problema 23.

146. Si posito $\partial z = p\partial x + q\partial y$, requiratur ut sit $q = pV + U$, existente tam V quam U functione quacunque binarum variabilium x et y, indolem functionis z in genere investigare.

Solutio.

Cum ob $q = pV + U$ sit

$$\partial z = p(\partial x + V\partial y) + U\partial y,$$

quaeratur primo multiplicator M formulam $\partial x + V\partial y$ reddens integrabilem, sitque

$$M(\partial x + V\partial y) = \partial S,$$

erunt M et S functiones ipsarum x et y, fietque

$$\partial z = \frac{p\partial S}{M} + U\partial y.$$

Cum jam sit S functio ipsarum x et y, inde x per y et S definiri potest, quo valore introducto fient U et M functiones ipsarum y et S. Nunc sumto S constante, integretur formula $U\partial y$, sitque

$$\int U\partial y = T + f : S,$$

ac posito

$$\partial T = U\partial y + W\partial S, \text{ fiet}$$

$$\frac{p}{M} = W + f' : S \quad \text{et} \quad z = T + f : S,$$

sicque omnia per binas variabiles y et S exprimentur.

Corollarium 1.

147. Datis ergo binarum variabilium x et y functionibus V et U, ut sit $q = pV + U$, solutio problematis primo postulat, ut multiplicator M investigetur formulam $\partial x + V\partial y$ integrabilem reddens, quo invento habetur functio S earundem variabilium x et y, ut sit

$$S = \int M(\partial x + V\partial y).$$

Corollarium 2.

148. In hunc finem considerari conveniet aequationem differentialem $\partial x + V\partial y = 0$, haec enim si integrari poterit, simul inde colligi potest multiplicator M, ut formula $M(\partial y + V\partial y)$ fiat verum differentiale cujusdam functionis S, quae propterea hinc invenietur.

Corollarium 3.

149. Inventa porro hac functione S, quantitas x per y et S exprimi debet, ita ut x aequetur functioni ipsarum y et S, quo valore in quantitate U substituto, quaeratur integrale $\int U \partial y = T$, spectata S ut constante, sicque obtinebitur T functio ipsarum y et S.

Corollarium 4.

150. Denique inventa hac functione T, sit $W = \left(\frac{\partial T}{\partial S}\right)$ unde tandem colligitur solutio problematis his duabus formulis contenta

$$\frac{p}{M} = W + f' : S, \quad \text{et} \quad z = T + f : S;$$

ubi cum S sit functio ipsarum x et y, pro z statim reperitur functio ipsarum x et y.

Corollarium 5.

151. Si U sit functio ipsius y tantum, non opus est illa expressione ipsius x per y et S, sed $T = \int U \partial y$ erit quoque functio ipsius y tantum, hinc $W = \left(\frac{\partial T}{\partial S}\right) = 0$. Hic autem casus manifesto reducitur ad praecedentem ponendo z loco $z - \int U \partial y$.

Exemplum 1.

152. *Si posito $\partial z = p \partial x + q \partial y$, debeat esse $q = \frac{px}{y} + \frac{y}{x}$, indolem functionis z investigare.*

Hic ergo est

$$V = \frac{x}{y} \quad \text{et} \quad U = \frac{y}{x};$$

unde ob

$$\partial x + V \partial y = \partial x + \frac{x \partial y}{y},$$

erit multiplicator $M = y$, et $\partial S = y\partial x + x\partial y$, hinc $S = xy$, sicque habebitur

$$x = \frac{S}{y} \quad \text{et} \quad U = \frac{yy}{S}.$$

Jam erit

$$T = \int U\partial y = \int \frac{yy\partial y}{S} = \frac{y^3}{3S}, \quad \text{et} \quad W = \frac{-y^3}{3SS}.$$

Quare pro solutione hujus exempli habebimus

$$\frac{p}{y} = \frac{-y^3}{3SS} + f' : S, \quad \text{et} \quad z = \frac{y^3}{3S} + f : S,$$

seu ob $S = xy$ erit

$$z = \frac{yy}{3x} + f : xy.$$

Exemplum 2.

153. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*

$$px + qy = n\sqrt{(xx + yy)},$$

indolem functionis z investigare.

Cum hic sit $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx + yy)}$, erit

$$V = \frac{-x}{y} \quad \text{et} \quad U = \frac{n}{y}\sqrt{(xx + yy)}.$$

Ergo $\partial S = M\left(\partial x - \frac{x\partial y}{y}\right)$, quare capiatur $M = \frac{1}{y}$, ut fiat

$$\partial S = \frac{\partial x}{y} - \frac{x\partial y}{yy}, \quad \text{et} \quad S = \frac{x}{y}.$$

Hinc oritur

$$x = Sy, \quad \text{et} \quad U = n\sqrt{(1 + SS)};$$

ideoque posito S constante erit

$$T = \int U\partial y = ny\sqrt{(1 + SS)}, \quad \text{et} \quad W = \left(\frac{\partial T}{\partial S}\right) = \frac{nyS}{\sqrt{(1 + SS)}};$$

ita ut solutio nostrae quaestionis sit

$$py = \frac{nyS}{\sqrt{(1 + SS)}} + f' : S, \quad \text{et} \quad z = ny\sqrt{(1 + SS)} + f : S.$$

Cum igitur sit $S = \frac{x}{y}$, erit

$$z = n\sqrt{(xx + yy)} + f : \frac{x}{y};$$

ubi $f : \frac{x}{y}$ denotat functionem quamcunque nullius dimensionis ipsarum x et y.

Exemplum 3.

154. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*

$$pxx + qyy = nxy,$$

functionis z indolem investigare.

Cum sit $q = \frac{-pxx}{yy} + \frac{nx}{y}$, erit

$$V = \frac{-xx}{yy} \text{ et } U = \frac{nx}{y}.$$

Quare ob $\partial S = M\left(\partial x - \frac{xx\partial y}{yy}\right)$, capiatur $M = \frac{1}{xx}$, ut fiat $S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$. Hinc erit

$$\frac{1}{x} = \frac{1}{y} - S, \text{ et } x = \frac{y}{1 - Sy};$$

ideoque $U = \frac{n}{1 - Sy}$. Sumto igitur S constante habebimus

$$T = \int \frac{n\partial y}{1 - Sy} = -\frac{n}{S}\, l\,(1 - Sy), \text{ et}$$

$$W = +\frac{n}{SS}\, l\,(1 - Sy) + \frac{ny}{S(1 - Sy)}.$$

Consequenter ob

$$S = \frac{x-y}{xy} \text{ et } 1 - Sy = \frac{y}{x},$$

solutio praebet

$$z = \frac{-nxy}{x-y}\, l\,\frac{y}{x} + f : \frac{x-y}{xy}.$$

Scholion.

155. Ex solutione hujus problematis etiam haec quaestio latius patens resolvi potest. Sint P, Q, item V, U functiones quaecunque datae ipsarum x et y, et quaeri oporteat functionem z, ut sit

$$\partial z = P\partial x + Q\partial y + L\,(V\partial x + U\partial y);$$

seu quod eodem redit, functio L investigari debet, ut ista formula differentialis integrationem admittat. Ad hoc praestandum quaeratur primo multiplicator M formulam $V\partial x + U\partial y$ integrabilem efficiens, ponaturque $\partial S = M(V\partial x + U\partial y)$, unde functio S reperitur per x et y expressa. Ex ea quaeratur valor ipsius x per y et S expressus; et cum sit

$$\partial z = P\partial x + Q\partial y + \frac{L\partial S}{M},$$

hic ubique loco x valor ille substituatur; sit autem inde $\partial x = E\partial y + F\partial S$, unde etiam E et F innotescent, eritque

$$\partial z = EP\partial y + Q\partial y + FP\partial S + \frac{L\partial S}{M}.$$

Sumatur quantitas S pro constante, sitque

$$T = \int (EP + Q)\,\partial y, \text{ erit}$$
$$z = T + f:S,$$

quod quidem ad solutionem sufficit; sed ad L inveniendum, differentietur haec expressio

$$\partial z = (EP + Q)\,\partial y + \partial S \cdot \left(\frac{\partial T}{\partial S}\right) + \partial S f':S,$$

ac necesse est fiat

$$FP + \frac{L}{M} = \left(\frac{\partial T}{\partial S}\right) + f':S,$$

ideoque

$$L = -FMP + M\left(\frac{\partial T}{\partial S}\right) + Mf':S.$$

Caeterum ob permutabilitatem ipsarum p, x et q, y, etiam hinc sequentia problemata resolvi possunt, quae propterea strictim percurram.

Problema 24.

156. Si posito $\partial z = p\partial x + q\partial y$ requiratur, ut sit $q = Vx + U$, existente tam V quam U functione quacunque data ipsarum p et y, investigare indolem functionis quaesitae z.

Solutio.

Utamur formula

$$z = px + \int (q\partial y - x\partial p),$$

et cum loco q valore substituto sit

$$\int (q\partial y - x\partial p) = \int (\mathrm{V}x\partial y - x\partial p + \mathrm{U}\partial y),$$

hanc formulam integrabilem reddi oportet. Sit ea brevitatis gratia $\mathfrak{P}$, et cum sit

$$\partial\mathfrak{P} = x(\mathrm{V}\partial y - \partial p) + \mathrm{U}\partial y,$$

quaeratur primo multiplicator M formulam $\mathrm{V}\partial y - \partial p$ integrabilem reddens, ponaturque

$$\mathrm{M}(\mathrm{V}\partial y - \partial p) = \partial \mathrm{S},$$

sicque S dabitur per y et p; unde p eliciatur per y et S expressum, quo valore ibi substituto erit

$$\partial\mathfrak{P} = \frac{x\partial \mathrm{S}}{\mathrm{M}} + \mathrm{U}\partial y.$$

Jam sumto S constante sumatur integrale

$$\int \mathrm{U}\partial y = \mathrm{T} + f : \mathrm{S}, \text{ eritque}$$

$$\frac{x}{\mathrm{M}} = \left(\frac{\partial \mathrm{T}}{\partial \mathrm{S}}\right) + f' : \mathrm{S}, \text{ et } \mathfrak{P} = \mathrm{T} + f : \mathrm{S}.$$

Solutio igitur per binas variabiles y et S ita se habebit

$$x = \mathrm{M}\left(\frac{\partial \mathrm{T}}{\partial \mathrm{S}}\right) + \mathrm{M}f' : \mathrm{S}, \text{ et } z = px + \mathrm{T} + f : \mathrm{S},$$

ubi nunc quidem S per p et y datur.

Problema 25.

157. Si posito $\partial z = p\partial x + q\partial y$ requiratur, ut sit $p = \mathrm{V}y + \mathrm{U}$, existentibus V et U functionibus datis ipsarum x et q, indolem functionis z investigare.

Solutio.

Utamur jam forma

$$z = qy + \int (p\partial x - y\partial q),$$

ponaturque formula ad integrationem perducenda:

$$\int (p\partial x - y\partial q) = \mathfrak{z}.$$

Hinc pro p valorem assumtum substituendo erit

$$\partial\mathfrak{z} = Vy\partial x + U\partial x - y\partial q = y(V\partial x - \partial q) + U\partial x.$$

Quaeramus multiplicatorem M, ut fiat

$$M(V\partial x - \partial q) = \partial S,$$

ac tam M quam S erunt functiones ipsarum x et q, ex quarum posteriori valor ipsius q per x et S expressus eliciatur, in sequenti operatione pro q substituendus. Scilicet cum nunc sit

$$\partial\mathfrak{z} = \frac{y\partial S}{M} + U\partial x,$$

sumto S constante quaeratur $T = \int U\partial x$, sitque

$$\mathfrak{z} = T + f:S,$$

unde colligitur

$$\frac{y}{M} = \left(\frac{\partial T}{\partial S}\right) + f':S, \text{ et } z = qy + T + f:S.$$

ac nunc quidem pro S valorem in x et q restituere licet.

Problema 26.

158. Si posito $\partial z = p\partial x + q\partial y$ requiratur, ut sit $y = Vx + U$, existentibus V et U functionibus quibuscunque datis ipsarum p et q, indolem functionis z in genere investigare.

Solutio.

Hic utendum est formula

$$z = px + qy - \int (x\partial p + y\partial q).$$

Statuatur $\int (x\partial p + y\partial q) = \mathfrak{z}$, eritque pro y valorem praescriptum substituendo

$$\partial\mathfrak{z} = x\partial p + Vx\partial q + U\partial q.$$

Quaeratur jam multiplicator M, formulam $\partial p + V\partial q$ integrabilem reddens, sitque

$$M(\partial p + V\partial q) = \partial S,$$

ubi M et S per p et q dabuntur; et ex posteriori eliciatur valor ipsius p per q et S expressus, quo deinceps uti oportet. Scilicet cum sit

$$\partial \mathfrak{p} = \frac{x\partial S}{M} + U\partial q,$$

sumto S constante integretur formula $U\partial q$, sitque $T = \int U\partial q$, erit $\mathfrak{p} = T + f : S$ hincque

$$\frac{x}{M} = \left(\frac{\partial T}{\partial S}\right) + f' : S, \quad \text{et } z = px + qy - T - f : S.$$

Omnia ergo per p et q, unde M, S et T cum $\left(\frac{\partial T}{\partial S}\right)$ dantur, ita determinabuntur ut sit

$$x = M\left(\frac{\partial T}{\partial S}\right) + Mf' : S, \quad y = Vx + U, \quad \text{et}$$
$$z = px + qy - T - f : S.$$

Exemplum.

159. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $px + qy = apq$, indolem functionis z investigare.*

Cum ergo sit

$$y = \frac{-px}{q} + ap, \quad \text{erit}$$
$$V = \frac{-p}{q}, \quad U = ap.$$

Quia nunc esse debet

$$M\left(\partial p - \frac{p\partial q}{q}\right) = \partial S,$$

capiatur $M = \frac{1}{q}$ fitque

$$S = \frac{p}{q} \quad \text{et} \quad p = Sq.$$

Hinc $U = aSq$, et sumto S constante

$$T = \int U\partial q = \tfrac{1}{2}aSqq,$$

ideoque $\left(\frac{\partial T}{\partial s}\right) = \frac{1}{2}aqq$. Quocirca pro solutione habebimus

$$x = \frac{1}{2}aq + \frac{1}{q}f' : \frac{p}{q}, \quad y = \frac{1}{2}ap - \frac{p}{qq}f' : \frac{p}{q}, \text{ et}$$

$$z = px + qy - \frac{1}{2}apq - f : \frac{p}{q} = \frac{1}{2}apq + f : \frac{p}{q}.$$

Per reductionem autem supra traditam habebimus

$$y = (aq - x)F' : (qx - \tfrac{1}{2}aqq), \text{ et}$$

$$z = qy + F : (qx - \tfrac{1}{2}aqq).$$

Scholion.

160. Quatuor problemata haec conjunctim considerata admodum late patent, atque pro formula $\partial z = p\partial x + q\partial y$ omnes relationes inter p, q, x et y complectuntur, in quibus vel x et y, vel p et y, vel x et q, vel p et q, nusquam unam dimensionem superant. Ex quo saepe fieri potest, ut eadem quaestio per duo plurave horum quatuor problematum resolvi possit; veluti evenit in exemplo hoc postremo, in quo cum non solum x et y, sed etiam x et q, itemque p et y, nusquam plus una dimensione occupant, id ad tria praecedentia problemata referri queat, haecque conditio primo tantum problemati adversatur. Quod si autem inter p, q, x et y haec relatio praescribatur, ut esse debeat

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

resolutio per omnia quatuor problemata aeque institui potest. Verum etiam resolutiones inde ortae, etiamsi forma discrepent, tamen per reductionem ante expositam ad consensum revocari possunt. At sequens casus latissime patens resolutionem quoque admittit, quem propterea evolvi conveniet.

Problema 27.

161. Si posito $\partial z = p\partial x + q\partial y$, inter p, q et x, y ejusmodi relatio detur, ut functio quaedam ipsarum p et x aequetur

functioni cuipiam ipsarum q et y, functionis z indolem in genere investigare.

Solutio.

Sit P functio illa ipsarum p et x, et Q functio illa ipsarum q et y, quae inter se aequales esse debent. Cum igitur sit $P = Q$, ponatur utraque $= v$, ut sit $P = v$ et $Q = v$. Ex priori ergo p definire licebit per x et v, ex posteriori vero q per y et v; quo facto in formula $\partial z = p\partial x + q\partial y$, cum p sit functio ipsarum x et v, integretur pars $p\partial x$ sumto v constante, sitque $\int p\partial x = R$, simili modo cum q sit functio ipsarum y et v, integretur quoque altera pars $q\partial y$ sumto v constante, sitque $\int q\partial y = S$; erit ergo $R =$ functioni ipsarum x et v, et $S =$ functioni ipsarum y et v. At sumto etiam v variabili sit

$$\partial R = p\partial x + V\partial v, \quad \text{et} \quad \partial S = q\partial y + U\partial v,$$

unde colligitur

$$\partial z = \partial R + \partial S - \partial v (V + U),$$

quae forma quia integrabilis esse debet, oportet sit $V + U = f' : v$. Quare solutio problematis his duabus aequationibus continebitur

$$V + U = f' : v \quad \text{et} \quad z = R + S - f : v.$$

Scilicet cum p, R et V dentur per x et v; atque q, S et U per y et v, per aequationem priorem definitur v ex x et y, qui valor in altera substitutus determinabit functionem quaesitam z per x et y.

Corollarium 1.

162. Quoties ergo q ejusmodi functioni ipsarum p, x, y aequari debet, ut inde aequatio formari possit, ex cujus altera

parte tantum binae litterae x et p, ex altera tantum binae reliquae y et q reperiantur, problema resolvi poterit.

Corollarium 2.

163. Si functio illa binarum litterarum p et x, quam posui P, ita sit comparata, ut posita ea $= v$ inde facilius x per p et v definiri possit, tum uti conveniet formula

$$z = px + \int (q\partial y - x\partial p),$$

et evolutio perinde se habebit atque ante.

Corollarium 3.

164. Simili modo si ex functione altera $Q = v$, quantitas y facilius per q et v definiatur, resolutio ex forma

$$z = qy + \int (p\partial x - y\partial q)$$

erit petenda. Sin autem utrumque eveniat, ut tam x per p et v, quam y per q et v definiatur, utendum erit formula

$$z = px + qy - \int (x\partial p + y\partial q).$$

Scholion.

165. Problema hoc innumerabiles complectitur casus in praecedentibus non comprehensos, atque etiam ejus solutio diverso nititur fundamento. Interim tamen longissime adhuc distamus a solutione problematis generalis, cui hoc caput est destinatum et quo in genere solutio desideratur, si inter quaternas quantitates p, q, x, y aequatio quaecunque proponatur; quae autem ob defectum Analyseos ne sperari quidem posse videtur. Contentos ergo nos esse oportet, si quam plurimos casus resolvere docuerimus. Quo autem vis hujus problematis magis perspiciatur aliquot exempla adjungamus.

Exemplum 1.

166. *Si posito $\partial z = p\partial x + q\partial y$, esse debeat $q = \frac{xxyy}{a^4 p}$, indolem functionis z investigare.*

Quia hic p, x, et q, y separare licet, cum sit $\frac{aaq}{yy} = \frac{xx}{aap}$, ponatur $\frac{xx}{aap} = v = \frac{aaq}{yy}$, unde p per x et v, et q per y et v ita definitur, ut sit

$$p = \frac{xx}{aav} \text{ et } q = \frac{vyy}{aa},$$

ideoque

$$\partial z = \frac{xx\partial x}{aav} + \frac{vyy\partial y}{aa}.$$

Hinc colligimus

$$z = \frac{x^3}{3aav} + \frac{vy^3}{3aa} + \frac{1}{3aa}\int\left(\frac{x^3\partial v}{vv} - y^3\partial v\right),$$

sicque $\frac{x^3}{vv} - y^3$ debet esse functio ipsius v. Ac posito

$$\frac{x^3}{vv} - y^3 = f' : v, \text{ seu } y^3 = \frac{x^3}{vv} - f' : v, \text{ erit}$$

$$z = \frac{1}{3aa}\left(\frac{x^3}{v} + vy^3 + f : v\right).$$

Corollarium.

167. Hinc facillime v eliminatur, si ponatur

$$f' : v = \frac{b^3}{vv} - c^3, \text{ hincque } f : v = \frac{-b^3}{v} - c^3 v.$$

Jam prior aequatio dat $y^3 - c^3 = \frac{x^3 - b^3}{vv}$, unde $vv = \frac{x^3 - b^3}{y^3 - c^3}$, et ob

$$3aaz = \frac{x^3 + vvy^3 - b^3 - c^3vv}{v} = 2v(y^3 - c^3), \text{ erit}$$

$$z = \frac{2}{3aa}\sqrt{(x^3 - b^3)(y^3 - c^3)}.$$

Exemplum 2.

168. *Si posito $\partial z = p\partial x + q\partial y$, debeat esse*

$$q = \frac{1}{b}\sqrt{(xx + yy - aapp)},$$

investigare indolem functionis z.

Conditio praescripta redit ad

$$bbqq - yy = xx - aapp = v,$$

unde elicimus

$$q = \tfrac{1}{b}\sqrt{(yy + v)}, \quad \text{et} \quad p = \tfrac{1}{a}\sqrt{(xx - v)}.$$

Nunc vero est

$$\int p\partial x = \tfrac{1}{a}\int \partial x \sqrt{(xx - v)} = \tfrac{1}{2a} x \sqrt{(xx - v)} - \tfrac{v}{2a}\int \frac{\partial x}{\sqrt{(xx - v)}}$$

seu $\int p\partial x = \frac{x}{2a}\sqrt{(xx - v)} - \frac{v}{2a} l[x + \sqrt{(xx - v)}] = R$;

simili modo est

$$\int q\partial y = \tfrac{y}{2b}\sqrt{(yy + v)} + \tfrac{v}{2b} l[y + \sqrt{(yy + v)}] = S.$$

Quare cum sit

$$V = \left(\frac{\partial R}{\partial v}\right) = \frac{-x}{4a\sqrt{(xx - v)}} - \frac{1}{2a} l[x + \sqrt{(xx - v)}]$$
$$+ \frac{v}{4a[x + \sqrt{(xx - v)}]\sqrt{(xx - v)}},$$

quae reducitur ad

$$V = -\tfrac{1}{4a} - \tfrac{1}{2a} l[x + \sqrt{(xx - v)}],$$

similique modo

$$U = \left(\frac{\partial S}{\partial v}\right) = + \tfrac{1}{4b} + \tfrac{1}{2b} l[y + \sqrt{(yy + v)}],$$

ubi cum $V + U = f' : v$, erit

$$\frac{a - b}{4ab} + l \cdot \frac{[y + \sqrt{(yy + v)}]^{\frac{1}{2b}}}{[x + \sqrt{(xx - v)}]^{\frac{1}{2a}}} = f' : v;$$

unde valor ipsius v per x et y determinatur. Ex quo tandem colligitur

$$z = \frac{x}{2a}\sqrt{(xx - v)} + \frac{y}{2b}\sqrt{(yy + v)} + v\, l \cdot \frac{[y + \sqrt{(yy + v)}]^{\frac{1}{2b}}}{[x + \sqrt{(xx - v)}]^{\frac{1}{2a}}} - f : v,$$

seu

$$z = \frac{x}{2a}\sqrt{(xx - v)} + \frac{y}{2b}\sqrt{(yy + v)} - \frac{(a-b)v}{4ab} + v f' : v - f : v.$$

Scholion.

169. Haec solutio a formulis logarithmicis liberari potest hoc modo. Ponatur

$$f' : v = lt + \frac{a-b}{4ab},$$

ut sit

$$t^{2ab} = \frac{[y + \sqrt{(yy + v)}]^a}{[x + \sqrt{(xx - v)}]^b},$$

unde v datur per t. Tum vero sit $v = tF' : t$, et ob

$$\partial v f'' : v = \frac{\partial t}{t} \text{ erit}$$

$$\int v \partial v f'' : v = v f' : v - f : v = \int \frac{v \partial t}{t} = F : t,$$

sicque erit

$$z = \frac{x}{2a}\sqrt{(xx - v)} + \frac{y}{2b}\sqrt{(yy + v)} - \frac{(a-b)v}{4ab} + F : t,$$

ubi est

$$v = tF' : t, \text{ et } t^{2ab} = \frac{[y + \sqrt{(yy + v)}]^a}{[x + \sqrt{(xx - v)}]^b},$$

unde t et v per x et y definiri potest. Hinc statim patet si capiatur $F' : t = 0$, fore $v = 0$, $F : t = 0$ et $z = \frac{xx}{2a} + \frac{yy}{2b}$; hincque $p = \frac{x}{a}$ et $q = \frac{y}{b}$, quo pacto utique conditioni praescriptae satisfit. Caeterum haec ratio quantitates logarithmicas elidendi maxime est notatu digna et in aliis casibus usum amplissimum habere potest.

Exemplum 3.

170. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $x^m y^n = Ap^\mu q^\nu$, indolem functionis z investigare.*

Statuatur ergo

$$\frac{x^m}{p^\mu} = \frac{Aq^\nu}{y^n} = v^{\mu\nu},$$

et hinc deducitur

$$p = \frac{x^{\frac{m}{\mu}}}{v^\nu} \text{ et } q = \frac{1}{a}\, y^{\frac{n}{\nu}}\, v^\mu,$$

posito $A = a^\nu$. Unde habebimus

$$\int p\partial x = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\mu\nu}{m+\mu}\int \frac{x^{\frac{m+\mu}{\mu}}\partial v}{v^{\nu+1}}, \text{ et}$$

$$\int q\partial y = \frac{\nu y^{\frac{n+\nu}{\nu}} v^\mu}{(n+\nu)\, a} - \frac{\mu\nu}{(n+\nu)\, a}\int y^{\frac{n+\nu}{\nu}} v^{\mu-1}\partial v.$$

Quocirca erit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^\mu}{(n+\nu)\, a} + \frac{\mu\nu}{(m+\mu)(n+\nu)a} \times$$
$$\int\partial v\left(\frac{(n+\nu)\, a x^{\frac{m+\mu}{\mu}}}{v^{\nu+1}} - (m+\mu)\, y^{\frac{n+\nu}{\nu}} v^{\mu-1}\right),$$

ita ut si statuamus

$$\frac{x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^{\nu+1}} - \frac{y^{\frac{n+\nu}{\nu}} v^{\mu-1}}{(n+\nu)\, a} = f' : v,$$

futurum sit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^\mu}{(n+\nu)\, a} + \mu\nu f : v.$$

Pro casu simplicissimo ponamus $f' : v = 0$ et $f : v = 0$, eritque

$$y^{\frac{n+\nu}{\nu}} v^{\mu+\nu} = \frac{(n+\nu)a}{m+\mu} x^{\frac{m+\mu}{\mu}} \quad \text{et} \quad v = \left(\frac{(n+\nu)ax^{\frac{m+\mu}{\mu}}}{(m+\mu)y^{\frac{n+\nu}{\nu}}}\right)^{\frac{1}{\mu+\nu}},$$

tum vero

$$z = \frac{1}{v^\nu}\left(\frac{\mu}{m+\mu} x^{\frac{m+\mu}{\mu}} + \frac{\nu}{(n+\nu)a} y^{\frac{n+\nu}{\nu}} v^{\mu+\nu}\right), \text{ seu}$$

$$z = \frac{(\mu+\nu)}{(m+\mu)v^\nu} x^{\frac{m+\mu}{\mu}} = (\mu+\nu)\left(\frac{x^{m+\mu} y^{n+\nu}}{(m+\mu)^\mu (n+\nu)^\nu A}\right)^{\frac{1}{\mu+\nu}}.$$

Problema 28.

171. Si posito $\partial z = p\partial x + q\partial y$, inter p, q et x, y ejusmodi detur relatio, ut p et q aequentur functionibus quibusdam ipsarum x, y et novae variabilis v, explorare casus, quibus indolem functionis z investigare licet.

Solutio.

Cum sit p functio ipsarum x, y et v, spectatis y et v ut constantibus, quaeratur integrale $\int p\partial x = \mathrm{P}$, sitque sumtis omnibus variabilibus

$$\partial \mathrm{P} = p\partial x + \mathrm{R}\partial y + \mathrm{M}\partial v,$$

unde si pro $p\partial x$ valor substituatur, erit

$$\partial z = \partial \mathrm{P} + (q - \mathrm{R})\,\partial y - \mathrm{M}\partial v.$$

Quodsi jam eveniat, ut $q - \mathrm{R}$ sit tantum functio ipsarum y et v, exclusa x, sumto v constante quaeratur $\int (q - \mathrm{R})\,\partial y = \mathrm{T}$, sitque deinceps

$$\partial \mathrm{T} = (q - \mathrm{R})\,\partial y + \mathrm{V}\partial v.$$

Hinc valor ipsius $(q - \mathrm{R})\,\partial y$ ibi substitutus dabit

$$\partial z = \partial \mathrm{P} + \partial \mathrm{T} - (\mathrm{M} + \mathrm{V})\,\partial v,$$

quae forma quia integrabilis esse debet, statuatur

$$M + V = f' : v, \quad \text{eritque} \quad z = P + T - f : v.$$

Ex operationibus autem susceptis dantur P, R, M, per V, x, y et v, at T et V per y et v tantum; ac resolutio succedit, si modo in forma $q - R$ non amplius x continetur. Pari ratione solutio succedet, si M tantum per y et v detur; tum enim ex y constante quaeratur $\int M \partial v = L$, sitque

$$\partial L = M \partial v + N \partial y, \text{ erit}$$

$$\partial z = \partial P + (q - R + N) \partial y - \partial L,$$

ponique conveniet

$$q - R + N = f' : y,$$

ut fiat

$$z = P - L + f : y.$$

Simili modo ab altera parte $\int q \partial y$ calculum incipere et prosequi licet.

Introducendo autem functionem ipsarum x, y et v indefinitam K, negotium generalius confici poterit. Sit enim

$$\partial K = F \partial x + G \partial y + H \partial v,$$

ac consideretur haec forma

$$\partial z + \partial K = (p + F) \partial x + (q + G) \partial y + H \partial v.$$

Nunc sumtis y et v constantibus, quaeratur

$$\int (p + F) \partial x = P,$$

sitque

$$\partial P = (p + F) \partial x + R \partial y + M \partial v,$$

unde habetur

$$\partial z + \partial K = \partial P + (q + G - R) \partial y + (H - M) \partial v.$$

Quod si jam eveniat, ut vel $q + G - R$ vel $H - M$ tantum binas variabiles y et v exclusa x contineat, resolutio ut ante est ostensum, absolvi poterit.

Problema 29.

172. Si posito $\partial z = p\partial x + q\partial y$, relatio detur inter binas formulas differentiales p, q et binas variabiles x et z, vel y et z, solutionem problematis quatenus fieri potest, perficere.

Solutio.

Ponamus relationem dari inter p, q et x, z, atque hunc casum facile ad praecedentem revocare licet. Consideretur enim haec formula

$$\partial y = \frac{\partial z - p\partial x}{q},$$

ex principali derivata; voceturque

$$\frac{1}{q} = m \quad \text{et} \quad \frac{p}{q} = -n,$$

ut habeatur

$$\partial y = m\partial z + n\partial x,$$

et ob $q = \frac{1}{m}$ et $p = -\frac{n}{m}$,

relatio proposita versabitur inter quaternas quantitates m, n, z et x, ideoque quaestio omnino similis est earum, quas antea tractavimus, hoc tantum discrimine, quod hic quantitas y definiatur, cum ante esset z investigata. Quoniam autem ista determinatio per aequationes absolvitur, perinde est utrum tandem inde z, an y elicere velimus. Quodsi ergo hac reductione facta quaestio in casus ante pertractatos incidat, methodis quoque expositis resolvi poterit.

Exemplum.

173. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $qxz = aap$, indolem functionis z investigare.*

Consideretur formula $\partial y = \frac{\partial z}{q} - \frac{p\partial x}{q}$. Jam quia $\frac{p}{q} = \frac{xz}{aa}$ erit

$$\partial y = \frac{\partial z}{q} - \frac{xz\partial x}{aa} \text{ et } y = \int\left(\frac{\partial z}{q} - \frac{xz\partial x}{aa}\right), \text{ at est}$$

$$\int \frac{xz\partial x}{aa} = \frac{xxz}{2aa} - \int \frac{xx\partial z}{2aa}, \text{ ergo}$$

$$y = \int \partial z\left(\frac{1}{q} + \frac{xx}{2aa}\right) - \frac{xxz}{2aa}.$$

Ponatur ergo

$$\frac{1}{q} + \frac{xx}{2aa} = f' : z, \text{ erit } y = \frac{-xxz}{2aa} + f : z,$$

ex qua aequatione utique z per x et y definitur. Si pro casu simpliciori sumamus $f : z = b + \alpha z$, erit

$$y - b = \left(\alpha - \frac{xx}{2aa}\right) z, \text{ et } z = \frac{2aa(y-b)}{2\alpha aa - xx},$$

et sumtis $\alpha = 0$ et $b = 0$ pro casu simplicissimo erit $z = \frac{-2aay}{xx}$.
Hinc autem fit

$$p = \frac{+4aay}{x^3} \text{ et } q = \frac{-2aa}{xx}. \text{ Ergo}$$

$$\frac{p}{q} = -\frac{2y}{x} \text{ et } \frac{xz}{aa} = \frac{-2y}{x}.$$

CAPUT VI.

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, ET OMNES TRES VARIABILES x, y, z QUAECUNQUE DATUR.

Problema 30.

174.

Si posito $\partial z = p\partial x + q\partial y$, debeat esse $nz = px + qy$, indolem functionis z in genere investigare.

Solutio.

Ope relationis datae elidatur vel p vel q. Scilicet cum sit $q = \frac{nz}{y} - \frac{px}{y}$, erit

$$\partial z = p\partial x + \frac{nz\partial y}{y} - \frac{px\partial y}{y},$$

quae aequatio in hanc formam transfundatur

$$\partial z - \frac{nz\partial y}{y} = p\left(\partial x - \frac{x\partial y}{y}\right) = py\partial \cdot \frac{x}{y}.$$

Ut prius membrum $\partial z - \frac{nz\partial y}{y}$ integrabile reddatur, multiplicetur aequatio per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, seu particulariter per $\frac{1}{y^n}$, eritque

$$\partial \cdot \frac{z}{y^n} = py^{1-n}\,\partial \cdot \frac{x}{y}.$$

Quo facto evidens est poni debere $py^{1-n} = f' : \frac{x}{y}$, ut fiat

$\frac{z}{y^n} = f : \frac{x}{y}$, seu $z = y^n f : \frac{x}{y}$. Unde patet fore z functionem homogeneam ipsarum x et y, dimensionum numero existente $= n$.

Si in genere aequatio multiplicetur per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, erit partis prioris integrale $F : \frac{z}{y^n}$, pro parte autem altera si ponatur $\frac{py}{z}$ funct. $\frac{z}{y^n} = f' : \frac{x}{y}$, erit $F : \frac{z}{y^n} = f : \frac{x}{y}$, atque ut ante $\frac{z}{y^n}$ aequabitur functioni cuicunque ipsius $\frac{x}{y}$.

Corollarium 1.

175. Cum z aequetur functioni homogeneae n dimensionum ipsarum x et y, erunt p et q functiones $n - 1$ dimensionum. Scilicet cum sit $z = y^n f : \frac{x}{y}$, erit

$$p = y^{n-1} f' : \frac{x}{y}, \text{ et } q = ny^{n-1} f : \frac{x}{y} - xy^{n-2} f' : \frac{x}{y},$$

unde fit manifesto $nz = px + qy$.

Corollarium 2.

176. Si p et q fuerint functiones $n - 1$ dimensionum ipsarum x et y, ac formula $p\partial x + q\partial y$ sit integrabilis seu $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial q}{\partial x}\right)$, tum integrale certo erit $\frac{px + qy}{n}$, quae proprietas nonnunquam insignem usum habere potest.

Scholion.

177. Fundamentum hujus solutionis in hoc consistit, quod aequatio integranda in duas partes resolvatur, quarum utraque ope certi multiplicatoris integrabilis reddi queat, unde deinceps una

quantitas variabilis, cujus differentiale in aequatione non occurrit determinetur. Hinc aequatio nostra

$$\partial z - \frac{nz\partial y}{y} = p\,(\partial x - \frac{x\partial y}{y})$$

etiam ita repraesentari potest

$$\frac{\partial x}{y} - \frac{x\partial y}{yy} = \frac{1}{py}(\partial z - \frac{nz\partial y}{y}) = \frac{y^{n-1}}{p}(\frac{\partial z}{y^n} - \frac{nz\partial y}{y^{n+1}}), \quad \text{seu}$$

$$\partial \cdot \frac{x}{y} = \frac{y^{n-1}}{p}\,\partial \cdot \frac{z}{y^n}.$$

Sit ergo

$$\frac{y^{n-1}}{p} = F' : \frac{z}{y^n}, \quad \text{eritque}$$

$$\frac{x}{y} = F : \frac{z}{y^n}, \quad \text{ac vicissim} \quad \frac{z}{y^n} = f : \frac{x}{y}, \quad \text{ut ante.}$$

Possumus etiam statim z ex calculo elidere; cum enim sit

$$nz = px + qy, \quad \text{erit}$$

$$n\partial z = p\partial x + q\partial y + x\partial p + y\partial q.$$

At est

$$n\partial z = np\partial x + nq\partial y,$$

$$(n-1)\,p\partial x - x\partial p + (n-1)\,q\partial y - y\partial q = 0, \quad \text{seu}$$

$$x^n\,(\frac{(n-1)\,p\partial x}{x^n} - \frac{\partial p}{x^{n-1}}) + y^n\,(\frac{(n-1)\,q\partial y}{y^n} - \frac{\partial q}{y^{n-1}}) = 0,$$

quae reducitur ad hanc formam

$$- x^n\,\partial \cdot \frac{p}{x^{n-1}} - y^n\,\partial \cdot \frac{q}{y^{n-1}} = 0, \quad \text{seu}$$

$$\partial \cdot \frac{q}{y^{n-1}} = - \frac{x^n}{y^n}\,\partial \cdot \frac{p}{x^{n-1}}.$$

Statuatur

$$\frac{x^n}{y^n} = -f' : \frac{p}{x^{n-1}}, \text{ erit } \frac{q}{y^{n-1}} = f : \frac{p}{x^{n-1}}.$$

Vel posito $\frac{x}{y} = v$, si ob. $v^n = -f' : \frac{p}{x^{n-1}}$ reciproce ponatur

$$\frac{p}{x^{n-1}} = u = \frac{1}{v^{n-1}} F' : v,$$

ut sit

$$f' : u = -v^n,$$

reperietur

$$\int \partial u f' : u = f : u = nF : v - vF' : v.$$

Hinc

$$p = \frac{x^{n-1}}{v^{n-1}} F' : v = y^{n-1} F' : \frac{x}{y}, \text{ et}$$

$$q = y^{n-1} f : u = ny^{n-1} F : \frac{x}{y} - xy^{n-2} F' : \frac{x}{y};$$

ideoque

$$nz = px + qy = ny^n F : \frac{x}{y}, \text{ seu } z = y^n F : \frac{x}{y},$$

ut ante.

Problema 31.

178. Si posito $\partial z = p\partial x + q\partial y$, debeat esse

$$\alpha px + \beta qy = nz,$$

indolem functionis z investigare.

Solutio.

Ex conditione praescripta eliciatur ut ante

$$q = \frac{nz}{\beta y} - \frac{\alpha px}{\beta y}, \text{ eritque}$$

$$\partial z - \frac{nz\partial y}{\beta y} = p\partial x - \frac{\alpha px\partial y}{\beta y},$$

quae aequatio per $y^{\frac{\beta}{n}}$ divisa dat

$$\partial \cdot \frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}}\left(\partial x - \frac{\alpha x \partial y}{\beta y}\right) = \frac{p y^{\alpha:\beta}}{y^{n:\beta}} \partial \cdot \frac{x}{y^{\alpha:\beta}}.$$

Quod si ergo ponamus

$$p y^{(\alpha - n):\beta} = f' : \frac{x}{y^{\alpha:\beta}},$$

habebimus solutionem

$$z = y^{n:\beta} \text{ f} : \frac{x}{y^{\alpha:\beta}}.$$

At functio ipsius $\frac{x}{y^{\alpha:\beta}}$ reducitur ad functionem ipsius $\frac{x^\beta}{y^\alpha}$, unde z etiam ita per x et y determinatur, ut sit

$$z = y^{n:\beta} \text{ f} : \frac{x^\beta}{y^\alpha},$$

vel etiam

$$z^{\frac{1}{n}} = y^{\frac{1}{\beta}} \text{ f} : \frac{x^{\frac{1}{\alpha}}}{y^{\frac{1}{\beta}}}.$$

Quodsi ergo quantitates $x^{\frac{1}{\alpha}}$ et $y^{\frac{1}{\beta}}$ unam dimensionem constituere censeantur, $z^{\frac{1}{n}}$ aequabitur earundem functioni unius dimensionis, ipsa autem quantitas z earundem functioni n dimensionum. Vel sumta pro z functione quacunque homogenea n dimensionum binarum variabilium t et u, scribatur deinde $t = x^{\frac{1}{\alpha}}$ et $u = y^{\frac{1}{\beta}}$, ac prodibit functio conveniens pro z.

Problema 32.

179. Si posito $\partial z = p\partial x + q\partial y$ debeat esse

$$Z = pX + qY,$$

denotante Z functionem ipsius z, X ipsius x, et Y ipsius y, indolem functionis z in genere investigare.

Solutio.

Ex conditione praescripta elicitur $q = \frac{Z}{Y} - \frac{pX}{Y}$, qui valor substitutus praebet

$$\partial z - \frac{Z\partial y}{Y} = p\left(\partial x - \frac{X\partial y}{Y}\right), \text{ hincque}$$

$$\frac{\partial z}{Z} - \frac{\partial y}{Y} = \frac{p}{Z}\left(\partial x - \frac{X\partial y}{Y}\right) = \frac{pX}{Z}\left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right),$$

ubi jam resolutio est manifesta. Statuatur scilicet

$$\frac{pX}{Z} = f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ eritque}$$

$$\int\frac{\partial z}{Z} - \int\frac{\partial y}{Y} = f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right),$$

unde valor ipsius z per x et y definitur.

Corollarium 1.

180. Hic ergo z ita per x et y definiri debet, ut si X, Y et Z datae sint functiones sigillatim ipsarum x, y et z, fiat

$$X\left(\frac{\partial z}{\partial x}\right) + Y\left(\frac{\partial z}{\partial y}\right) = Z;$$

cujus ergo aequationis resolutionem hic invenimus hac aequatione finita contentam

$$\int\frac{\partial z}{Z} = \int\frac{\partial y}{Y} + f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

Corollarium 2.

181. Quemadmodum autem hic valor conditioni problematis satisfaciat, ex ejus differentiatione statim patet. Cum enim sit

$$\frac{\partial z}{Z} = \frac{\partial y}{Y} + \left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right) f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ erit}$$

$$\left(\frac{\partial z}{\partial x}\right) = \frac{Z}{X} f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{Z}{Y} - \frac{Z}{Y} f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right),$$

unde fit

$$X\left(\frac{\partial z}{\partial x}\right) + Y\left(\frac{\partial z}{\partial y}\right) = Z.$$

Scholion.

182. Solutio ergo, eodem modo ut fecimus, sine introductione novarum litterarum p et q absolvi potest, retinendo earum loco valores differentiales $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$; facilius autem singulae litterae scribuntur, calculusque fit brevior. Caeterum ex hoc problematum genere, ubi omnes tres variabiles x, y et z praeter binos valores differentiales p et q in determinationem ingrediuntur, paucissima resolvere licet; ac praeter hoc, quod tractavimus vix unum aut alterum insuper adjungere poterimus. Unde hic insignia adhuc calculi incrementa desiderantur. Quo autem hujus problematis vis penitius inspiciatur, nonnulla exempla subjungamus.

Exemplum 1.

183. *Si posito* $\partial z = p\partial x + q\partial y$ *debeat esse*

$$zz = pxx + qyy,$$

indolem functionis z *in genere investigare.*

Hic ergo est $Z = zz$, $X = xx$, et $Y = yy$; unde habemus

$$\int\frac{\partial x}{X} = -\frac{1}{x}, \quad \int\frac{\partial y}{Y} = -\frac{1}{y}, \quad \text{et } \int\frac{\partial z}{Z} = -\frac{1}{z},$$

quibus valoribus substitutis pro solutione adipiscimur

$$-\frac{1}{z} = -\frac{1}{y} + f : \left(\frac{1}{y} - \frac{1}{x}\right), \text{ seu}$$

$$z = \frac{y}{1 - yf : \left(\frac{1}{y} - \frac{1}{x}\right)}.$$

Sumatur ergo functio quaecunque quantitatis

$$\frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy},$$

quae si ponatur V, erit $z = \frac{y}{1 - Vy}$.

Veluti si ponamus $V = \frac{n}{y} - \frac{n}{x}$, erit

$$\frac{1}{z} = \frac{1}{y} - \frac{n}{y} + \frac{n}{x} = \frac{ny - (n-1)x}{xy},$$

hincque $z = \frac{xy}{ny - (n-1)x}$, unde

$$p = \left(\frac{\partial z}{\partial x}\right) = \frac{nyy}{[ny - (n-1)x]^2}, \quad \text{et} \quad q = \left(\frac{\partial z}{\partial y}\right) = \frac{-(n-1)xx}{[ny - (n-1)x]^2},$$

sicque

$$pxx + qyy = \frac{xxyy}{[ny - (n-1)x]^2} = zz.$$

Exemplum 2.

184. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $\frac{n}{z} = \frac{p}{x} + \frac{q}{y}$, indolem functionis z investigare.*

Cum hic sit

$$X = \frac{1}{x}, \quad Y = \frac{1}{y} \quad \text{et} \quad Z = \frac{n}{z}, \quad \text{erit}$$

$$\int \frac{\partial x}{X} = \tfrac{1}{2}xx, \quad \int \frac{\partial y}{Y} = \tfrac{1}{2}yy \quad \text{et} \quad \int \frac{\partial z}{Z} = \frac{1}{2n}zz;$$

unde solutio ita erit comparata

$$\frac{1}{2n}zz = \tfrac{1}{2}yy + f:(xx - yy), \quad \text{sive}$$

$$zz = nyy + f:(xx - yy),$$

non enim est necesse functionem per $2n$ multiplicari, cum ea omnes operationes jam per se involvat.

Si pro hac functione sumatur $\alpha(xx - yy)$, habebitur solutio particularis

$$zz = axx + (n - \alpha) yy \text{ et } z = \sqrt{[axx + (n - \alpha) yy]},$$

hincque

$$p = \left(\frac{\partial z}{\partial x}\right) = \frac{\alpha x}{\sqrt{[\alpha xx + (n - \alpha) yy]}}, \text{ et}$$

$$q = \left(\frac{\partial z}{\partial y}\right) = \frac{(n - \alpha) y}{\sqrt{[\alpha xx + (n - \alpha) yy]}},$$

seu $\frac{p}{x} = \frac{\alpha}{z}$ et $\frac{q}{y} = \frac{n - \alpha}{z}$, ideoque $\frac{p}{x} + \frac{q}{y} = \frac{n}{z}$.

Problema 33.

185. Si posito $\partial z = p\partial x + q\partial y$ debeat esse $q = pT + V$, existente T functione quacunque ipsarum x et y, ac V functione ipsarum y et z, investigare indolem functionis z.

Solutio.

Substituto loco q valore praescripto, huic aequationi inducatur forma

$$\partial z - V\partial y = p (\partial x + T\partial y).$$

Cum jam V tantum binas variabiles y et z involvat, dabitur multiplicator M prius membrum $\partial z - V\partial y$ integrabile reddens; ponatur ergo

$$M (\partial z - V\partial y) = \partial S.$$

Simili modo quia T tantum x et y continet, dabitur multiplicator L membrum quoque posterius $\partial x + T\partial y$ integrabile efficiens; sit igitur

$$L (\partial x + T\partial y) = \partial R,$$

ita ut nunc sint R et S functiones cognitae, illa ipsarum x et y, haec vero ipsarum y et z. Hinc nostra aequatio induet hanc formam

$$\frac{\partial S}{M} = \frac{p\partial R}{L} \text{ seu } \partial S = \frac{pM\partial R}{L},$$

cujus integrabilitas necessario postulat ut sit $\frac{pM}{L}$ functio ipsius R. Ponamus ergo

$$\frac{pM}{L} = f' : R, \text{ eritque } S = f : R$$

qua aequatione relatio inter z et x, y definitur.

Corollarium 1.

186. In hoc problemate praecedens tanquam casus particularis continetur: cum enim ibi esset $Z = pX + qY$, erit $q = -\frac{X}{Y}p + \frac{Z}{Y}$, ideoque hujus problematis applicatione facta fit

$$T = \frac{-X}{Y} \text{ et } V = \frac{Z}{Y}.$$

Corollarium 2.

187. Quanquam autem hoc problema infinite latius patet quam praecedens, arctissimis tamen adhuc limitibus continetur, neque ejus ope vel hunc casum simplicissimum $z = py + qx$ resolvere licet.

Scholion.

188. Omnino est haec forma $z = py + qx$ digna notatu, quod nulla ratione hactenus cognita resolvi posse videtur. Sive enim inde eliciatur $q = \frac{z - py}{x}$, unde fit

$$\partial z - \frac{z\partial y}{x} = p\left(\partial x - \frac{y\partial y}{x}\right);$$

sive simili modo p, nulla via ad solutionem patet; cujus difficultatis causa in hoc manifesto est posita, quod formula $\partial z - \frac{z\partial y}{x}$ nullo multiplicatore integrabilis reddi potest; seu quod haec aequatio $\partial z - \frac{z\partial y}{x} = 0$ plane est impossibilis, cum x perinde sit variabilis atque y et z. Supra scilicet jam notavi non omnes aequationes differentiales inter ternas variabiles esse possibiles, simulque characterem possibilitatis exhibui, qui pro tali forma

$$\partial x + P\partial x + Q\partial y = 0,$$

huc reducitur, ut sit

$$P\left(\frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial P}{\partial x}\right) = \left(\frac{\partial Q}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right),$$

nostro jam casu est $P = 0$ et $Q = \frac{-z}{x}$, unde hic character dat $0 = \frac{z}{xx}$, quod cum sit falsum, etiam aequatio illa $\partial z - \frac{z\partial y}{x} = 0$ est impossibilis, quod quidem per se est manifestum. Verum tamen pro hoc casu $z = py + qx$ solutio particularis est obvia scilicet $z = n(x + y)$, unde fit $p = q = n$. Deinceps autem methodum dabimus ex hujusmodi solutione particulari generalem eruendi.

Exemplum 1.

189. *Si posito* $\partial z = p\partial x + q\partial y$ *debeat esse*

$$py + qx = \frac{nxz}{y},$$

indolem functionis z *investigare.*

Cum hinc sit $q = -\frac{py}{x} + \frac{nz}{y}$, erit

$$T = \frac{-y}{x} \text{ et } V = \frac{nz}{y},$$

unde fit

$$\partial S = M\left(\partial z - \frac{nz\partial y}{y}\right) \text{ et } \partial R = L\left(\partial x - \frac{y\partial y}{x}\right).$$

Sumatur ergo $M = \frac{1}{y^n}$, ut fiat $S = \frac{z}{y^n}$, et $L = 2x$, ut fiat $R = xx - yy$. Quocirca hanc adipiscimur solutionem

$$\frac{z}{y^n} = f:(xx - yy), \text{ seu } z = y^n f:(xx - yy).$$

Exemplum 2.

190. *Si posito* $\partial z = p\partial x + q\partial y$ *debeat esse*

$$pxx + qyy = nyz,$$

definire indolem functionis z.

Cum ergo sit $q = \frac{-pxx}{yy} + \frac{nz}{y}$, erit

$$T = \frac{-xx}{yy} \text{ et } V = \frac{nz}{y},$$

sicque hic casus in nostro problemate continetur. Unde colligi oportet

$$\partial R = L\left(\partial x - \frac{xx\partial y}{yy}\right) \text{ et } \partial S = M\left(\partial z - \frac{nz\partial y}{y}\right).$$

Quare sumto $L = \frac{1}{xx}$ fit $R = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$; et sumto $M = \frac{1}{y^n}$, fit $S = \frac{z}{y^n}$, ideoque solutio prodit ista

$$\frac{z}{y^n} = f : \frac{x-y}{xy} \text{ et } z = y^n f : \frac{x-y}{xy}.$$

Problema 34.

191. Si posito $\partial z = p\partial x + q\partial y$, debeat esse $p = qT + V$, existente T functione ipsarum x et y, at V functione ipsarum x et z, indolem functionis z investigare.

Solutio.

Simili modo ut ante si loco p valor praescriptus substituatur, obtinebitur

$$\partial z - V\partial x = q(\partial y + T\partial x),$$

Jam ob indolem functionum V et T sequentes integrationes instituere licebit

$$M(\partial z - V\partial x) = \partial S, \; N(\partial y + T\partial x) = \partial R,$$

unde fit

$$\frac{\partial S}{M} = \frac{q\partial R}{N}, \text{ seu } \partial S = \frac{Mq}{N}\partial R.$$

Atque hinc facillime colligitur haec solutio

$$\frac{Mq}{N} = f' : R, \text{ et } S = f : R.$$

Problema 35.

192. Si posito $\partial z = p \partial x + q \partial y$ debeat esse $z = Mp + Nq$, existentibus M et N functionibus quibusvis binarum variabilium x et y; ex quadam solutione particulari, qua constat esse $z = V$, indolem functionis z in genere determinare.

Solutio.

Valor iste particularis V, qui est functio ipsarum x et y differentietur, sitque

$$\partial V = P \partial x + Q \partial y,$$

qui valor quia loco z substitutus satisfacit, ubi fit $p = P$ et $q = Q$, erit per hypothesin

$$V = MP + NQ.$$

Jam generatim ponatur $z = V f:T$, sitque

$$\partial T = R \partial x + S \partial y,$$

et nunc quaeri oportet hanc functionem T. Ex differentiatione autem eruimus

$$p = \left(\frac{\partial z}{\partial x}\right) = P f:T + V R f':T, \text{ et}$$

$$q = \left(\frac{\partial z}{\partial y}\right) = Q f:T + V S f':T.$$

Quare cum sit

$$z = Mp + Nq = V f:T, \text{ erit}$$

$$V f:T = (MP + NQ) f:T + V(MR + NS) f':T,$$

et ob $V = MP + NQ$ per hypothesin habebitur

$$MR + NS = 0, \text{ hinc}$$

$$\partial T = R\left(\partial x - \frac{M \partial y}{N}\right).$$

Jam nosse non oportet R, sed sufficit considerari formulam $N \partial x - M \partial y$, quae ope multiplicatoris cujusdam integrabilis reddi potest. Solutio ergo facillime huc redit, ut ex conditione praescripta $z = Mp + Nq$ formetur aequatio realis

$$\partial T = R(N \partial x - M \partial y),$$

invento enim multiplicatore idoneo R, per integrationem reperitur quantitas T, qua inventa erit $z = V f : T$.

Aliter.

Facilius valor generalis hoc modo invenitur; ob valorem ipsius z cognitum V, statuatur $z = V v$, sitque

$$\partial v = r \partial x + s \partial y; \text{ erit}$$

$$p = P v + V r \text{ et } q = Q v + V s,$$

ideoque

$$z = M p + N q = (M P + N Q) v + V (M r + N s) = V v.$$

At est $V = M P + N Q$; ergo

$$M r + N s = 0, \text{ seu } s = - \frac{M r}{N}.$$

Unde fit

$$\partial v = r \left(\partial x - \frac{M \partial y}{N}\right) = \frac{r}{N} (N \partial x - M \partial y).$$

Statuatur ergo, idoneum multiplicatorem investigando,

$$R (N \partial x - M \partial y) = \partial T, \text{ erit } \partial v = \frac{r}{NR} . \partial T,$$

ex quo colligitur

$$\frac{r}{NR} = f' : T \text{ et } v = f : T,$$

ita ut in genere sit ut ante $z = V v$.

Corollarium 1.

193. Proposita ergo conditione $z = M p + N q$, ut sit $\partial z = p \partial x + q \partial y$, statim consideretur aequatio differentialis $R (N \partial x - M \partial y) = \partial T$, unde tam multiplicator R quam inde integrale T reperitur; haecque operatio non pendet a valore particulari cognito V.

Corollarium 2.

194. Inventa autem quantitate T, si undecunque innotuerit solutio particulariter satisfaciens $z = V$, erit solutio generalis $z = V f : T$. Probe autem notetur ex solutione particulari generalem elici non posse, nisi conditio praescripta sit hujusmodi $z = Mp + Nq$.

Exemplum 1.

195. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $z = py + qx$, ex valore particulari $z = x + y$ generalem definire.*

Cum hic sit $M = y$ et $N = x$, habebimus hanc aequationem

$$R(x\partial x - y\partial y) = \partial T, \text{ hincque}$$

$$T = f:(xx - yy);$$

ergo solutio generalis erit

$$z = (x + y)\, f:(xx - yy).$$

Exemplum 2.

196. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*

$$z = p(x + y) + q(y - x),$$

ex valore particulari $z = \sqrt{(xx + yy)}$ generalem invenire.

Ob $M = x + y$ et $N = y - x$ formula $N\partial x - M\partial y$ deducit ad hanc aequationem

$$R(y\partial x - x\partial x - x\partial y - y\partial y) = \partial T.$$

Sumatur $R = \frac{1}{xx + yy}$, ut sit

$$\partial T = \frac{y\partial x - x\partial y}{xx + yy} - \frac{x\partial x - y\partial y}{xx + yy}, \text{ erit}$$

$$T = \text{Ang. tang. } \frac{x}{y} - \tfrac{1}{2} l(xx + yy).$$

Atque ex valore hoc dupliciter transcendente erit

$$z = \sqrt{(xx+yy)}\, f : T,$$

simulque patet nullum alium dari valorem particularem, qui sit algebraicus, praeter datum $z = \sqrt{(xx+yy)}$.

Exemplum 3.

197. *Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse*

$$z = p(\alpha x + \beta y) + q(\gamma x + \delta y),$$

ex invento valore particulari $z = V$, indolem functionis z in genere definire.

Hic est $M = \alpha x + \beta y$ et $N = \gamma x + \delta y$, unde deducimur ad hanc aequationem

$$R[(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y] = \partial T,$$

ubi ob formam homogeneam debet esse

$$R = \frac{1}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ut sit

$$\partial T = \frac{(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ad quod integrale inveniendum ponatur $y = ux$, ac prodibit

$$\partial T = \frac{\partial x}{x} - \frac{(\alpha + \beta u)\partial u}{\gamma + (\delta - \alpha)u - \beta uu}, \text{ sit}$$

$$\int \frac{(\alpha + \beta u)\partial u}{\gamma + (\delta - \alpha)u - \beta uu} = lU, \text{ erit } T = lx - lU,$$

et cum functio ipsius T sit etiam functio ipsius $\frac{x}{U}$, erit in genere $z = V f : \frac{x}{U}$ Patet autem, cum U sit functio ipsius $u = \frac{y}{x}$, fore U functionem homogeneam nullius dimensionis ipsarum x et y, ideoque $\frac{x}{U}$ functionem unius dimensionis.

Scholion.

198. Hoc ergo exemplo difficultas restat, quomodo solutio particularis $z = V$ obtineri queat; nisi enim una saltem hujusmod

solutio particularis constet, solutio generalis ne absolvi quidem potest. Pro hoc autem casu solutionem particularem sequenti modo elicere licet, qui cum aliquid singulare habeat, nullum est dubium, quin ejus ope hoc calculi genus haud parum adjumenti sit consecuturum.

Problema 36.

199. Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse

$$z = p(\alpha x + \beta y) + q(\gamma x + \delta y),$$

valorem particularem investigare, qui loco z substitutus huic conditioni satisfaciat.

Solutio.

Negotium hoc succedet, si pro z ejusmodi valorem quaeramus, qui sit functio nullius dimensionis ipsarum x et y, seu posito $y = ux$, qui sit functio ipsius u tantum. Ponamus ergo

$$z = \mathrm{f}:u = \mathrm{f}:\tfrac{y}{x}, \text{ eritque } \mathrm{f}':u = \tfrac{\partial z}{\partial u};$$

at ob $\partial u = \frac{\partial y}{x} - \frac{y\partial x}{xx}$, erit

$$\partial z = \left(\tfrac{\partial y}{x} - \tfrac{u\partial x}{x}\right)\mathrm{f}':u, \text{ hinc}$$

$$p = -\tfrac{u}{x}\mathrm{f}':u = -\tfrac{u\partial z}{x\partial u} \text{ et } q = \tfrac{1}{x}\mathrm{f}':u = \tfrac{\partial z}{x\partial u}.$$

Quibus valoribus pro p et q substitutis, conditio praescripta praebet

$$z = x(\alpha + \beta u)p + x(\gamma + \delta u)q = \frac{-u\partial z(\alpha+\beta u) + \partial z(\gamma + \delta u)}{\partial u},$$

unde fit

$$\frac{\partial z}{z} = \frac{\partial u}{\gamma + (\delta - \alpha)u - \beta uu}.$$

Ponamus

$$\int \frac{\partial u}{\gamma + (\delta - \alpha)u - \beta uu} = l\,\mathrm{V},$$

ut fiat $z = \mathrm{V}$, eritque V valor particularis pro z satisfaciens.

Corollarium 1.

200. Invento hoc valore V, praecedentis exempli ope solutio generalis facile invenitur. Erit scilicet $z = \mathrm{V}\,f : \frac{x}{\mathrm{U}}$ existente

$$\frac{\partial \mathrm{U}}{\mathrm{U}} = \frac{(\alpha + \beta u)\partial u}{\gamma + (\delta - \alpha)u - \beta uu};$$

unde patet quantitatem U ex ipso valore particulari V inveniri posse.

Corollarium 2.

201. Erit enim

$$l\mathrm{U} = -l\sqrt{[\gamma + (\delta - \alpha)u - \beta uu]} + \int \frac{\frac{1}{2}(\delta + \alpha)\partial u}{\gamma + (\delta - \alpha)u - \beta uu},$$

ideoque

$$l\mathrm{U} = -l\sqrt{[\gamma + (\delta - \alpha)u - \beta uu]} + \tfrac{1}{2}(\alpha + \delta)l\mathrm{V},$$

sive

$$\mathrm{U} = \frac{\mathrm{V}^{\frac{1}{2}(\alpha + \delta)}}{\sqrt{[\gamma + (\delta - \alpha)u - \beta uu]}};\ \text{hinc}$$

$$\frac{x}{\mathrm{U}} = \frac{\sqrt{[\gamma xx + (\delta - \alpha)xy - \beta yy]}}{\mathrm{V}^{\frac{1}{2}(\alpha + \delta)}}.$$

Corollarium 3.

202. Quocirca invento valore particulari $z = \mathrm{V}$, ut sit $\frac{\partial \mathrm{V}}{\mathrm{V}} = \frac{\partial u}{\gamma + (\delta - \alpha)u - \beta uu}$, existente $u = \frac{y}{x}$, erit valor generaliter satisfaciens

$$z = \mathrm{V}\,f : \frac{\gamma xx + (\delta - \alpha)xy - \beta yy}{\mathrm{V}^{\alpha + \delta}} = \mathrm{V}\,f : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{\mathrm{V}^{\alpha + \delta}}$$

Corollarium 4.

203. Hinc colligitur alius valor particularis, qui semper est algebraicus, erit is scilicet

$$z = [x(\gamma x + \delta y) - y(\alpha x + \beta y)]^{\frac{1}{\alpha + \delta}},$$

vel ejus multiplum quodcunque. Nisi autem V sit quantitas algebraica, omnes reliqui valores erunt transcendentes, et in hac forma contenti

$$z = [x(\gamma x + \delta y) - y(\alpha x + \beta y)]^{\frac{1}{\alpha + \delta}} f : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha + \delta}}.$$

Scholion.

204. Unicus casus, quo $\delta = -\alpha$ et conditio proposita

$$z = p(\alpha x + \beta y) + q(\gamma x - \alpha y),$$

peculiarem evolutionem postulat. Primo autem posito $u = \frac{y}{x}$, pro valore particulari $z = V$ erit

$$l V = \int \frac{\partial u}{\gamma - 2\alpha u - \beta u u}.$$

Tum vero ob

$$\frac{\partial U}{U} = \frac{(\alpha + \beta u)\partial u}{\gamma - 2\alpha u - \beta u u}, \text{ erit}$$

$$U = \frac{1}{\sqrt{(\gamma - 2\alpha u - \beta u u)}} \text{ et } \frac{x}{U} = \sqrt{(\gamma x x - 2\alpha x y - \beta y y)},$$

ita ut jam valor generalis sit

$$z = V f : (\gamma x x - 2\alpha x y - \beta y y).$$

Per se enim manifestum est, formam $f : \sqrt{T}$ exprimi posse per $f : T$. Nisi ergo V sit functio algebraica, hoc casu nulla solutio particularis algebraica locum habet.

Exemplum 1.

205. *Si posito $\partial z = p\partial x + q\partial y$ esse debeat $nz = py - qy$, indolem functionis z investigare.*

Comparatione cum forma nostra generali instituta fit

$$\alpha = 0,\ \beta = \tfrac{1}{n},\ \gamma = -\tfrac{1}{n},\ \delta = 0.$$

Hic ergo casus ob $\delta = -\alpha$ pertinet ad §. praecedentem, unde fit

$$l\,V = \int \frac{n\,\partial u}{-1 - uu} = -n \text{ Ang. tang. } u.$$

Cum igitur sit $u = \frac{y}{x}$, forma generalis est

$$z = e^{-n \text{ Ang. tang. } \frac{y}{x}}\, f:(xx + yy).$$

Exemplum 2.

206. *Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse*

$$z = p\,(x + y) - q\,(x + y),$$

indolem functionis z investigare.

Comparatione facta fit

$$\alpha = 1,\ \beta = 1,\ \gamma = -1,\ \delta = -1,$$

hincque

$$l\,V = \int \frac{\partial u}{-1 - 2u - uu} = \frac{1}{1+u},\ \text{et}\ V = e^{\frac{1}{1+u}},$$

et solutio generalis est

$$z = e^{\frac{x}{x+y}}\, f:(x + y).$$

Exemplum 3.

207. *Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse*

$$z = p\,(x - 2y) + q\,(2x - 3y),$$

indolem functionis z investigare.

Cum ergo hic sit

$$\alpha = 1,\ \beta = -2,\ \gamma = 2,\ \text{et}\ \delta = -3,$$

erit primo

$$lV = \int \frac{\partial u}{2 - 4u + 2uu} = \frac{+1}{2(1-u)} = \frac{x}{2(x-y)},$$

et quia non est $\delta = -\alpha$, solutio generalis statim prodit

$$z = (2xx - 4xy + 2yy)^{\frac{-1}{2}} f : \frac{(2xx - 4xy + 2yy)}{V^{-2}},$$

et ob

$$V = e^{\frac{x}{2(x-y)}}, \text{ erit}$$

$$z = \frac{1}{x-y} f : (x-y)^2 \, e^{\frac{x}{x-y}}.$$

Unde solutio simplicissima est $z = \frac{1}{x-y}$.

Scholion.

208. Hic merito quaerimus, quo pacto haec solutio generalis statim sine adjumento solutionis specialis inveniri potuisset? sequenti autem modo ista investigatio instituenda videtur. Cum sit

$$p(\alpha x + \beta y) = z - q(\gamma x + \delta y) \text{ et}$$
$$q(\gamma x + \delta y) = z - p(\alpha x + \beta y),$$

si uterque valor seorsim in forma

$$\partial z = p \partial x + q \partial y$$

substituatur, prodibunt binae sequentes aequationes

$$(\alpha x + \beta y)\partial z = z\partial x - q(\gamma x + \delta y)\partial x + q(\alpha x + \beta y)\partial y,$$
$$(\gamma x + \delta y)\partial z = z\partial y + p(\gamma x + \delta y)\partial x - p(\alpha x + \beta y)\partial y.$$

Multiplicetur prior indefinite per M posterior per N, et productorum summa dabit

$$\partial z[M(\alpha x + \beta y) + N(\gamma x + \delta y)] - z(M\partial x + N\partial y)$$
$$= (Np - Mq)[(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y],$$

ubi jam M et N ita capi debent, ut prius membrum integrationem admittat, tum enim ejus integrale aequabitur functioni cuicunque quantitatis

$$\int \frac{(\gamma x+\delta y)\partial x-(\alpha x+\beta y)\partial y}{\gamma xx+(\delta-\alpha)xy-\beta yy},$$

quam supra (§. 197.) definire docuimus: unde patet illud integrale fieri $= f:\frac{z}{U}$. Manifestum autem est, M et N ejusmodi functiones esse oportere ut haec aequatio fiat possibilis

$$\frac{\partial z}{z}=\frac{M\partial x+N\partial y}{M(\alpha x+\beta y)+N(\gamma x+\delta y)},$$

seu ut membrum posterius integrationem admittat; quod si enim ejus integrale sit $= lV$, erit $\frac{z}{V}=f:\frac{z}{U}$. Pro hac integrabilitate ponamus $y=ux$, et M et N functionis ipsius u, erit

$$\frac{\partial z}{z}=\frac{(M+Nu)\partial x+Nx\partial u}{Mx(\alpha+\beta u)+Nx(\gamma+\delta u)}.$$

Ubi integratio succedit sumendo $M=-Nu$, ut sit

$$\frac{\partial z}{z}=\frac{+\partial u}{\gamma+(\delta-\alpha)u-\beta uu}, \text{ seu}$$

$$lV=\int\frac{\partial u}{\gamma+(\delta-\alpha)u-\beta uu},$$

prorsus ut ante.

Problema 36.

209. Si posito $\partial z=p\partial x+q\partial y$ debeat esse $Z=pP+qQ$, existente Z functione ipsius z tantum, P et Q autem functionibus ipsarum x et y quibusvis datis, indolem functionis z investigare.

Solutio.

Formentur sequentes aequationes ex propositis

$$L\partial z=Lp\partial x+Lq\partial y,\ MZ\partial x=MpP\partial x+MqQ\partial x,$$
$$NZ\partial y=NpP\partial y+NqQ\partial y,$$

quae in unam summam collectae dabunt

$$L\partial z+Z(M\partial x+N\partial y)=p[(L+MP)\partial x+NP\partial y]$$
$$+q[(L+NQ)\partial y+MQ\partial x].$$

Ut jam pars posterior habeat factorem a litteris p et q liberum, fiat

$$L + MP : NP = MQ : L + NQ,$$

unde fit

$$LL + LNQ + LMP = 0, \text{ seu } L = -MP - NQ,$$

quo valore inducto erit

$$-\partial z(MP + NQ) + Z(M\partial x + N\partial y) = (Mq - Np)(Q\partial x - P\partial y).$$

Cum nunc P et Q sint functiones datae ipsarum x et y, dabitur multiplicator R, ut fiat

$$R(Q\partial x - P\partial y) = \partial U, \text{ ideoque}$$

$$-\partial z(MP + NQ) + Z(M\partial x + N\partial y) = \frac{Mq - Np}{R}.\partial U.$$

Pro parte priori capiantur functiones indefinitae M et N ita ut formula $\frac{M\partial x + N\partial y}{MP + NQ}$ integrabilis evadat, id quod semper fieri licet, sitque

$$\frac{M\partial x + N\partial y}{MP + NQ} = \partial V,$$

et ob

$$M\partial x + N\partial y = (MP + NQ)\partial V,$$

aequatio nostra hanc induet formam

$$(MP + NQ)(-\partial z + Z\partial V) = \frac{Mq - Np}{R}.\partial U, \text{ seu}$$

$$\frac{\partial z}{Z} - \partial V = \frac{Np - Mq}{RZ(MP + NQ)}.\partial U.$$

Statuatur jam

$$\frac{Np - Mq}{RZ(MP + NQ)} = f' : U,$$

atque habebitur

$$\int\frac{\partial z}{Z} - V = f : U, \text{ seu } \int\frac{\partial z}{Z} = V + f : U,$$

unde z determinatur per x et y.

Corollarium 1.

210. Pro solutione ergo problematis quaeratur primo ad formulam $Q\partial x - P\partial y$ multiplicator R eam reddens integrabilem, statuaturque

$$R(Q\partial x - P\partial y) = \partial U,$$

unde colligitur quantitas U per binas variabiles x et y expressa.

Corollarium 2.

211. Deinde quantitates M et N ita capiantur, ut formula $\frac{M\partial x + N\partial y}{MP + NQ}$ fiat integrabilis, cujus integrale si statuatur $= V$ statim habetur solutio generalis problematis, quae dat

$$\int\frac{\partial z}{z} = V + f:U.$$

Exemplum.

212. *Si P et Q sint functiones homogeneae ipsarum x et y utraque dimensionum numeri $= n$, solutionem problematis perficere.*

Ponatur $y = ux$, et tam P quam Q fiet productum ex potestate x^n in functionem quandam ipsius u. Sit ergo

$$P = x^n S \text{ et } Q = x^n T,$$

eruntque S et T functiones datae ipsius u. Tum vero ob

$$\partial y = u\partial x + x\partial u,$$

formula

$$Q\partial x - P\partial y$$

abit in

$$x^n T\partial x - x^n Su\partial x - x^{n+1} S\partial u = x^n[(T - Su)\partial x - Sx\partial u].$$

Sumatur ergo

$$R = \frac{1}{x^{n+1}(T - Su)}, \text{ fietque}$$

$$\partial U = \frac{\partial x}{x} - \frac{S\partial u}{T - Su}, \text{ unde colligitur } U.$$

Deinde pro altera quantitate V habebimus hanc aequationem

$$\partial V = \frac{(M + Nu)\partial x + Nx\partial u}{x^n(MS + NT)},$$

ubi jam facile est, pro M et N ejusmodi functiones ipsius u assumere, ut haec formula integrationem admittat. Integrale scilicet erit

$$V = \frac{-M - Nu}{(n-1)x^{n-1}(MS + NT)},$$

at M et N seu $\frac{M}{N} = K$ ita accipi debet, ut fiat

$$\frac{1}{(n-1)x^{n-1}} \partial \cdot \frac{K+u}{KS+T} = \frac{1}{x^{n-1}} \cdot \frac{\partial u}{KS+T}, \text{ seu}$$

$$-KK\partial S + KS\partial u - uK\partial S - uS\partial K + T\partial K - K\partial T$$
$$+ T\partial u - u\partial T + (n-1)\partial u(KS+T) = 0,$$

quae ad hanc formam reducitur

$$(T - Su)\partial K + K(nS\partial u - u\partial S - \partial T) - KK\partial S + nT\partial u - u\partial T = 0.$$

Ex qua, concessa aequationum resolutione, cognoscitur quantitas K, qua inventa erit

$$V = \frac{-K - u}{(n-1)x^{n-1}(KS+T)}.$$

Cum autem illa aequatio solutu difficilis videatur, ponatur statim $\frac{K+u}{KS+T} = v$, eritque

$$K = \frac{Tv - u}{1 - Sv} \text{ et } KS + T = \frac{T - Su}{1 - Sv},$$

unde fit

$$\partial v + \frac{(n-1)\partial u(1 - Sv)}{T - Su} = 0,$$

qua resoluta erit $V = \frac{-v}{(n-1)x^{n-1}}$.

Corollarium.

213. Casus autem quo $n = 1$ singulari evolutione eget. Facile autem patet tum sumi debere $M = -Nu$, ut fiat $\partial V = \frac{\partial u}{T - Su}$, unde postquam quantitas V fuerit inventa, erit semper

$$\int \frac{\partial z}{Z} = V + f : U.$$

Scholion.

214. Cum ternae variabiles x, y, z, sint inter se permutabiles patet hoc problema multo latius extendi posse. Scilicet si conditio proposita hac continetur aequatione $pP + qQ + R = 0$, non solum solvendi methodus adhibita succedit, si R sit functio ipsius z, et P cum Q functiones ipsarum x et y, sed etiam si fuerit P functio ipsius x et Q et R functiones ipsarum y et z; tum vero etiam si Q functio ipsius y, at P et R functiones binarum reliquarum x et z. Haec vero conditio cum ante tractatis eo redit, ut binae formulae differentiales p et q sint a se invicem separatae, neque plus una dimensione occupent, etiamsi et his casibus ingens restrictio accedat. Quodsi autem conditio magis sit complicata, solutio vix unquam sperari posse videtur, interim tamen casum ejusmodi proferam, quo solutionem expedire licet.

Problema 37.

215. *Si posito $\partial z = p\,\partial x + q\,\partial y$, debeat esse*
$$q = A p^n x^\lambda y^\mu z^\nu,$$
indolem functionis z in genere investigare.

Solutio.

Posito hoc valore loco q, habebimus
$$\partial z = p\,\partial x + A p^n x^\lambda y^\mu z^\nu \partial y, \text{ unde fit}$$
$$A y^\mu \partial y = p^{-n} x^{-\lambda} z^{-\nu} (\partial z - p\,\partial x).$$
Ponatur $p^{-n} x^{-\lambda} z^{-\nu} = t$, ut sit
$$p = t^{-\frac{1}{n}} x^{-\frac{\lambda}{n}} z^{-\frac{\nu}{n}}, \text{ eritque}$$
$$A y^\mu \partial y = t\,\partial z - t^{\frac{n-1}{n}} x^{-\frac{\lambda}{n}} z^{-\frac{\nu}{n}} \partial x.$$
Statuatur porro

$t^{n-1} z^{-\nu} = u^n$, seu $t = z^{\frac{\nu}{n-1}} u^{\frac{n}{n-1}}$, erit

$$A y^{\mu} \partial y = u^{\frac{n}{n-1}} z^{\frac{\nu}{n-1}} \partial z - u x^{-\frac{\lambda}{n}} \partial x.$$

Jam partibus quoad fieri licet integratis adipiscimur

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{n-1}{n+\nu-1} u^{\frac{n}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{nu}{n-\lambda} x^{\frac{n-\lambda}{n}} - \int \partial u \left(\frac{n}{n+\nu-1} u^{\frac{1}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{n}{n-\lambda} x^{\frac{n-\lambda}{n}}\right),$$

ac nunc solutionem per praecepta supra data expedire licet; scilicet statuatur

$$\frac{1}{n+\nu-1} u^{\frac{1}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{1}{n-\lambda} x^{\frac{n-\lambda}{n}} = f' : u,$$

eritque

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{n-1}{n+\nu-1} u^{\frac{n}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{n}{n-\lambda} u x^{\frac{n-\lambda}{n}} - n f : u,$$

atque ex his binis aequationibus si elidatur u, dabitur utique z per x et y.

Corollarium 1.

216. Casus $n = 1$ peculiarem postulat tractationem, cum enim posito $p = \frac{1}{t} x^{-\lambda} z^{-\nu}$ sit

$$A y^{\mu} \partial y = t \partial z - x^{-\lambda} z^{-\nu} \partial x,$$ erit

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{1}{\lambda-1} x^{1-\lambda} z^{-\nu} + \int \partial z \left(t - \frac{\nu}{\lambda-1} x^{1-\lambda} z^{-\nu-1}\right),$$

atque hinc statim concluditur

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{1}{\lambda-1} x^{1-\lambda} z^{-\nu} + f : z,$$ existente

$$t - \frac{\nu}{\lambda-1} x^{1-\lambda} z^{-\nu-1} = f' : z.$$

Corollarium 2.

217. Casus autem $n + \nu - 1 = 0$ et $n - \lambda = 0$ nullam facessunt molestiam, cum sit priori casu

$$\frac{n-1}{n+\nu-1} z^{\frac{n+\nu-1}{n-1}} = l z,$$

posteriori autem

$$\frac{n}{n-\lambda}\, x^{\frac{n-\lambda}{n}} = l\,x$$

quos valores in solutionem introduci oportet.

Exemplum.

218. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $pqxy = az$, seu $q = \frac{az}{pxy}$, functionem z investigare.*

Erit ergo

$$\partial z = p\,\partial x + \frac{az\partial y}{pxy}, \text{ seu } \frac{a\partial y}{y} = \frac{px}{z}(\partial z - p\,\partial x).$$

Ponatur $\frac{px}{z} = t$, seu $p = \frac{tz}{x}$, erit $\frac{a\partial y}{y} = t\partial z - \frac{ttz\partial x}{x}$.
Statuatur porro $ttz = uu$, seu $t = \frac{u}{\sqrt{z}}$, ut sit

$$\frac{a\partial y}{y} = \frac{u\partial z}{\sqrt{z}} - \frac{uu\partial x}{x},$$

et quoad fieri potest integrando

$$a\,l\,y = 2u\sqrt{z} - uu\,l\,x - \int \partial u\,(2\sqrt{z} - 2u\,l\,x),$$

ita ut jam post signum integrale unicum differentiale ∂u reperiatur. Posito ergo

$$\sqrt{z} - u\,l\,x = f' : u, \text{ erit}$$

$$a\,l\,y = 2u\sqrt{z} - uu\,l\,x - 2\,f:u = uu\,l\,x + 2u\,f':u - 2\,f:u.$$

Pro casu simplicissimo sumatur $f':u = 0$ et $f:u = 0$, erit $u = \frac{\sqrt{z}}{lx}$, ideoque

$$a\,l\,y = \frac{2z}{lx} - \frac{z}{lx} = \frac{z}{lx},$$

ita ut pro casu simplicissimo sit $z = a\,l\,x\,.\,l\,y$. Si ponatur

$$f':u = u\,l\,c \text{ et } f:u = \tfrac{1}{2}uu\,l\,c, \text{ erit}$$

$$u = \frac{\sqrt{z}}{lx + lc} = \frac{\sqrt{z}}{l.cx} \text{ et}$$

$$a\,l\,y = \frac{2z}{l.cx} - \frac{z\,lx}{(lcx)^2} - \frac{z\,lc}{(lcx)^2} = \frac{lz}{lcx},$$

ita ut sit

$$z = a\,l\,y\,(lc + lx).$$

magis generaliter autem erit

$$z = a(lb + ly)(lc + lx).$$

Scholion.

219. Methodi hactenus traditae haud mediocriter amplificabuntur, si loco binarum variabilium x et y, quarum functio esse debet z, binae aliae variabiles t et u introducantur, quarum relatio ad illas detur. Ita si z sit functio binarum variabilium x et y, ut inde prodeat

$$\partial z = p\,\partial x + q\,\partial y;$$

ac loco x et y aliae novae variabiles t et u introducantur, ut jam differentiatione instituta prodeat

$$\partial z = r\,\partial t + s\,\partial u;$$

quaeritur quomodo r et s per p et q determinentur, pro relatione inter pristinas variabiles x, y et novas t et u stabilita. Hinc ergo tam x quam y certae cuidam functioni ipsarum t et u aequabitur, quae cum detur sit

$$\partial x = P\,\partial t + Q\,\partial u \text{ et } \partial y = R\,\partial t + S\,\partial u,$$

ita ut facta hac substitutione z jam sit functio ipsarum t et u. Cum igitur esset

$$\partial z = p\,\partial x + q\,\partial y,$$

erit nunc

$$\partial z = (Pp + Rq)\,\partial t + (Qp + Sq)\,\partial u.$$

Est vero per hypothesin

$$\partial z = r\,\partial t + s\,\partial u,$$

unde habebitur

$$r = Pp + Rq \text{ et } s = Qp + Sq.$$

Quare facta hac substitutione valores differentiales novi ex praecedentibus ita determinabuntur, ut sit

$$\left(\frac{\partial z}{\partial t}\right) = P\left(\frac{\partial z}{\partial x}\right) + R\left(\frac{\partial z}{\partial y}\right) \text{ et } \left(\frac{\partial z}{\partial u}\right) = Q\left(\frac{\partial z}{\partial x}\right) + S\left(\frac{\partial z}{\partial y}\right).$$

Unde etiam cum sit vicissim

$$Q r - P s = (Q R - P S)\, q \text{ et } S r - R s = (P S - Q R)\, p,$$

concludimus fore

$$\left(\frac{\partial z}{\partial x}\right) = \frac{S}{PS - QR}\left(\frac{\partial z}{\partial t}\right) - \frac{R}{PS - QR}\left(\frac{\partial z}{\partial u}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{-Q}{PS - QR}\left(\frac{\partial z}{\partial t}\right) + \frac{P}{PS - QR}\left(\frac{\partial z}{\partial u}\right).$$

Vel cum x et y perinde ac z sint functiones ipsarum t et u haec relatio ita exprimi potest, ut sit

$$\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial z}{\partial y}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial y}\right).$$

Hinc efficitur, ut quae problemata pro data quadam relatione inter p, q, x, y, z resolvi possunt, ea quoque pro relatione inde resultante inter r, s, t, u et z resolvi queant; unde saepe problemata nascuntur, quae solutu vehementer difficilia videantur, ex quo non contemnenda subsidia in hanc Analyseos partem inferri possent; sed quia usus praecipue in formulis differentialibus secundi gradus spectatur, his non fusius immorans ad eas evoluendas progredior.

CALCULI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA,

SEU

INVESTIGATIO FUNCTIONUM DUARUM VARIABILIUM EX DATA DIFFERENTIALIUM CUJUSVIS GRADUS RELATIONE.

SECTIO SECUNDA,

INVESTIGATIO DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM SECUNDI GRADUS RELATIONE.

CAPUT I.

DE

FORMULIS DIFFERENTIALIBUS SECUNDI GRADUS IN GENERE.

Problema 38.

220.

Si z sit functio quaecunque binarum variabilium x et y, ejus formulas differentiales secundi gradus exhibere.

Solutio.

Cum z sit functio binarum variabilium x et y, ejus differentiale hujusmodi habebit formam

$$\partial z = p\partial x + q\partial y,$$

ex qua p et q sunt formulae differentiales primi gradus, quas ita denotare solemus

$$p = \left(\frac{\partial z}{\partial x}\right) \text{ et } q = \left(\frac{\partial z}{\partial y}\right).$$

Cum nunc sint quoque p et q functiones ipsarum x et y, formulae differentiales inde natae erunt formulae differentiales secundi gradus ipsius z, unde intelligitur quatuor hujusmodi formulas nasci

$$\left(\frac{\partial p}{\partial x}\right), \left(\frac{\partial p}{\partial y}\right), \left(\frac{\partial q}{\partial x}\right), \left(\frac{\partial q}{\partial y}\right),$$

quarum autem secundam ac tertiam inter se congruere in calculo differentiali est demonstratum. Sed cum sit $p = \left(\frac{\partial z}{\partial x}\right)$, simili scribendi ratione erit $\left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial \partial z}{\partial x^2}\right)$, cujus scripturae significatus hinc sponte patet. Deinde eodem modo erit $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial\partial z}{\partial x \partial y}\right)$, atque ob $q = \left(\frac{\partial z}{\partial y}\right)$ habebimus

$$\left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial\partial z}{\partial y \partial x}\right) \text{ et } \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial\partial z}{\partial y^2}\right).$$

Quia ergo est $\left(\frac{\partial\partial z}{\partial y \partial x}\right) = \left(\frac{\partial\partial z}{\partial x \partial y}\right)$, functioni z convenient tres formulae differentiales secundi gradus, quae sunt

$$\left(\frac{\partial\partial z}{\partial x^2}\right), \left(\frac{\partial\partial z}{\partial x \partial y}\right) \text{ et } \left(\frac{\partial\partial z}{\partial y^2}\right).$$

Corollarium 1.

221. Ut ergo functio z duarum variabilium x et y duas habet formulas differentiales primi gradus

$$\left(\frac{\partial z}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right),$$

ita habet tres formulas differentiales secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right), \left(\frac{\partial\partial z}{\partial x \partial y}\right) \text{ et } \left(\frac{\partial\partial z}{\partial y^2}\right).$$

Corollarium 2.

222. Haec ergo formulae per duplicem differentiationem nascuntur, unicam tantum quantitatem pro variabili accipienda. In prima scilicet bis eadem x variabilis sumitur, in secunda vero in altera differentiatione x, in altera autem y variabilis accipitur; in tertia autem bis y.

Corollarium 3.

223. Simili modo patet, ejusdem functionis z quatuor dari formulas differentiales tertii gradus, scilicet

$$\left(\frac{\partial^3 z}{\partial x^3}\right); \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right), \left(\frac{\partial^3 z}{\partial x \partial y^2}\right); \left(\frac{\partial^3 z}{\partial y^3}\right),$$

quarti autem gradus quinque; quinti, sex, etc.

Scholion.

224. Formulae hae differentiales secundi gradus ope substitutionis saltem ad formam primi gradus revocari possunt. Veluti

formula $\left(\frac{\partial\partial z}{\partial x^2}\right)$, si ponatur $\left(\frac{\partial z}{\partial x}\right) = p$, transformabitur in $\left(\frac{\partial p}{\partial x}\right)$; formula autem $\left(\frac{\partial\partial z}{\partial x \partial y}\right)$ eadem substitutione in hanc $\left(\frac{\partial p}{\partial y}\right)$. At posito $\left(\frac{\partial z}{\partial y}\right) = q$, formula $\left(\frac{\partial\partial z}{\partial x \partial y}\right)$ transmutatur in hanc $\left(\frac{\partial q}{\partial x}\right)$; formula autem $\left(\frac{\partial\partial z}{\partial y^2}\right)$ in hanc $\left(\frac{\partial q}{\partial y}\right)$. Vicissim autem uti ex aequalitate $p = \left(\frac{\partial z}{\partial x}\right)$ deduximus

$$\left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ et } \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial\partial z}{\partial x \partial y}\right),$$

ita ex his ulterius progrediendo colligemus

$$\left(\frac{\partial\partial p}{\partial x^2}\right) = \left(\frac{\partial^3 z}{\partial x^3}\right), \ \left(\frac{\partial\partial p}{\partial x \partial y}\right) = \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right), \ \left(\frac{\partial\partial p}{\partial y^2}\right), \ \left(\frac{\partial^3 z}{\partial x \partial y^2}\right).$$

Tum vero etiam si ponamus $\left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial\partial z}{\partial x \partial y}\right)$, hinc sequentur istae aequalitates

$$\left(\frac{\partial\partial q}{\partial x^2}\right) = \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) \text{ et } \left(\frac{\partial\partial q}{\partial x \partial y}\right) = \left(\frac{\partial^3 z}{\partial x \partial y^2}\right).$$

Hicque est quasi novus algorithmus, cujus principia per se ita sunt manifesta, ut majore illustratione non indigeant.

Exemplum 1.

225. *Si sit $z = xy$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit $\left(\frac{\partial z}{\partial x}\right) = y$ et $\left(\frac{\partial z}{\partial y}\right) = x$, erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = 0, \ \left(\frac{\partial\partial z}{\partial x \partial y}\right) = 1 \text{ et } \left(\frac{\partial\partial z}{\partial y^2}\right) = 0.$$

Exemplum 2.

226. *Si sit $z = x^m y^n$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit $\left(\frac{\partial z}{\partial x}\right) = m x^{m-1} y^n$ et $\left(\frac{\partial z}{\partial y}\right) = n x^m y^{n-1}$, erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = m(m-1) x^{m-2} y^n, \ \left(\frac{\partial\partial z}{\partial x \partial y}\right) = m n x^{m-1} y^{n-1},$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = n(n-1) x^m y^{n-2}.$$

Exemplum 3.

227. *Si sit $z = \sqrt{(xx+yy)}$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{x}{\sqrt{(xx+yy)}} \text{ et } \left(\frac{\partial z}{\partial y}\right) = \frac{y}{\sqrt{(xx+yy)}}, \text{ erit}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}};$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

Scholion.

228. Quemadmodum binae formulae differentiales primi gradus cujusque functionis z ita sunt comparatae, ut sit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right),$$

et integrando

$$z = \int \left[\partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right)\right],$$

ita quoque in formulis secundi gradus erit

$$\left(\frac{\partial z}{\partial x}\right) = \int \left[\partial x \left(\frac{\partial\partial z}{\partial x^2}\right) + \partial y \left(\frac{\partial\partial z}{\partial x\,\partial y}\right)\right] \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \int \left[\partial x \left(\frac{\partial\partial z}{\partial x\,\partial y}\right) + \partial y \left(\frac{\partial\partial z}{\partial y^2}\right)\right].$$

Tres igitur formulae secundi gradus semper ita sunt comparatae, ut geminam integrationem praebeant, si scilicet cum differentialibus ∂x et ∂y rite combinentur, haecque proprietas quae probe notetur, in sequentibus insigne adjumentum afferet.

Problema 39.

229. Si z sit functio binarum variabilium x et y, loco x et y introducantur binae novae variabiles t et u, ita ut tam x

quam y aequetur certae functioni ipsarum t et u, formulas differentiales secundi gradus ipsius z respectu harum novarum variabilium definire.

Solutio.

Quatenus z per x et y datur, datae sunt ejus formulae differentiales tam primi gradus $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, quam secundi gradus $\left(\frac{\partial \partial z}{\partial x^2}\right)$, $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$, $\left(\frac{\partial \partial z}{\partial y^2}\right)$, ex quibus quomodo formulae differentiales respectu novarum variabilium t et u determinentur definiri oportet. Pro primo gradu autem cum sit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right),$$

quia tam x quam y datur per t et u erit

$$\partial x = \partial t \left(\frac{\partial x}{\partial t}\right) + \partial u \left(\frac{\partial x}{\partial u}\right) \text{ et } \partial y = \partial t \left(\frac{\partial y}{\partial t}\right) + \partial u \left(\frac{\partial y}{\partial u}\right),$$

quibus valoribus substitutis habebitur ipsius z differentiale plenum ex variatione utriusque t et u ortum

$$\partial z = \partial t \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial z}{\partial x}\right) + \partial u \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \partial t \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial z}{\partial y}\right) + \partial u \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial y}\right).$$

Quodsi jam vel sola t variabilis sumatur, vel sola u, prodibunt formulae differentiales primi gradus

$$\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial z}{\partial y}\right), \quad \left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial y}\right).$$

Simili modo ulterius progrediendo, differentiemus formulas

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = q$$

primo generaliter, tum vero loco x et y etiam t et u introducamus; hincque nanciscemur

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial p}{\partial y}\right), \quad \left(\frac{\partial p}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial p}{\partial y}\right),$$

$$\left(\frac{\partial q}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial q}{\partial y}\right), \quad \left(\frac{\partial q}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial q}{\partial y}\right),$$

unde poterimus formulas $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ pro variabilitate tam solius t quam solius u assignare; scilicet cum sit

$$\left(\frac{\partial z}{\partial t}\right) = p\left(\frac{\partial x}{\partial t}\right) + q\left(\frac{\partial y}{\partial t}\right) \text{ et } \left(\frac{\partial z}{\partial u}\right) = p\left(\frac{\partial x}{\partial u}\right) + q\left(\frac{\partial y}{\partial u}\right),$$

inveniemus

$$\left(\frac{\partial\partial z}{\partial t^2}\right) = \left(\frac{\partial\partial x}{\partial t^2}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial t^2}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial t}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ 2\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)^2\left(\frac{\partial\partial z}{\partial y^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) = \left(\frac{\partial\partial x}{\partial t\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial t\partial u}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial y^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial u^2}\right) = \left(\frac{\partial\partial x}{\partial u^2}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial u^2}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ 2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial\partial z}{\partial y^2}\right).$$

Corollarium 1.

230. Proposita ergo conditione quadam inter formulas differentiales functionis z, quatenus per variabiles t et u definitur, eadem conditio pro eadem functione z transfertur ad alias binas variabiles x et y, ab illis utcunque pendentes.

Corollarium 2.

231. Formulae quidem

$$\left(\frac{\partial x}{\partial t}\right),\ \left(\frac{\partial y}{\partial t}\right),\ \left(\frac{\partial x}{\partial u}\right),\ \left(\frac{\partial y}{\partial u}\right), \text{ etc.}$$

per t et u exprimuntur, ex relatione, quae inter x, y et t, u assumitur, verum indidem eaedem formulae ad variabiles x et y revocari possunt.

Scholion.

232. Quemadmodum hic variabilitas quantitatum t et u per formulas differentiales ex variabilibus x et y natas est expressa, ita vicissim si variabiles t et u proponantur, ex quibus certo modo alterae x et y determinentur, sequentes reductiones habebuntur, facta tantum variabilium permutatione. Primo scilicet pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right), \quad \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial z}{\partial u}\right).$$

Pro formulis autem differentialibus secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial t}{\partial x^2}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial\partial u}{\partial x^2}\right)\left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial t}{\partial x}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+ 2\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \left(\frac{\partial\partial t}{\partial x\,\partial y}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial\partial u}{\partial x\,\partial y}\right)\left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+ \left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+ 2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right),$$

ubi determinatio litterarum t et u per alteras x et y considerari debet. Quoniam scilicet in conditionibus praescriptis binis variabilibus x et y uti solemus, earum loco alias quascunque t et u introducendo, loco illarum formularum differentialium has novas formas ad variabiles t et u relatas adhibere poterimus, ubi deinceps relatio inter variabiles x, y et t, u ita est constituenda, ut quaestio solutu facilior evadat. Pro variis igitur hujusmodi relationibus exempla evoluamus.

Exemplum 1.

233. *Si inter variabiles x, y et t, u haec relatio constituatur, ut sit*

$$t = \alpha x + \beta y \quad et \quad u = \gamma x + \delta y,$$

reductionem formularum differentialium exhibere.

Cum sit

$$\left(\frac{\partial t}{\partial x}\right) = \alpha, \quad \left(\frac{\partial t}{\partial y}\right) = \beta, \quad \left(\frac{\partial u}{\partial x}\right) = \gamma, \quad \left(\frac{\partial u}{\partial y}\right) = \delta,$$

hincque formulae pro secundo gradu evanescant, habebimus pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = \alpha\left(\frac{\partial z}{\partial t}\right) + \gamma\left(\frac{\partial z}{\partial u}\right), \quad \left(\frac{\partial z}{\partial y}\right) = \beta\left(\frac{\partial z}{\partial t}\right) + \delta\left(\frac{\partial z}{\partial u}\right).$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \alpha\alpha\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\alpha\gamma\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \gamma\gamma\left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \alpha\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \gamma\delta\left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \beta\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\beta\delta\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \delta\delta\left(\frac{\partial\partial z}{\partial u^2}\right).$$

Corollarium 1.

234. Si sumatur $t = x$ et $u = x + y$, erit $\alpha = 1$, $\beta = 0$, $\gamma = 1$ et $\delta = 1$, ergo

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial z}{\partial u}\right), \quad \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial u}\right), \text{ atque}$$
$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial u^2}\right).$$

Corollarium 2.

235. Etsi ergo hic est $t = x$, tamen non est $\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial z}{\partial x}\right)$, cujus rei ratio est, quod in forma $\left(\frac{\partial z}{\partial x}\right)$ quantitas y sumitur constans, in $\left(\frac{\partial z}{\partial t}\right)$ vero quantitas $u = x + y$, id quod in genere notasse iuvat, ne ex aequalitate $t = x$ ad aequalitatem formularum $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial t}\right)$ concludamus.

Exemplum 2.

236. *Si inter variabiles* t, u *et* x, y *haec relatio constituatur, ut sit* $t = \alpha x^m$ et $u = \beta y^n$, *reductionem exhibere.*

Hic ergo erit

$$\left(\frac{\partial t}{\partial x}\right) = m\alpha x^{m-1}, \quad \left(\frac{\partial t}{\partial y}\right) = 0, \quad \left(\frac{\partial\partial t}{\partial x^2}\right) = m(m-1)\alpha x^{m-1},$$
$$\left(\frac{\partial u}{\partial x}\right) = 0, \quad \left(\frac{\partial u}{\partial y}\right) = n\beta y^{n-1}, \quad \left(\frac{\partial\partial u}{\partial y^2}\right) = n(n-1)\beta y^{n-2},$$

unde obtinemus pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = m\,\alpha\,x^{m-1}\left(\frac{\partial z}{\partial t}\right),\quad \left(\frac{\partial z}{\partial y}\right) = n\,\beta\,y^{n-1}\left(\frac{\partial z}{\partial u}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = m\,(m-1)\,\alpha\,x^{m-2}\left(\frac{\partial z}{\partial t}\right) + m\,m\,\alpha\,\alpha\,x^{2m-2}\left(\frac{\partial\partial z}{\partial t^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = m\,n\,\alpha\,\beta\,x^{m-1}\,y^{n-1}\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = n\,(n-1)\,\beta\,y^{n-2}\left(\frac{\partial z}{\partial u}\right) + n\,n\,\beta\,\beta\,y^{2n-2}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

in quibus formulis jam loco x et y earum valores per t et u induci debent.

Exemplum 3.

237. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit $x = t$ et $\frac{x}{y} = u$, formularum differentialium reductionem exhibere.*

Cum sit $t = x$ et $u = \frac{x}{y}$, erit

$$\left(\frac{\partial t}{\partial x}\right) = 1,\quad \left(\frac{\partial t}{\partial y}\right) = 0,$$

hincque formulae involuentes $\partial\partial t$ evanescunt. Porro

$$\left(\frac{\partial u}{\partial x}\right) = \frac{1}{y} = \frac{u}{t},\quad \left(\frac{\partial u}{\partial y}\right) = \frac{-x}{yy} = \frac{-uu}{t},$$

$$\left(\frac{\partial\partial u}{\partial x^2}\right) = 0,\quad \left(\frac{\partial\partial u}{\partial x\partial y}\right) = \frac{-1}{yy} = \frac{-uu}{tt},\quad \left(\frac{\partial\partial u}{\partial y^2}\right) = \frac{2x}{y^3} = \frac{2u^3}{tt},$$

unde pro formulis primi gradus habebimus

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial t}\right) + \frac{u}{t}\left(\frac{\partial z}{\partial u}\right),\quad \left(\frac{\partial z}{\partial y}\right) = \frac{-uu}{t}\left(\frac{\partial z}{\partial u}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial z}{\partial t^2}\right) + \frac{2u}{t}\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{uu}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{-uu}{tt}\left(\frac{\partial z}{\partial u}\right) - \frac{uu}{t}\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{u^3}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{2u^3}{tt}\left(\frac{\partial z}{\partial u}\right) + \frac{u^4}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right).$$

Exemplum 4.

238. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit $t = e^x$ et $u = e^x y$, seu $x = lt$ et $y = \frac{u}{t}$, reductionem formularum differentialium exhibere.*

Hic ergo est

$$\left(\frac{\partial t}{\partial x}\right) = e^x = t,\ \left(\frac{\partial t}{\partial y}\right) = 0,\ \left(\frac{\partial\partial t}{\partial x^2}\right) = e^x = t,\ \left(\frac{\partial\partial t}{\partial x \partial y}\right) = 0.$$

Deinde

$$\left(\frac{\partial u}{\partial x}\right) = e^x y = u,\ \left(\frac{\partial u}{\partial y}\right) = e^x = t,$$

tum vero

$$\left(\frac{\partial\partial u}{\partial x^2}\right) = e^x y = u,\ \left(\frac{\partial\partial u}{\partial x \partial y}\right) = e^x = t,\ \left(\frac{\partial\partial u}{\partial y^2}\right) = 0.$$

Quare pro formulis primi gradus habebimus

$$\left(\frac{\partial z}{\partial x}\right) = t\left(\frac{\partial z}{\partial t}\right) + u\left(\frac{\partial z}{\partial u}\right),\ \left(\frac{\partial z}{\partial y}\right) = t\left(\frac{\partial z}{\partial u}\right).$$

Pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = t\left(\frac{\partial z}{\partial t}\right) + u\left(\frac{\partial z}{\partial u}\right) + tt\left(\frac{\partial\partial z}{\partial t^2}\right) + 2tu\left(\frac{\partial\partial z}{\partial t \partial u}\right) + uu\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) = t\left(\frac{\partial z}{\partial u}\right) + tt\left(\frac{\partial\partial z}{\partial t \partial u}\right) + tu\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = tt\left(\frac{\partial\partial z}{\partial u^2}\right).$$

Scholion.

239. In formulis generalibus §. 232. datis assumsimus valores variabilium t et u per x et y expressos dari, et universa evolutione facta tum demum pro x et y variabiles t et u restitui. Commodius ergo videatur, si statim variabilium x et y valores per t et u expressi habeantur; verum inde valores formularum $\left(\frac{\partial t}{\partial x}\right)$, $\left(\frac{\partial t}{\partial y}\right)$, etc. nimis complicate exprimerentur, quam ut eas in calculum introducere liceret. Scilicet si x et y per t et u dentur, inde fit

$$\left(\frac{\partial t}{\partial x}\right) = \frac{\left(\frac{\partial y}{\partial u}\right)}{\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right) - \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial t}\right)},$$

ac formulae secundi gradus multo magis prodituraе sunt perplexae. Quovis ergo casu, quo hujusmodi reductione utendum videtur, conjectura potius quam certa ratione idoneam variabilium immutationem colligi conveniet. Alia vero etiam datur reductio saepe insignem utilitatem afferens, dum ipsius functionis z quaesitae forma mutatur; veluti si ponatur $z = Vv$, denotante V functionem datam ipsarum x et y, ita ut jam v sit functio quaesita; quin etiam haec nova quaesita v alio modo cum datis implicari potest.

Problema 40.

240. Proposita functione z binarum variabilium x et y, ac posita $z = Pv$, ita ut P sit data quaedam functio ipsarum x et y, formulas differentiales novae functionis v exprimere.

Solutio.

Cum sit $z = Pv$, ex regulis differentiandi traditis habebimus primo formulas differentiales primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial P}{\partial x}\right) v + P\left(\frac{\partial v}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial P}{\partial y}\right) v + P\left(\frac{\partial v}{\partial y}\right).$$

Atque hinc deinceps formulae differentiales secundi ordinis ita prodibunt expressae

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial P}{\partial y^2}\right) v + 2\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + P\left(\frac{\partial\partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \left(\frac{\partial\partial P}{\partial x\partial y}\right) v + \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + P\left(\frac{\partial\partial v}{\partial x\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial P}{\partial y^2}\right) v + 2\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + P\left(\frac{\partial\partial v}{\partial y^2}\right),$$

ubi cum P sit functio data ipsarum x et y, ejus formulae differentiales simul habentur.

Corollarium 1.

241. Si P esset functio ipsius x tantum, puta X, tum posito $z = Xv$ erit

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial X}{\partial x}\right) . v + X\left(\frac{\partial v}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right) = X\left(\frac{\partial v}{\partial y}\right), \text{ tum}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial X}{\partial x^2}\right) . v + \frac{2\partial X}{\partial x}\left(\frac{\partial v}{\partial x}\right) + X\left(\frac{\partial\partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{\partial X}{\partial x}\left(\frac{\partial v}{\partial y}\right) + X\left(\frac{\partial\partial v}{\partial x\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = X\left(\frac{\partial\partial v}{\partial y^2}\right).$$

Corollarium 2.

242. Transformatio haec easdem variabiles x et y servat, et tantum loco functionis z alia v introducitur; cum ante manente eadem functione z, binae variabiles x et y ad alias t et u sint reductae. Ex quo hae duae transformationes genere sunt diversae.

Scholion 1.

243. Casus simplicior fuisset, si per additionem posuissemus $z = P + v$, ut esset P functio quaedam data ipsarum x et y; verum tum transformatio ita fit obvia, ut investigatione non indigeat; est enim manifesto

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial P}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial P}{\partial x^2}\right) + \left(\frac{\partial\partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \left(\frac{\partial\partial P}{\partial x\partial y}\right) + \left(\frac{\partial\partial v}{\partial x\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial P}{\partial y^2}\right) + \left(\frac{\partial\partial v}{\partial y^2}\right).$$

Neque vero etiam formas magis compositas evolvi necesse est, veluti, si ponamus $z = \sqrt{(PP + vv)}$, quandoquidem talis forma vix unquam usum foret habitura.

Scholion 2.

241. Praemissis his principiis et transformationibus, negotium aggrediamur, et methodos aperiamus, ex data relatione inter formulas differentiales secundi gradus, et primi gradus, itemque ipsas quantitates principales, harum ipsarum relationem investigandi. Hic scilicet praeter ipsas quantitates x, y et z, earumque formulas differentiales primi gradus $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ considerandae veniunt tres formulae differentiales secundi gradus $\left(\frac{\partial\partial z}{\partial x^2}\right)$, $\left(\frac{\partial\partial z}{\partial x\partial y}\right)$ et $\left(\frac{\partial\partial z}{\partial y^2}\right)$; quarum vel una, vel binae; vel omnes tres in relationem propositam ingredi possunt, ubi insuper ingens discrimen formulae primi gradus, sive in relationem ingrediantur, sive secus, constituunt. Non solum autem nimis longum foret omnes combinationes, uti in praecedente sectione fecimus, prosequi, sed etiam defectus idonearum methodorum impedit, quo minus singula quaestionum huc pertinentium genera percurramus. Capita igitur pertractanda ita instituamus, prout methodus solvendi patietur, ea, ubi nihil praestare licet, penitus praetermissuri.

CAPUT II.

DE

UNA FORMULA DIFFERENTIALI SECUNDI GRADUS PER RELIQUAS QUANTITATES UTCUNQUE DATA.

Problema 41.

245.

Si z debeat esse ejusmodi functio ipsarum x et y, ut formula secundi gradus $\left(\frac{\partial\partial z}{\partial x^2}\right)$ aequetur functioni datae ipsarum x et y; indolem functionis z investigare.

Solutio.

Sit P functio ista data ipsarum x et y, ita ut esse debeat $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$. Sumatur jam y constans, et cum sit

$$\partial \,.\, \left(\frac{\partial z}{\partial x}\right) = \partial x \left(\frac{\partial\partial z}{\partial x^2}\right), \quad \text{erit} \quad \partial \,.\, \left(\frac{\partial z}{\partial x}\right) = P\partial x,$$

unde integrando prodit

$$\left(\frac{\partial z}{\partial x}\right) = \int P\partial x + \text{Const.}$$

Ubi in integratione $\int P\partial x$ quantitas y pro constante habetur, et constans adjicienda functionem quamcunque ipsius y denotabit, ita ut haec prima integratio praebeat

$$\left(\frac{\partial z}{\partial x}\right) = \int P\partial x + f : y.$$

Nunc iterum quantitate y ut constante spectata, erit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) \quad \text{seu} \quad \partial z = \partial x \int P\partial x + \partial x f : y,$$

ubi cum $\int P\partial x$ sit functio ipsarum x et y, quarum haec y constans assumitur, integratio denuo instituta dabit

$$z = \int\partial x\int P\partial x + xf:y + F:y,$$

quod est integrale completum aequationis differentio - differentialis propositae $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$; propterea quod duas functiones arbitrarias $f:y$ et $F:y$ complectitur, quarum utramque ita pro lubitu accipere licet, ut etiam functiones discontinuae non excludantur.

Corollarium 1.

246. Quodsi ergo proponatur haec conditio $\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$ ejus integratio completa dabit

$$z = xf:y + F:y,$$

ob $P = 0$, cujus veritas ex differentiatione perspicitur, unde fit primo $\left(\frac{\partial z}{\partial x}\right) = f:y$, tum vero $\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$.

Corollarium 2.

247. Eodem modo in genere integrale inventum per differentiationem comprobatur. Cum enim invenerimus

$$z = \int\partial x\int P\partial x + xf:y + F:y,$$

prima differentiatio praebet

$$\left(\frac{\partial z}{\partial x}\right) = \int P\partial x + f:y,$$

repetita vero $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$.

Corollarium 3.

248. Simili modo si haec proponatur conditio $\left(\frac{\partial\partial z}{\partial y^2}\right) = Q$, existente Q functione quacunque ipsarum x et y, integrale completum reperitur

$$z = \int\partial y\int Q\partial y + yf:x + F:x,$$

ubi in geminato integrali $\int \partial y \int Q \partial y$ quantitas x pro constante habetur.

Scholion.

249. Hinc ratio integralium completorum, quae ex formulis differentialibus secundi gradus nascuntur, in genere perspicitur, quae in hoc est sita, ut duae functiones arbitrariae invehantur, ubi iterum notandum est, has functiones tam discontinuas quam continuas esse posse. Nisi ergo per totam hanc sectionem integralia duas hujusmodi functiones arbitrarias involvant, ea pro completis haberi nequeunt. Quotiescunque enim problema ad hujusmodi aequationem $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$ perducit, ejus indoles semper ita est comparata, ut tributo ipsi x certo quodam valore $x = a$, tam formula $\left(\frac{\partial z}{\partial x}\right)$ quam ipsa quantitas z datae cuipiam functioni ipsius y aequari possit. Quare si tam integrale $\int P \partial x$ quam hoc $\int \partial x \int P \partial x$ ita accipiatur, ut posito $x = a$ evanescat, erit pro eodem casu $x = a$, valor

$$\left(\frac{\partial z}{\partial x}\right) = f : y \quad \text{et} \quad z = a f : y + F : y,$$

unde ex problematis natura utraque functio $f : y$ et $F : y$ definitur. Haec autem applicatio ad omnes casus fieri non posset, nisi integrale completum haberetur; quamobrem in hoc praecipue est incumbendum, ut omnium hujusmodi problematum integralia completa habeantur. Caeterum hic in perpetuum monendum duco, quoties hujusmodi formula integralis $\int P \partial x$ occurrit, semper solam quantitatem x variabilem accipi esse intelligendam; siquidem si etiam y variabilis acciperetur, formula $\int P \partial x$ ne significatum quidem admitteret. Simili modo in formula $\int \partial x \int P \partial x$ intelligi debet, in utraque integratione solam x variabilem assumi. Sin autem talis forma $\int \partial y \int P \partial x$ occurrat, intelligendum est, integrale $\int P \partial x$ ex variabilitate solius x colligi debere, quod si ponatur $= R$, ut habeatur $\int R \partial y$, hic jam sola y pro variabili erit habenda.

Exemplum 1.

250. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{xy}{a}$.

Cum hic sit $P = \frac{xy}{a}$, erit

$$\int P\partial x = \frac{xxy}{2a} \quad \text{et} \quad \int\partial x\int P\partial x = \frac{x^3y}{6a},$$

sicque habebitur ex prima integratione

$$\left(\frac{\partial z}{\partial x}\right) = \frac{xxy}{2a} + f:y,$$

ita ut posito $x = a$, formula $\left(\frac{\partial z}{\partial x}\right)$ functioni cuicunque ipsius y aequari possit, seu applicatae curvae cujuscunque respondenti abscissae y. Tum vero altera integratione instituta, erit

$$z = \frac{x^3y}{6a} + xf:y + F:y,$$

qui valor casu $x = a$ denuo functioni cuicunque ipsius y aequari potest.

Exemplum 2.

251. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{ax}{\sqrt{(xx+yy)}}$.

Ob $P = \frac{ax}{\sqrt{(xx+yy)}}$, erit

$$\int P\partial x = a\sqrt{(xx+yy)}, \text{ et}$$

$$\int\partial x\int P\partial x = a\int\partial x\sqrt{(xx+yy)} = \tfrac{1}{2}ax\sqrt{(xx+yy)} + \tfrac{1}{2}ayyl[x+\sqrt{(xx+yy)}];$$

unde prima integratio praebet

$\left(\frac{\partial z}{\partial x}\right) = a\sqrt{(xx+yy)} + f:y$ altera vero

$$z = \tfrac{1}{2}ax\sqrt{(xx+yy)} + \tfrac{1}{2}ayyl[x+\sqrt{(xx+yy)}] + xf:y + F:y.$$

Exemplum 3.

252. *Quaeratur binarum variabilium x et y ejusmodo functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{1}{\sqrt{(aa - xx - yy)}}$,

Cum sit $P = \frac{1}{\sqrt{(aa - xx - yy)}}$, erit

$$\int P\partial x = \text{Ang. sin.} \frac{x}{\sqrt{(aa - yy)}},$$

tum vero

$$\int \partial x \int P\partial x = x \,\text{Ang. sin.} \frac{x}{\sqrt{(aa - yy)}} - \int \frac{x\partial x}{\sqrt{(aa - xx - yy)}}.$$

Quare integratio prima praebet

$$\left(\frac{\partial z}{\partial x}\right) = \text{Ang. sin.} \frac{x}{\sqrt{(aa - yy)}} + f:y,$$

hincque ipsa functio quaesita erit

$$z = x \,\text{Ang. sin} \frac{x}{\sqrt{(aa - yy)}} + \sqrt{(aa - xx - yy)} + x f:y + F:y.$$

Exemplum 4.

253. *Quaeratur binarum variabilium x et y ejusmodi functio. z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = x \sin.(x + y)$.

Ob $P = x \sin.(x + y)$, erit

$$\int P\partial x = \int x\partial x \sin.(x + y) = -x\cos.(x + y) + \int \partial x \cos.(x + y)$$

$$\text{seu } \int P\partial x = -x\cos.(x + y) + \sin.(x + y).$$

Tum vero est

$$\int x\partial x \cos.(x + y) = x\sin.(x + y) + \cos.(x + y),$$

ideoque

$$\int \partial x \int P\partial x = -2\cos.(x + y) - x\sin.(x + y).$$

Quocirca ambo nostra integralia erunt

$$\left(\frac{\partial z}{\partial x}\right) = \sin.(x + y) - x\cos.(x + y) + f:y \quad \text{et}$$

$$z = -2\cos.(x + y) - x\sin.(x + y) + x f:y + F:y.$$

Problema 42.

254. Si z debeat esse ejusmodi functio variabilium x et y, ut sit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = P\left(\frac{\partial z}{\partial x}\right) + Q,$$

existentibus P et Q functionibus quibusvis ipsarum x et y, indolem functionis z in genere investigare.

Solutio.

Ponamus hic $\left(\frac{\partial z}{\partial x}\right) = v$, ut sit $\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial v}{\partial x}\right)$, erit nostra aequatio integranda

$$\left(\frac{\partial v}{\partial x}\right) = Pv + Q.$$

Spectetur ergo sola x ut variabilis, et ob $\partial v = \partial x\left(\frac{\partial v}{\partial x}\right)$, erit

$$\partial v = Pv\partial x + Q\partial x,$$

quae per $e^{-\int P\partial x}$ multiplicata et integrata dat

$$e^{-\int P\partial x} v = \int e^{-\int P\partial x} Q\partial x + f:y,$$

ideoque

$$\left(\frac{\partial z}{\partial x}\right) = e^{\int P\partial x}\int e^{-\int P\partial x} Q\partial x + e^{\int P\partial x} f:y.$$

Retineatur sola x variabilis, spectata y ut constante, et ob

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) \text{ erit}$$

$$z = \int e^{\int P\partial x}\partial x \int e^{-\int P\partial x} Q\partial x + f:y \int e^{\int P\partial x}\partial x + F:y,$$

quod ob binas functiones arbitrarias $f:y$ et $F:y$ est integrale completum.

Corollarium 1.

255. Problema hoc multo latius patet praecedente, cum conditio proposita etiam formulam primi gradus $\left(\frac{\partial z}{\partial x}\right)$ involvat, nihilo vero minus solutio feliciter successit.

Corollarium 2.

256. Hic ergo quadruplici integratione est opus, primo scilicet quaeri debet integrale $\int P\partial x$, quod si ponatur $= lR$, quaeri porro debet integrale

$$\int e^{\int P\partial x}\partial x = \int R\partial x,$$

quod si ponamus $= S$, restat integrale

$$\int R\partial x \int \frac{Q\partial x}{R} = \int \partial S \int \frac{Q\partial x}{R},$$

quod abit in

$$S\int \frac{Q\partial x}{R} - \int \frac{QS\partial x}{R},$$

ita ut insuper hae duae formae integrari debeant.

Corollarium 3.

257. Eodem omnino modo resolvitur problema, quo esse debet

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial z}{\partial y}\right) + Q,$$

si P et Q fuerint functiones quaecunque datae ipsarum x et y. Reperitur enim

$$\left(\frac{\partial z}{\partial y}\right) = e^{\int P\partial y}\int e^{-\int P\partial y} Q\partial y + e^{\int P\partial y}.\,f:x \text{ et}$$

$$z = \int e^{\int P\partial y}\partial y \int e^{-\int P\partial y} Q\partial y + f:x.\int e^{\int P\partial y}\partial y + F:x.$$

Exemplum 1.

258. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{n}{x}\left(\frac{\partial z}{\partial x}\right)$.

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, sumtoque solo x variabili, erit $\frac{\partial v}{\partial x} = \frac{nv}{x}$, ideoque $\frac{\partial v}{v} = \frac{n\partial x}{x}$, cujus integrale dat

$$v = \left(\frac{\partial z}{\partial x}\right) = x^n\, f:y.$$

Jam iterum sola x pro variabili habita, erit

$$\partial z = x^n \partial x f : y,$$

cujus integrale completum est

$$z = \frac{1}{n+1} x^{n+1} f : y + F : y.$$

Casu autem $n = -1$, seu $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{-1}{x}\left(\frac{\partial z}{\partial x}\right)$, erit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x} f : y, \text{ et } z = lx . f : y + F : y.$$

Exemplum 2.

259. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{n}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{a}{xy}$.

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, sumtoque solo x variabili, erit

$$\partial v = \frac{nv\partial x}{x} + \frac{a\partial x}{xy},$$

quae aequatio per x^n divisa et integrata praebet

$$\frac{v}{x^n} = \frac{a}{y}\int\frac{\partial x}{x^{n+1}} = \frac{-a}{nx^n y} + f : y, \text{ seu}$$

$$v = \left(\frac{\partial z}{\partial x}\right) = \frac{-a}{ny} + x^n f : y.$$

Sit iterum sola x variabilis, ut habeatur

$$\partial z = \frac{-a\partial x}{ny} + x^n \partial x f : y,$$

prodibitque integrale completum

$$z = \frac{-ax}{ny} + \frac{1}{n+1} x^{n+1} f : y + F : y.$$

Exemplum 3.

260. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{2nx}{xx+yy}\left(\frac{\partial z}{\partial x}\right) + \frac{x}{ay}$.

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, erit sumendo y constans

$$\partial v = \frac{2nxv\partial x}{xx+yy} + \frac{x\partial x}{ay},$$

cunque per quantitates principales x, y et z ac praeterea formulam $\left(\frac{\partial z}{\partial x}\right)$ determinetur, ita ut etiam hujus formulae $\left(\frac{\partial z}{\partial x}\right)$ potestates aliaeve functiones quaecunque ingrediantur, solutio semper ad librum superiorem revocabitur; quia ponendo y constans fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial x} \text{ et } \left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial\partial z}{\partial x^2},$$

ideoque resultat aequatio differentialis secundi gradus formae consuetae duas tantum variabiles x et z involvens. Hoc tantum teneatur, loco constantium per utramque integrationem ingredientium scribi oportere formas $f:y$ et $F:y$. Satis igitur notabilem partem propositi nostri expedivimus, scilicet cum vel $\left(\frac{\partial\partial z}{\partial x^2}\right)$ utcunque per x, y, z et $\left(\frac{\partial z}{\partial x}\right)$, vel $\left(\frac{\partial\partial z}{\partial y^2}\right)$ utcunque per x, y, z et $\left(\frac{\partial z}{\partial y}\right)$ determinatur, ibi nempe excluditur formula primi gradus $\left(\frac{\partial z}{\partial y}\right)$, hic vero formula $\left(\frac{\partial z}{\partial x}\right)$. Quae si accederet, quaestio hac methodo neutiquam tractari posset; quemadmodum vel ex hoc casu simplicissimo $\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial z}{\partial y}\right)$ intelligere licet, cujus resolutio maxime ardua est putanda.

Scholion 2.

266. Cum igitur trium formularum differentialium secundi gradus $\left(\frac{\partial\partial z}{\partial x^2}\right)$, $\left(\frac{\partial\partial z}{\partial x\partial y}\right)$, $\left(\frac{\partial\partial z}{\partial y^2}\right)$ primam ac tertiam hactenus sim contemplatus, quatenus earum per reliquas quantitates, determinatio resolutionem admittit methodo quidem hic adhibita: superest ut formulam quoque secundam $\left(\frac{\partial\partial z}{\partial x\partial y}\right)$ consideremus, et quibusnam determinationibus per reliquas quantitates x, y, z, $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$ solutio absolvi queat, investigemus, in quo negotio a casibus simplicissimis exordiri conveniet.

Problema 44.

267. Si z ejusmodi debeat esse functio binarum variabilium x et y, ut fiat $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = P$, existente P functione quacunque data ipsarum x et y, indolem functionis z generaliter determinare.

Solutio.

Ponatur $\left(\frac{\partial z}{\partial x}\right) = v$, eritque $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \left(\frac{\partial v}{\partial y}\right)$, ideoque habebitur $\left(\frac{\partial v}{\partial y}\right) = P$. Jam spectetur quantitas x ut constans, ita ut P solam variabilem y contineat, eritque $\partial v = P\partial y$, unde in hypothesi quantitatis x constantis integrando prodit

$$v = \left(\frac{\partial z}{\partial x}\right) = \int P\partial y + f' : x,$$

ubi $\int P\partial y$ erit functio data ipsarum x et y. Nunc porro spectetur x ut variabilis, y vero ut constans, ut adipiscamur hanc aequationem differentialem

$$\partial z = \partial x \int P\partial y + \partial x f' : x,$$

quae integrata dat

$$z = \int \partial x \int P\partial y + f : x + F : y,$$

ubi cum habeantur duae functiones arbitrariae, id indicio est, hoc integrale esse completum.

Corollarium 1.

268. Si ordine inverso primum y tum vero x constans posuissemus, invenissemus

$$\left(\frac{\partial z}{\partial y}\right) = \int P\partial x + f' : y, \text{ et } z = \int \partial y \int P\partial x + f : y + F : x,$$

qui valor aeque satisfacit ac praecedens.

Corollarium 2.

269. Patet ergo vel fore

$$\int \partial x \int P\partial y = \int \partial y \int P\partial x,$$

vel differentiam saltem exprimi per aggregatum ex functione ipsius x et functione ipsius y. Quod etiam inde patet quod posito

$$\int \partial x \int P \partial y = \int \partial y \int P \partial x = V,$$

fiat utrinque $P = \left(\frac{\partial \partial V}{\partial x \partial y}\right)$.

Corollarium 3.

270. Si sit $P = 0$, seu debeat esse $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = 0$, reperitur pro indole functionis z haec forma

$$z = f : x + F : y.$$

Scholion.

271. Hic casus in doctrina solidorum frequenter occurrit, si enim natura superficiei exprimatur aequatione inter ternas coordinatas x, y et u, erit soliditas $= \int \partial x \int u \partial y$, quare si soliditas exprimatur per z, erit $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = u$, ordinatae scilicet ad binas x et y normali. Tum vero si ponatur

$$\partial u = p \partial x + q \partial y,$$

superficies hujus solidi erit

$$= \int \partial x \int \partial y \sqrt{(1 + pp + qq)},$$

quae superficies si exprimatur littera z, erit

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \sqrt{(1 + pp + qq)}.$$

Quando ergo in nostro problemate ejusmodi functio z ipsarum x et y quaeritur, ut sit $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = P$, idem est ac si quaeratur soliditas respondens superficiei, cujus natura aequatione inter ternas coordinatas x, y et P exprimitur. Exemplis igitur aliquot hunc calculum illustremus.

Exemplum 1.

272. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \alpha x + \beta y$.

Cum hic sit $P = \alpha x + \beta y$, erit

$$\int P \partial y = \alpha x y + \tfrac{1}{2}\beta y y \text{ et}$$

$$\int \partial x \int P \partial y = \tfrac{1}{2}\alpha x x y + \tfrac{1}{2}\beta x y y = \tfrac{1}{2} x y (\alpha x + \beta y),$$

unde functio quaesita z ita exprimitur, ut sit

$$z = \tfrac{1}{2} x y (\alpha x + \beta y) + f : x + F : y.$$

Exemplum 2.

273. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial y}{\partial x \partial y}\right) = \sqrt{(a a - y y)}$.

Hic est $P = \sqrt{(a a - y y)}$, ergo

$$\int P \partial x = x \sqrt{(a a - y y)},$$

ubi quia perinde est, a variabilitate ipsius x incipio. Hinc igitur fit

$$\int \partial y \int P \partial x = x \int \partial y \sqrt{(a a - y y)}$$

$$= \tfrac{1}{2} x y \sqrt{(a a - y y)} + \tfrac{1}{2} a a x \int \frac{\partial y}{\sqrt{(a a - y y)}},$$

ex quo integrale completum erit

$$z = \tfrac{1}{2} x y \sqrt{(a a - y y)} + \tfrac{1}{2} a a x \text{ Ang. sin. } \frac{y}{a} + f : x + F : y.$$

Exemplum 3.

274. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \frac{a}{\sqrt{(a a - x x - y y)}}$.

Ob $P = \frac{a}{\sqrt{(a a - x x - y y)}}$, erit

$$\int P \partial y = a \text{ Ang. sin. } \frac{y}{\sqrt{(a a - x x)}}, \text{ hinc}$$

$$\int \partial x \int P \partial y = a \int \partial x \text{ Ang. sin. } \frac{y}{\sqrt{(a a - x x)}}.$$

Ponatur brevitatis gratia

$$\text{Ang. sin.} \frac{y}{\sqrt{(aa-xx)}} = \Phi, \text{ erit}$$

$$\int \partial x \int P \partial y = a \int \Phi \partial x = a x \Phi - a \int x \partial x \left(\frac{\partial \Phi}{\partial x}\right),$$

in hac enim integratione y pro constante habetur. Quare ob $\frac{y}{\sqrt{(aa-xx)}} = \sin. \Phi$, erit

$$\frac{yx}{(aa-xx)^{\frac{3}{2}}} = \left(\frac{\partial \Phi}{\partial x}\right) \cos. \Phi.$$

At vero est

$$\cos. \Phi = \frac{\sqrt{(aa-xx-yy)}}{\sqrt{(aa-xx)}}, \text{ hincque}$$

$$\left(\frac{\partial \Phi}{\partial x}\right) = \frac{yx}{(aa-xx)\sqrt{(aa-xx-yy)}}, \text{ et}$$

$$\int x \partial x \left(\frac{\partial \Phi}{\partial x}\right) = y \int \frac{xx \partial x}{(aa-xx)\sqrt{(aa-xx-yy)}};$$

quo integrali invento, erit

$$z = ax \text{Ang. sin.} \frac{y}{\sqrt{(aa-xx)}} - ay \int \frac{xx \partial x}{(aa-xx)\sqrt{(aa-xx-yy)}} + f:x + F:y,$$

quae forma per integrationem evoluta reducitur ad hanc

$$z = ax \text{ Ang. sin.} \frac{y}{\sqrt{(aa-xx)}} + ay \text{ Ang. sin.} \frac{x}{\sqrt{(aa-yy)}}$$
$$- aa \text{ Ang. sin.} \frac{xy}{\sqrt{(aa-xx)(aa-yy)}} + f:x + F:y.$$

Formulae enim $\int \frac{aa \partial x}{(aa-xx)\sqrt{(aa-xx-yy)}}$ integrale ita facillime elicitur. Ponatur $\frac{x}{\sqrt{(aa-xx-yy)}} = p$, erit $xx = \frac{pp(aa-yy)}{1+pp}$, et ob y constans per logarithmos differentiando

$$\frac{\partial x}{x} = \frac{\partial p}{p} - \frac{p \partial p}{1+pp} = \frac{\partial p}{p(1+pp)},$$

tum per illam formulam multiplicando

$$\frac{\partial x}{\sqrt{(aa-xx-yy)}} = \frac{\partial p}{1+pp}.$$

Porro est

$$aa - xx = \frac{aa + ppyy}{1+pp},$$

unde formula integralis fit

$$\int \frac{aa\partial x}{(aa-xx)\sqrt{(aa-xx-yy)}} = \int \frac{aa\partial p}{aa+ppyy} = \frac{aa}{yy}\int \frac{\partial p}{\frac{aa}{yy}+pp}$$

$$= \frac{a}{y}\text{ Ang. tang. }\frac{py}{a} = \frac{a}{y}\text{ Ang. tang. }\frac{xy}{a\sqrt{(aa-xx-yy)}}$$

$$= \frac{a}{y}\text{ Ang. sin. }\frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

Problema 45.

275. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut sit

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = P\left(\frac{\partial z}{\partial x}\right) + Q,$$

existentibus P et Q functionibus quibuscunque ipsarum x et y, investigare indolem functionis z.

Solutio.

Ponatur $\left(\frac{\partial z}{\partial x}\right) = v$, ut oriatur ista aequatio

$$\left(\frac{\partial v}{\partial y}\right) = Pv + Q,$$

quae continet quantitates x, y et v; statuatur ergo x constans, eritque

$$\partial v = Pv\partial y + Q\partial y,$$

quae per $e^{-\int P\partial y}$ multiplicata praebet

$$e^{-\int P\partial y}v = \int e^{-\int P\partial y}Q\partial y + f':x,$$

ideoque

$$v = \left(\frac{\partial z}{\partial x}\right) = e^{\int P\partial y}\int e^{-\int P\partial y}Q\partial y + e^{\int P\partial y}f':x.$$

Nunc cum haec integralia determinate contineant x et y, spectetur y ut constans, et sequens integratio praebet

$$z = \int e^{\int P \partial y} \partial x \int e^{-\int P \partial y} Q \partial y + \int e^{\int P \partial y} \partial x f' : x + F : y,$$

quae integralia quovis casu evoluta fiunt manifesta.

Corollarium 1.

276. Ad hoc ergo problema resolvendum, per integrationem primo quaeratur R, ut sit $\int P \partial y = l R$; deinde quaeratur S, ut sit $\int \frac{Q \partial y}{R} = S$. Denique sit $\int R S \partial x = T$; ita ut in illis sola quantitas y, hic vero sola x pro variabili habeatur. Quo facto erit nostrum integrale completum

$$z = T + \int R \partial x f' : x + F : y.$$

Corollarium 2.

277. Hic ergo functio arbitraria $f : x$ in formula integrali est involuta, quae tamen si per applicatam curvae cujuscunque respondentem abscissae x exhibeatur, hoc integrale $\int R \partial x f' : x$ pro quovis valore ipsius y seorsim construi poterit, siquidem in hac integratione quantitas y ut constans spectatur.

Scholion.

278. Eodem plane modo resolvitur permutandis variabilibus x et y hoc problema, quo functio z quaeritur, ut sit

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = P \left(\frac{\partial z}{\partial y}\right) + Q,$$

dummodo P et Q sint functiones ipsarum x et y tantum, ipsam functionem z non implicantes. Solutio enim ita se habebit

$$z = \int e^{\int P \partial x} \partial y \int e^{-\int P \partial x} Q \partial x + \int e^{\int P \partial x} \partial y f' : y + F : x.$$

Quin etiam utrinque problema latius extendi potest, ac prius resolutionem admittet, si formula $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ aequetur functioni cuicunque trium quantitatum x, y et $\left(\frac{\partial z}{\partial x}\right)$, posterius vero si $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ aequetur functioni cuicunque harum trium quantitatum x, y et $(\frac{\partial z}{\partial x})$, utroque enim casu res reducitur ad aequationem differentialem primi gradus. Neque vero haec solvendi methodus succedit, si utraque formula primi gradus $(\frac{\partial z}{\partial x})$ et $(\frac{\partial z}{\partial y})$ simul ingrediatur, vel si functiones P et Q etiam ipsam quantitatem z complectantur.

Exemplum 1.

279. *Quaeratur binarum variabilium x et y functio z, ut sit* $(\frac{\partial \partial z}{\partial x \partial y}) = \frac{n}{y}(\frac{\partial z}{\partial x}) + \frac{m}{x}$.

Sit $(\frac{\partial z}{\partial x}) = v$, erit

$$(\frac{\partial v}{\partial y}) = \frac{nv}{y} + \frac{m}{x},$$

et spectata x ut constante, erit

$$\partial v = \frac{nv\partial y}{y} + \frac{m\partial y}{x},$$

unde per y^n dividendo prodit

$$\frac{v}{y^n} = \frac{m}{x}\int\frac{\partial y}{y^n} = \frac{-m}{(n-1)\,x\,y^{n-1}} + f':x,$$

ita ut sit

$$v = (\frac{\partial z}{\partial x}) = \frac{-my}{(n-1)x} + y^n f':x:$$

sumatur jam y constans, et denuo integrando obtinetur

$$z = \frac{-m}{n-1}\,y\,l\,x + y^n\,f:x + F:y.$$

Exemplum 2.

280. *Quaeratur binarum x et y functio z, ut sit*

$$(\frac{\partial \partial z}{\partial x \partial y}) = \frac{y}{xx+yy}(\frac{\partial z}{\partial x}) + \frac{a}{xx+yy}.$$

Posito $(\frac{\partial z}{\partial x}) = v$ et sumto x constante, erit

$$\partial v = \frac{vy\partial y}{xx+yy} + \frac{a\partial y}{xx+yy},$$

quae aequatio per $\sqrt{(xx+yy)}$ divisa dat

$$\frac{v}{\sqrt{(xx+yy)}} = a\int\frac{\partial y}{(xx+yy)^{\frac{3}{2}}} = \frac{ay}{xx\sqrt{(xx+yy)}} + f:x.$$

Ergo

$$v = \left(\frac{\partial z}{\partial x}\right) = \frac{ay}{xx} + \sqrt{(xx+yy)}\,.\,f:x,$$

sit jam y constans, reperieturque

$$z = -\frac{ay}{x} + \int f:x \times \partial x\sqrt{(xx+yy)} + F:y,$$

ubi quidem integrale

$$\int f:x \times \partial x\sqrt{(xx+yy)},$$

ob functionem indeterminatam $f:x$, etsi y constans ponitur, in genere exprimi nequit, ita ut explicate per y et functiones ipsius x exhiberi possit.

Scholion.

281. Formula ergo secundi gradus $\left(\frac{\partial\partial z}{\partial x\partial y}\right)$ non tam largam casuum resolubilium copiam admittit, quam binae reliquae $\left(\frac{\partial\partial z}{\partial x^2}\right)$ et $\left(\frac{\partial\partial z}{\partial y^2}\right)$, cum in his solutio succedat, etiamsi ipsa quantitas z quoque in earum determinationem ingrediatur, quod hic secus evenit, cum methodus non pateat hujusmodi aequationem $\left(\frac{\partial\partial z}{\partial x\partial y}\right) = P\left(\frac{\partial z}{\partial x}\right) + Q$, quando litterae P et Q quantitatem z continent resolvendi; neque etiam solutio locum habet, quando praeter formulam primi gradus $\left(\frac{\partial z}{\partial x}\right)$ simul quoque altera $\left(\frac{\partial z}{\partial y}\right)$ adest. Interim tamen dantur casus, quibus solutiones particulares exhiberi possunt, eaeque adeo infinitae, quae junctim sumtae solutioni generali aequivalere videntur,

etiamsi in applicatione ad usum practicum parum subsidii plerumque afferant, formas tamen hujusmodi solutionum notasse juvabit.

Problema 46.

282. Si z ejusmodi debeat esse functio binarum variabilium x et y, ut fiat $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = a z$, indolem hujus functionis z particulariter saltem investigare.

Solutio.

Cum quantitas z unam ubique teneat dimensionem evidens est, si statuatur $z = e^p q$, quantitatem exponentialem e^p ex calculo evanescere. Ponamus igitur $z = e^{\alpha x} Y$, ita ut Y functionem ipsius y tantum contineat, eritque

$$\left(\frac{\partial z}{\partial x}\right) = \alpha e^{\alpha x} Y \text{ et } \left(\frac{\partial\partial z}{\partial x \partial y}\right) = \alpha e^{\alpha x} \frac{\partial Y}{\partial y} = a e^{\alpha x} Y,$$

unde fit

$$\frac{\alpha \partial Y}{Y} = a \partial y \text{ et } Y = e^{\frac{a y}{\alpha}},$$

sicque jam solutionem particularem habemus

$$z = A e^{\alpha x + \frac{a y}{\alpha}};$$

quae autem satis late patet, cum tam A quam α pro lubitu assumi possit. Plures autem valores ipsius x seorsim satisfacientes, etiam junctim sumti satisfaciunt, unde hujusmodi expressionem multo generaliorem deducimus

$$z = A e^{\alpha x + \frac{a}{\alpha} y} + B e^{\beta x + \frac{a}{\beta} y} + C e^{\gamma x + \frac{a}{\gamma} y} + D e^{\delta x + \frac{a}{\delta} y},$$

ubi cum A, B, C, etc. item α, β, γ, etc. omnes valores possibiles recipere queant, haec forma pro maxime universali est habenda, neque si ad amplitudinem spectamus, quicquam cedere videtur superioribus solutionibus, quae binas functiones arbitrarias involvunt, propterea quod hic duplicis generis coëfficientes arbitrarii occurrunt, interim tamen haud liquet, quomodo functiones discontinuae hac relatione repraesentari queant.

Corollarium 1.

283. Pro solutione ergo particulari invenienda, sumantur bini numeri m et n, ut eorum productum sit $mn = a$, eritque $z = A e^{mx+ny}$. Atque etiam ex iisdem numeris permutatis erit $z = A e^{nx+my}$.

Corollarium 2.

284. Ex tali numerorum m et n pari, ut sit $mn = -a$, solutiones quoque per sinus et cosinus angulorum exhiberi possunt; erit enim

$$z = B \sin.(mx - ny), \text{ vel } z = B \cos.(mx - ny),$$

vel etiam permutando

$$z = B \sin.(nx - my), \text{ vel } z = B \cos.(nx - my).$$

Corollarium 3.

285. Cum igitur hujusmodi formulae innumerabiles exhiberi queant, singulae per constantes quascunque multiplicatae et in unam summam collectae dabunt solutionem generalem problematis.

Scholion.

286. Neque tamen haec solutio, etsi infinities infinitas determinationes recipit, ita est comparata, ut ejusmodi solutionibus, quae binas functiones arbitrarias involvunt, aequivalens aestimari possit; propterea quod non patet, quomodo singulas litteras assumi opor-

teat, ut pro dato casu, verbi gratia $y = 0$, quantitas z vel $\left(\frac{\partial z}{\partial x}\right)$ seu $\left(\frac{\partial z}{\partial y}\right)$ data functioni ipsius x aequalis evadat, cujuscunque etiam indolis fuerit haec functio. Semper autem solutio generalis duplicis hujusmodi determinationis capax esse debet. Quando autem talem solutionem impetrare non licet, utique ejusmodi solutionibus, uti hic invenimus, contenti esse debemus. Ac tales quidem solutiones simili modo obtinere possumus, si proponatur ejusmodi aequatio:

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + Rz = 0,$$

si modo litterae P, Q, R denotent functiones ipsius x tantum. Posito enim $z = e^{\alpha y} X$, ut X sit functio solius x, ob

$$\left(\frac{\partial z}{\partial x}\right) = e^{\alpha y} \frac{\partial X}{\partial x}, \quad \left(\frac{\partial z}{\partial y}\right) = \alpha e^{\alpha y} X, \text{ et ob } \left(\frac{\partial\partial z}{\partial x \partial y}\right) = \alpha e^{\alpha y} \left(\frac{\partial X}{\partial x}\right),$$

erit

$$\frac{\alpha \partial X}{\partial x} + \frac{P \partial X}{\partial x} + \alpha Q X + R X = 0,$$

unde reperitur

$$\frac{\partial X}{X} = \frac{-\partial x (\alpha Q + R)}{\alpha + P};$$

sicque elicitur pro quovis nnmero α idoneus valor ipsius X. Quare sumendis infinitis numeris α, hoc modo expressio infinities infinitas determinationes recipiens colligitur

$$z = A e^{\alpha y} X + B e^{\beta y} X' + C e^{\gamma y} X'' + \text{etc.}$$

Verumtamen dantur etiam casus ejusmodi aequationum, quae solutiones vere completas admittunt, quarum rationem in sequente problemate indagemus.

Problema 47.

287. Proposita aequatione resolvenda

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

investigare cujusmodi functiones ipsarum x et y esse debeant quantitates P, Q, R et S, ut haec aequatio solutionem vere completam admittat.

Solutio.

Sit V functio quaecunque ipsarum x et y, ac ponatur $z = e^V v$, ita ut jam v sit quantitas incognita, cujus valorem investigari oporteat. Cum igitur sit

$$\left(\frac{\partial z}{\partial x}\right) = e^V \left[\left(\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial V}{\partial x}\right)\right], \quad \left(\frac{\partial z}{\partial y}\right) = e^V \left[\left(\frac{\partial v}{\partial y}\right) + v\left(\frac{\partial V}{\partial y}\right)\right],$$

facta substitutione totaque aequatione per e^V divisa prodibit sequens aequatio

$$\left.\begin{aligned} e^{-V} S + \left(\frac{\partial\partial v}{\partial x\partial y}\right) + \left(\frac{\partial V}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) v \\ + P\left(\frac{\partial v}{\partial x}\right) + Q\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial\partial V}{\partial x\partial y}\right) v \\ + P\left(\frac{\partial V}{\partial x}\right) v \\ + Q\left(\frac{\partial V}{\partial y}\right) v \\ + R\, v \end{aligned}\right\} = 0.$$

Efficiendum jam est, ut haec aequatio resolutionem completam admittat. Cum igitur ante viderimus, talem aequationem

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) + T\left(\frac{\partial v}{\partial x}\right) + e^{-V} S = 0$$

generaliter resolvi posse, qualescunque etiam functiones ipsarum x et y pro S, T et V accipiantur, ad hanc aequationem illam redigamus. Necesse igitur est statui

$$P + \left(\frac{\partial V}{\partial y}\right) = T, \quad Q + \left(\frac{\partial V}{\partial x}\right) = 0 \quad \text{et}$$

$$R + Q\left(\frac{\partial V}{\partial y}\right) + P\left(\frac{\partial V}{\partial x}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) + \left(\frac{\partial\partial V}{\partial x\partial y}\right) = 0,$$

unde obtinemus

$$P = T - \left(\frac{\partial V}{\partial y}\right), \quad Q = -\left(\frac{\partial V}{\partial x}\right) \quad \text{et}$$

$$R = \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) - T\left(\frac{\partial V}{\partial x}\right) - \left(\frac{\partial\partial V}{\partial x\partial y}\right).$$

Cum igitur per §. 275. reperietur

$$v = -\int e^{-\int T\partial y}\,\partial y \int e^{\int T\partial y - V} S\partial y + \int e^{-\int T\partial y}\,\partial x\, f : x + F : y,$$

erit aequationis propositae

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

si modo litterae P, Q, R assignatos teneant valores, integrale completum

$$z = -e^{V}\int e^{-\int T\partial y}\partial x \int e^{\int T\partial y - V} S\partial y + e^{V}\int e^{-\int T\partial y}\partial x\, f:x + e^{V} F:y,$$

quandoquidem hic formae $f:x$ et $F:y$ functiones quascunque ipsius x et y denotant.

Corollarium 1.

288. Quaecunque ergo functiones ipsarum x et y pro litteris T et V accipiantur, inde oriuntur valores idonei pro litteris P, Q, R assumendi, ut aequatio resolutionem completam admittat, functio autem S arbitrio nostro relinquitur.

Corollarium 2.

289. Possunt etiam in aequatione proposita functiones P et Q indefinitae relinqui, eritque tum

$$V = -\int Q\partial x \text{ et } \left(\frac{\partial V}{\partial y}\right) = -\int \partial x \left(\frac{\partial Q}{\partial y}\right), \text{ atque}$$

$$\left(\frac{\partial\partial V}{\partial x\partial y}\right) = -\left(\frac{\partial Q}{\partial y}\right);$$

unde tantum quantitas R ita determinari debet, ut sit

$$R - PQ - \left(\frac{\partial Q}{\partial y}\right) = 0, \text{ seu}$$

$$R = PQ + \left(\frac{\partial Q}{\partial y}\right).$$

Corollarium 3.

290. Quia hic pro $\int Q\partial x$ scribi potest $\int Q\partial x + Y$, denotante Y functionem quamcunque ipsius y, ob

$$V = -\int Q\partial x - Y,$$

complete integrabilis erit haec aequatio:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0,$$

cujus integrale est

$$z = e^{-\int Q\partial x - Y}\, v, \text{ existente}$$

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + \left[P - \int \partial x \left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}\right] \left(\frac{\partial v}{\partial x}\right) + e^{-V} S = 0,$$

ubi

$$T = P - \int \partial x \left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y},$$

ac propterea

$$\int T \partial y = \int P \partial y - \int Q \partial x - Y,$$

unde valor ipsius v facile definitur.

Scholion.

291. In hoc calculo, quo differentialia formularum integralium capi oportet, dum alia quantitas variabilis assumitur, atque in integratione supponitur, haec regula est tenenda, quod si fuerit $V = \int Q \partial x$, fore $\left(\frac{\partial V}{\partial y}\right) = \int \partial x \left(\frac{\partial Q}{\partial y}\right)$. Cum enim sit $\left(\frac{\partial V}{\partial x}\right) = Q$, erit $\left(\frac{\partial\partial V}{\partial x \partial y}\right) = \left(\frac{\partial Q}{\partial y}\right)$. Quodsi ergo statuatur $\left(\frac{\partial V}{\partial y}\right) = S$, erit $\left(\frac{\partial S}{\partial x}\right) = \left(\frac{\partial Q}{\partial y}\right)$, et

$$S = \left(\frac{\partial V}{\partial y}\right) = \int \partial x \left(\frac{\partial Q}{\partial y}\right);$$

unde vicissim colligitur, si fuerit $S = \int \partial x \left(\frac{\partial Q}{\partial y}\right)$, fore ob $\int S \partial y = V$, integrando $\int S \partial y = \int Q \partial x$; quod cum ex principiis ante stabilitis per se sit manifestum, non opus esse judico, pro hoc quasi novo algorithmi genere praecepta seorsim tradere. Videamus autem in aliquot exemplis, cujusmodi aequationes ope hujus methodi complete resolvere liceat.

Exemplum 1.

292. *Proposita aequatione differentio-differentiali*

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + a \left(\frac{\partial z}{\partial x}\right) + b \left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

definire indolem functionis R, ut haec aequatio resolutionem admittat, existente S functione quacunque ipsarum x et y.

Cum sit $P = a$ et $Q = b$, erit $R = ab$ et $V = -bx$; tuto enim functio Y omitti potest, quia in sequente integratione jam binae functiones arbitrariae introducuntur, erit ergo $T = a$; unde posito $z = e^{-bx} v$, habebitur haec aequatio:

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + a\left(\frac{\partial v}{\partial x}\right) + e^{bx} S = 0,$$

ac posito $\left(\frac{\partial v}{\partial x}\right) = u$, fit

$$\left(\frac{\partial u}{\partial y}\right) + au + e^{bx} S = u,$$

et sumto x constante

$$e^{ay} u = -\int e^{ay+bx} S\partial y + f' : y, \text{ ergo}$$

$$u = \left(\frac{\partial v}{\partial x}\right) = -e^{-ay}\int e^{ay+bx} S\partial y + e^{-ay} f' : x,$$

et sumto jam y constante

$$v = -e^{-ay}\int\partial x\int e^{ay+bx} S\partial y + e^{-ay} f : x + F : y,$$

sumendo

$$\int\partial x f' : x = f : x.$$

Quod si jam pro $e^{-bx} f : x$ scribatur $f : x$, erit

$$z = -e^{-ay-bx}\int\partial x\int e^{ay+bx} S\partial y + e^{-ay} f : x + e^{-bx} F : y.$$

Aliter.

Si sumsissemus $V = -bx - ay$, prodiisset $T = a - a = 0$; ideoque posito $z = e^{-bx-ay} v$, quantitas v ex hac aequatione:

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + e^{bx+ay} S = 0$$

definiri deberet, quae dat

$$\left(\frac{\partial v}{\partial x}\right) = -\int e^{bx+ay} S\partial y + f' : x, \text{ et}$$

$$v = -\int\partial x\int e^{bx+ay} S\partial y + f : x + F : y, \text{ et}$$

$$z = e^{-bx-ay}\left(-\int\partial x\int e^{bx+ay} S\partial y + f : x + F : y\right),$$

quae forma simplicior est praecedente, etiamsi eodem redeat, estque hoc integrale completum aequationis

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + a\left(\frac{\partial z}{\partial x}\right) + b\left(\frac{\partial z}{\partial y}\right) + abz + S = 0.$$

Exemplum 2.

293. *Proposita aequatione differentio-differentiali*

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{a}{y}\left(\frac{\partial z}{\partial x}\right) + \frac{b}{x}\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

definiri indolem functionis R, ut haec aequatio resolutionem admittat, existente S functione quacunque ipsarum x et y.

Cum sit $P = \frac{a}{y}$ et $Q = \frac{b}{x}$, erit $V = -blx - Y$, hincque $R = \frac{ab}{xy}$, et aequatio integrabilis erit

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{a}{y}\left(\frac{\partial z}{\partial x}\right) + \frac{b}{x}\left(\frac{\partial z}{\partial y}\right) + \frac{ab}{xy}z + S = 0.$$

Quoniam igitur fit

$$T = P + \left(\frac{\partial V}{\partial y}\right) = \frac{a}{y} - \frac{\partial Y}{\partial y},$$

sumamus $Y = +aly$, ut fiat $T = 0$, ac posito:

$$z = e^{-blx - aly}v = x^{-b}y^{-a}v,$$

quantitas v ex hac aequatione definiri debet

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) + x^b y^a S = 0,$$

unde fit

$$\left(\frac{\partial v}{\partial x}\right) = -x^b\int y^a S\partial y + f' : x \quad \text{et}$$

$$v = -\int x^b\partial x\int y^a S\partial y + f : x + F : y,$$

ideoque

$$z = \frac{-\int x^b\partial x\int y^a S\partial y + f : x + F : y}{x^b y^a}.$$

Scholion 1.

294. Hinc igitur patet ope istius methodi in genere integrari posse hanc aequationem:

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0,$$

quaecunque functiones ipsarum x et y pro P, Q et S accipiantur. Ac resolutio quidem ita se habet, ut posito

$$z = e^{-\int Q\partial x - Y} v,$$

haec quantitas v determinetur hac aequatione:

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + \left[P - \int \partial x \left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}\right]\left(\frac{\partial v}{\partial x}\right) + e^{\int Q\partial x + Y} S = 0,$$

ubi jam pro Y talis functio ipsius y accipi potest, ut hujus aequationis forma simplicissima evadat; id quod potissimum evenit, si expressio

$$P - \int \partial x \left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}$$

ad nihilum redigi queat. In genere autem reperitur

$$v = -\int e^{-\int P\partial y + \int Q\partial x + Y} \partial x \int e^{\int P\partial y} S\partial y$$
$$+ \int e^{-\int P\partial y + \int Q\partial x + Y} \partial x \, f:x + F:y,$$

qui valor ergo per $e^{-\int Q\partial x - Y}$ multiplicatus praebet formam functionis z. Hoc modo autem functio Y ab arbitrio nostro pendens penitus e calculo egreditur, fitque

$$z = -e^{-\int Q\partial x} \int e^{-\int P\partial y + \int Q\partial x} \partial x \int e^{\int P\partial y} S\partial y$$
$$+ e^{-\int Q\partial x} \int e^{-\int P\partial y + \int Q\partial x} \partial x \, f:x + e^{-\int Q\partial x} F:y,$$

quod est integrale completum hujus aequationis:

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0.$$

Scholion 2.

295. Permutandis autem variabilibus x et y etiam haec aequatio complete integrari potest:

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial P}{\partial x}\right)\right] z + S = 0,$$

cujus integrale erit

$$z = - e^{-\int P\partial y} \int e^{-\int Q\partial x + \int P\partial y} \partial y \int e^{\int Q\partial x} S\partial x$$
$$+ e^{-\int P\partial y} \int e^{-\int Q\partial x + \int P\partial y} \partial y \, f:y + e^{-\int P\partial y} F:x,$$

ubi praecipue hic casus in utraque forma contentus notari meretur, si fuerit $P = Y$ et $Q = X$, existente X functione ipsius x et Y ipsius y tantum; tum enim hujus aequationis

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + Y\left(\frac{\partial z}{\partial x}\right) + X\left(\frac{\partial z}{\partial y}\right) + XYz + S = 0,$$

integrale completum erit

$$z = - e^{-\int X\partial x - \int Y\partial y} \int e^{\int X\partial x} \partial x \int e^{\int Y\partial y} S\partial y$$
$$+ e^{-\int X\partial x - \int Y\partial y} (f:x + F:y),$$

quod etiam ita exhiberi potest:

$$e^{\int X\partial x + \int Y\partial y} z = f:x + F:y - \int e^{\int X\partial x} \partial x \int e^{\int Y\partial y} S\partial y,$$

vel etiam hoc modo:

$$e^{\int X\partial x + \int Y\partial y} z = f:x + F:y - \int e^{\int Y\partial y} \partial y \int e^{\int X\partial x} S\partial x.$$

CAPUT III.

SI DUAE VEL OMNES FORMULAE SECUNDI GRADUS PER RELIQUAS QUANTITATES DETERMINANTUR.

Problema 48.

296.

Si z ejusmodi debeat esse functio ipsarum x et y, ut fiat

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = aa\left(\frac{\partial\partial z}{\partial x^2}\right),$$

indolem functionis z determinare.

Solutio.

Introducantur binae novae variabiles t et u, ut sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, atque ex §. 231. omnes formulae differentiales sequentes mutationes subibunt:

$$\left(\frac{\partial z}{\partial x}\right) = \alpha\left(\frac{\partial z}{\partial t}\right) = \gamma\left(\frac{\partial z}{\partial u}\right);\ \left(\frac{\partial z}{\partial y}\right) = \beta\left(\frac{\partial z}{\partial t}\right) + \delta\left(\frac{\partial z}{\partial u}\right),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \alpha\alpha\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\alpha\gamma\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \gamma\gamma\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \alpha\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \gamma\delta\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \beta\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\beta\delta\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \delta\delta\left(\frac{\partial\partial z}{\partial u^2}\right),$$

unde nostra aequatio transibit in hanc:

$$(\beta\beta - \alpha\alpha aa)\left(\frac{\partial\partial z}{\partial t^2}\right) + 2(\beta\delta - \alpha\gamma aa)\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ (\delta\delta - \gamma\gamma aa)\left(\frac{\partial\partial z}{\partial u^2}\right) = 0.$$

Ponatur ergo

$$\beta\beta = \alpha\alpha aa \text{ et } \delta\delta = \gamma\gamma aa, \text{ seu}$$
$$\alpha = 1,\ \gamma = 1,\ \beta = a \text{ et } \delta = -a,$$

ut binae formulae extremae evanescant, quod fit ponendo

$$t = x + ay \quad \text{et} \quad u = x - ay,$$

eritque

$$-2(aa + aa)\left(\frac{\partial\partial z}{\partial t \partial u}\right) = 0, \quad \text{seu} \quad \left(\frac{\partial\partial z}{\partial t \partial u}\right) = 0,$$

unde per §. 269. colligitur integrale completum

$$z = f : t + F : u,$$

ac pro t et u restitutis valoribus

$$z = f : (x + ay) + F : (x - ay),$$

quae forma manifesto satisfacit, cum sit

$$\left(\frac{\partial z}{\partial x}\right) = f' : (x + ay) + F' : (x - ay),$$

$$\left(\frac{\partial z}{\partial y}\right) = af' : (x + ay) - aF' : (x - ay),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = f'' : (x + ay) + F'' : (x - ay),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = aaf'' : (x + ay) + aaF'' : (x - ay).$$

Corollarium 1.

297. Valor igitur ipsius z aequatur aggregato duarum functionum arbitrariarum, alterius ipsius $x + ay$, alterius ipsius $x - ay$, atque ambae hae functiones ita ad arbitrium assumi possunt, ut etiam functiones discontinuas earum loco capere liceat.

Corollarium 2.

298. Pro lubitu ergo binae curvae quaecunque etiam libero manus tractu descriptae ad hunc usum adhiberi possunt. Scilicet si in una curva abscissa capiatur $= x + ay$, in altera vero abscissa $= x - ay$, summa applicatarum semper valorem idoneum pro functione z suppeditabit.

Scholion 1.

299. Hoc fere primum est problema, quod in hoc novo calculi genere solvendum occurrit; perduxerat autem solutio generalis problematis de cordis vibrantibus ad hanc ipsam aequationem, quam hic tractavimus. Celeb. *Alembertus*, qui hoc problema primus felici successu est aggressus, methodo singulari aequationem integravit; scilicet cum esse oporteat $\left(\frac{\partial\partial z}{\partial y^2}\right) = a^2 \left(\frac{\partial\partial z}{\partial x^2}\right)$, posito $\partial z = p\partial x + q\partial y$, indeque

$$\partial p = r\partial x + s\partial y \quad \text{et} \quad \partial q = s\partial x + t\partial y,$$

illa aequatio postulat ut sit $t = aar$. Consideratis porro istis aequationibus:

$$\left.\begin{array}{l}\partial p = r\partial x + s\partial y\\ \partial q = s\partial x + aar\partial y\end{array}\right| \begin{array}{l}\text{elicitur combinando}\\ a\partial p + \partial q = ar(\partial x + a\partial y) + s(a\partial y + \partial x),\end{array}$$

seu $a\partial p + \partial q = (ar + s)(\partial x + a\partial y)$,
unde patet $ar + s$ functioni ipsius $x + ay$ aequari debere, ex quo etiam $ap + q$ tali functioni aequatur. Atque quia a aeque negative ac positive accipi potest, habentur duae hujusmodi aequationes:

$$ap + q = 2af' : (x + ay) \quad \text{et} \quad q - ap = 2aF' : (x - ay),$$

unde colligitur

$$q = af' : (x + ay) + aF' : (x - ay), \quad \text{et}$$
$$p = f' : (x + ay) - F' : (x - ay),$$

hincque aequatio $\partial z = p\partial x + q\partial y$ sponte integratur, fitque

$$z = f : (x + ay) - F : (x - ay).$$

Hoc modo sagacissimus Vir integrale completum est adeptus, sed non animadvertit, loco functionum harum introductarum, non solum omnis generis functiones continuas, sed etiam omni continuitatis lege destitutas, accipi licere.

Scholion 2.

300. Cum plurimum intersit, in hoc novo calculi genere quam plurimas methodos persequi. ab aliis solutio nostrae aequationis ita est tentata, ut ponerent $\left(\frac{\partial z}{\partial y}\right) = k\left(\frac{\partial z}{\partial x}\right)$, unde fit primo $\left(\frac{\partial\partial z}{\partial x\partial y}\right) = k\left(\frac{\partial\partial z}{\partial x^2}\right)$, tum vero $\left(\frac{\partial\partial z}{\partial y^2}\right) = k\left(\frac{\partial\partial z}{\partial x\partial y}\right)$, ex quo colligitur $\left(\frac{\partial\partial z}{\partial y^2}\right) = kk\left(\frac{\partial\partial z}{\partial x^2}\right)$. Evidens ergo est pro nostro casu capi debere $kk = aa$, seu $k = \pm a$. Sit ergo $k = a$, et ob $\left(\frac{\partial z}{\partial y}\right) = a\left(\frac{\partial z}{\partial x}\right)$, fiet

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial x}\right)(\partial x + a\partial y),$$

hincque manifestum est fore $z = f:(x + ay)$, et ob a ambiguum, quoniam bini valores seorsim satisfacientes etiam juncti satisfacient, concluditur ipsa solutio inventa. Hoc etiam modo negotium confici potest: Statuatur

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = aa\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial v}{\partial x\partial y}\right), \text{ eritque}$$

$$\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial v}{\partial x}\right) \text{ et } aa\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial v}{\partial y}\right).$$

Inventis nunc formulis primi gradus $\left(\frac{\partial v}{\partial x}\right)$ et $\left(\frac{\partial v}{\partial y}\right)$, ob

$$\partial v = \partial x\left(\frac{\partial v}{\partial x}\right) + \partial y\left(\frac{\partial v}{\partial y}\right),$$

habebimus has aequationes:

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right) \text{ et}$$

$$\partial v = \partial x\left(\frac{\partial z}{\partial y}\right) + aa\partial y\left(\frac{\partial z}{\partial x}\right),$$

ex quarum combinatione colligimus

$$\partial v + a\partial z = (\partial x + a\partial y)\left[\left(\frac{\partial z}{\partial y}\right) + a\left(\frac{\partial z}{\partial x}\right)\right],$$

hincque

$$v + az = f:(x + ay) \text{ et } v - az = F:(x - ay),$$

sicque pro z eadem forma exsurgit. Methodus vero, quam in solutione sum secutus, ad naturam rei magis videtur accommodata, cum etiam in aliis problematibus magis complicatis insignem utilitatem afferat.

Scholion 3.

301. Nostra autem solutio hoc habet incommodi, quod pro hac aequatione $\left(\frac{\partial\partial z}{\partial y^2}\right) + aa\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$, ad expressionem imaginariam deducit, scilicet

$$z = \text{f}:(x + ay\sqrt{-1}) + \text{F}:(x - ay\sqrt{-1}).$$

Quoties autem functiones f et F sunt continuae, cujuscunque demum fuerint indolis, semper earum valores ad hanc formam $P \pm Q\sqrt{-1}$ reduci possunt, unde sequens forma, ex illa facile deducenda, semper valorem realem exhibebit:

$$z = \tfrac{1}{2}\text{f}:(x + ay\sqrt{-1}) + \tfrac{1}{2}\text{f}:(x - ay\sqrt{-1})$$
$$+ \frac{1}{2\sqrt{-1}}\text{F}:(x + ay\sqrt{-1}) - \frac{1}{2\sqrt{-1}}\text{F}:(x - ay\sqrt{-1})$$

pro cujus ad realitatem reductione notasse juvabit, posito

$x = s\cos.\Phi$ et $ay = s\sin.\Phi$, fore

$$(x \pm ay\sqrt{-1})^n = s^n(\cos.n\Phi \pm \sqrt{-1}\,.\sin.n\Phi).$$

Quare quoties functiones propositae per operationes analyticas sunt conflatae, hoc est, continuae, earum valores realiter per cosinus et sinus ipsius Φ exhiberi possunt. Quando autem functiones illae sunt discontinae, talis reductio neutiquam locum habet, etiamsi certum videatur, etiam tunc formam allatam valorem realem esse adepturam. Quis autem in curva quacunque, libero manus ductu descripta, applicatas abscissis $x + ay\sqrt{-1}$ et $x - ay\sqrt{-1}$ respondentes animo saltem imaginari, ac summam earum realem assignare valuerit, aut differentiam, quae per $\sqrt{-1}$ divisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet; atque ob hunc ipsum defectum hujusmodi solutiones universales plurimum de sua vi perdunt.

Problema 49.

302. Proposita aequatione $\left(\frac{\partial\partial z}{\partial y^2}\right) = PP\left(\frac{\partial\partial z}{\partial x^2}\right)$, inquirere, quales functiones ipsarum x et y pro P assumere liceat, ut integratio ope reductionis succedat.

Solutio.

Reductionem hanc ita fieri assumo, ut loco x et y binae aliae variabiles t et u introducantur, qua substitutione secundum §. 231. in genere facta prodit haec aequatio:

$$\left.\begin{aligned}&+\left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right)+\left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)+2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right)\\&-PP\left(\frac{\partial\partial t}{\partial x^2}\right)-PP\left(\frac{\partial\partial u}{\partial x^2}\right)-PP\left(\frac{\partial t}{\partial x}\right)^2-2PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)-PP\left(\frac{\partial u}{\partial x}\right)^2\end{aligned}\right\} = 0.$$

Jam relatio inter binas variabiles t, u et praecedentes x, y ejusmodi statuatur, ut binae formulae $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ ex calculo egrediantur, id quod fiet ponendo

$$\left(\frac{\partial t}{\partial y}\right)+P\left(\frac{\partial t}{\partial x}\right)=0 \quad\text{et}\quad \left(\frac{\partial u}{\partial y}\right)-P\left(\frac{\partial u}{\partial x}\right)=0.$$

Tum autem erit

$$\left(\frac{\partial\partial t}{\partial y^2}\right)=-P\left(\frac{\partial\partial t}{\partial x\partial y}\right)-\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial t}{\partial x}\right);$$

at cum sit indidem

$$\left(\frac{\partial\partial t}{\partial x\partial y}\right)=-P\left(\frac{\partial\partial t}{\partial x^2}\right)-\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right),\quad\text{erit}$$

$$\left(\frac{\partial\partial t}{\partial y^2}\right)=PP\left(\frac{\partial\partial t}{\partial x^2}\right)+P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right)-\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial t}{\partial x}\right),$$

similique modo sumendo P negative

$$\left(\frac{\partial\partial u}{\partial y^2}\right)=PP\left(\frac{\partial\partial u}{\partial x^2}\right)+P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right).$$

His substitutis nostra aequatio hanc induet formam:

$$\left[P\left(\frac{\partial P}{\partial x}\right)-\left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\left[P\left(\frac{\partial P}{\partial x}\right)+\left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right)$$
$$-4PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right)=0,$$

quae cum unicam formulam secundi gradus $\left(\frac{\partial\partial z}{\partial t\partial u}\right)$ contineat, integrationem admittit, si vel $(\frac{\partial z}{\partial t})$ vel $(\frac{\partial z}{\partial u})$ e calculo excesserit. Ponamus ergo insuper

$$P(\frac{\partial P}{\partial x}) - (\frac{\partial P}{\partial y}) = 0,$$

qua aequatione indoles quaesitae functionis P definitur; quo facto aequatio integranda, per $2P(\frac{\partial u}{\partial x})$ divisa, erit

$$(\frac{\partial P}{\partial x})(\frac{\partial z}{\partial u}) - 2P(\frac{\partial t}{\partial x})(\frac{\partial \partial z}{\partial t \partial u}) = 0,$$

cujus integrale, posito $(\frac{\partial z}{\partial u}) = v$, fit

$$2 lv = \int \frac{\partial t (\frac{\partial P}{\partial x})}{P(\frac{\partial t}{\partial x})} = l(\frac{\partial z}{\partial u}).$$

Verum prius ipsam functionem P per x et y definiri oportet. Cum igitur sit $(\frac{\partial P}{\partial y}) = P(\frac{\partial P}{\partial x})$, erit

$$\partial P = \partial x (\frac{\partial P}{\partial x}) + P\partial y (\frac{\partial P}{\partial x}),$$

hineque ponendo brevitatis ergo $(\frac{\partial P}{\partial x}) = p$, fit

$$\partial x = \frac{\partial P}{p} - P\partial y, \text{ atque}$$

$$x = -Py + \int \partial P (y + \frac{1}{p}).$$

Statuatur ergo $y + \frac{1}{p} = f' : P$, ac reperitur

$$x + Py = f : P \text{ et } p = (\frac{\partial P}{\partial x}) = \frac{1}{f' : P - y},$$

ac $(\frac{\partial P}{\partial y}) = \frac{P}{f' : P - y}$, unde ratio determinationis quantitatis P per x et y definitur. Pro novis autem variabilibus t et u, ob

$$(\frac{\partial t}{\partial y}) = -P(\frac{\partial t}{\partial x}), \text{ erit}$$

$$\partial t = (\frac{\partial t}{\partial x})(\partial x - P\partial y)$$

et ob $x = -Py + f:P$, fit

$$\partial t = (\frac{\partial t}{\partial x})(\partial P f' : P - 2P\partial y - y\partial P)$$

$$= P^{\frac{1}{2}}(\frac{\partial t}{\partial x})(\frac{\partial P}{\sqrt{P}} f' : P - 2\partial y\sqrt{P} - \frac{y\partial P}{\sqrt{P}}),$$

cujus postremae formulae cum integrale sit

$$\int \frac{\partial P}{\sqrt{P}} f' : P - 2y\sqrt{P}, \text{ erit}$$
$$t = F : \left(\int \frac{\partial P}{\sqrt{P}} f' : P - 2y\sqrt{P}\right).$$

Deinde ob $\left(\frac{\partial u}{\partial y}\right) = P\left(\frac{\partial u}{\partial x}\right)$, habetur

$$\partial u = \left(\frac{\partial u}{\partial x}\right)(\partial x + P\partial y) = \left(\frac{\partial u}{\partial x}\right)(\partial P f' : P - y\partial P),$$

ideoque

$$\partial u = \left(\frac{\partial u}{\partial x}\right)(f' : P - y)\,\partial P;$$

quare u aequabitur functioni ipsius P. In hoc autem negotio functiones quascunque accipere licet, quia sequente demum integratione universalitas solutionis obtinetur. Quare ponamus

$$t = \int \frac{\partial P}{\sqrt{P}} f' : P - 2y\sqrt{P} \text{ et } u = P, \text{ existente}$$
$$x + Py = f : P.$$

Denique ad ipsum integrale inveniendum, quia est

$$2l\left(\frac{\partial z}{\partial u}\right) = \int \frac{\partial t \left(\frac{\partial P}{\partial x}\right)}{P\left(\frac{\partial t}{\partial x}\right)},$$

in qua integratione u seu P sumitur constans, pur superiora erit

$$\frac{\partial t}{\left(\frac{\partial t}{\partial x}\right)} = \partial P f' : P - 2P\partial y - y\partial P = -2P\partial y,$$

ob P constans, et $\left(\frac{\partial P}{\partial x}\right) = \frac{1}{f' : P - y}$, unde fit

$$2l\left(\frac{\partial z}{\partial P}\right) = \int \frac{-2\partial y}{f' : P - y} = 2l(f' : P - y) + 2lF : P, \text{ seu}$$
$$\left(\frac{\partial z}{\partial P}\right) = (f' : P - y)\,F : P,$$

hincque porro

$$z = \int \partial P\,(f' : P - y)\,F : P,$$

sumendo hinc t constans. Cum igitur sit

$$y = +\frac{1}{2\sqrt{P}} \int \frac{\partial P}{\sqrt{P}} f' : P - \frac{t}{2\sqrt{P}},$$

ideoque

$$f:P - y = f':P - \frac{1}{2\surd P}\int\frac{\partial P}{\surd P}f':P + \frac{f}{2\surd P},$$

unde conficitur

$$z = \int\partial P\,(f':P - \frac{1}{2\surd P}\int\frac{\partial P}{\surd P}f':P)\,F:P$$
$$+ (\tfrac{1}{2}\int\frac{\partial P}{\surd P}f':P - y\surd P)\int\frac{\partial P}{\surd P}F:P + \Phi:(\int\frac{\partial P}{\surd P}f':P - 2y\surd P),$$

quae expressio duas continet functiones arbitrarias F et Φ.

Corollarium 1.

303. Primum hujus formae membrum ita transformari potest:

$$\int\frac{\partial P}{\surd P}(\surd P\,.\,f':P - \tfrac{1}{2}\int\frac{\partial P}{\surd P}f':P)\,F:P, \text{ at}$$
$$\surd P\,.\,f':P - \tfrac{1}{2}\int\frac{\partial P}{\surd P}f':P = \int\partial P\surd P\,.\,f'':P,$$

unde primum membrum erit

$$\int\frac{\partial P}{\surd P}F:P\,.\int\partial P\surd P\,.\,f'':P.$$

Corollarium 2.

304. Cum autem hoc primum membrum sit functio indefinita ipsius P, si ea indicetur per $\Pi:P$, erit

$$\frac{\partial P}{\surd P}F:P = \frac{\partial P\,\Pi':P}{\int\partial P\surd P\,.\,f'':P},$$

unde forma integralis fit

$$z = \Pi:P + \Phi:(\int\frac{\partial P}{\surd P}f':P - 2y\surd P)$$
$$+ (\int\frac{\partial P}{\surd P}f':P - 2y\surd P)\int\frac{\partial P\,\Pi':P}{2\int\partial P\surd P\,.\,f'':P}.$$

Corollarium 3.

305. Solutio magis particularis nascitur sumendo $\Pi:P = 0$, hincque z aequabitur functioni cuicunque quantitatis $\int\frac{\partial P}{\surd P}f':P - 2y\surd P$, quae ob $x + Py = f:P$ per x et y exhiberi censenda est.

Scholion.

306. Quanquam hic eadem methodo sum usus atque in problemate praecedente, tamen, quod mirum videatur, casus praecedentis problematis, quo erat $P = a$, in hac solutione non continetur. Ratio hujus paradoxi in resolutione aequationis $\left(\frac{\partial P}{\partial y}\right) = P\left(\frac{\partial P}{\partial x}\right)$ est sita, cui manifesto satisfacit valor $P = a$, etiamsi in forma inde derivata $x + Py = f:P$ non contineatur. Hic scilicet simile quiddam usu venit, quod jam supra observavimus, saepe aequationi differentiali valorem quendam satisfacere posse, qui in integrali non contineatur. Veluti aequationi $\partial y \surd(a-x) = \partial x$ satisfacere videmus valorem $x = a$, quem tamen integrale $y = C - 2\surd(a-x)$ excludit. Quare etiam nostro casu valor $P = a$ peculiarem evolutionem postulat, in priore problemate peractam. De reliquis, ubi pro $f:P$ certa quaedam functio ipsius P assumitur, exempla quaedam evolvamus.

Exemplum 1.

307. *Sumto* $f:P = 0$, *ut sit* $P = -\frac{x}{y}$, *integrale completum hujus aequationis:*

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{xx}{yy}\left(\frac{\partial\partial z}{\partial x^2}\right),$$

investigare.

Cum sit $f':P = 0$, solutio inventa, ob $\int\frac{\partial P}{\surd P}f':P = C$, praebet

$$z = \frac{-C}{2}\int\frac{\partial P}{\surd P}F:P + (\tfrac{1}{2}C - y\surd P)\int\frac{\partial P}{\surd P}F:P + \Phi:(C - 2y\surd P).$$

Statuatur $\int\frac{\partial P}{\surd P}F:P = \Pi:P$, prodibitque

$$z = -y\surd P.\Pi:P + \Phi:y\surd P.$$

Restituatur pro P valor $\frac{-x}{y}$, et ob $y\surd P = \surd -xy$, imaginarium $\surd -1$ in functiones involvendo erit

$$z = \sqrt{xy} \,.\, \Pi : \frac{x}{y} + \Phi : \sqrt{xy},$$

quae forma facile in hanc transfunditur:

$$z = x \Gamma : \frac{x}{y} + \Theta : xy,$$

ubi $x \Gamma : \frac{x}{y}$ denotat functionem quamcunque homogeneam unius dimensionis ipsarum x et y. Resolutio autem instituetur loco x et y has novas variabiles t et u introducendo, ut sit $t = C - 2\sqrt{-xy}$ et $u = -\frac{x}{y}$, vel etiam simplicius $t = 2\sqrt{xy}$ et $u = \frac{x}{y}$, unde fit

$$\left(\frac{\partial t}{\partial x}\right) = \frac{\sqrt{y}}{\sqrt{x}}, \left(\frac{\partial t}{\partial y}\right) = \frac{\sqrt{x}}{\sqrt{y}}, \left(\frac{\partial\partial t}{\partial x^2}\right) = \frac{-\sqrt{y}}{2x\sqrt{x}}, \left(\frac{\partial\partial t}{\partial y^2}\right) = \frac{-\sqrt{x}}{2y\sqrt{y}}.$$

$$\left(\frac{\partial u}{\partial x}\right) = \frac{1}{y}, \left(\frac{\partial u}{\partial y}\right) = \frac{-x}{yy}, \left(\frac{\partial\partial u}{\partial x^2}\right) = 0, \left(\frac{\partial\partial u}{\partial y^2}\right) = \frac{2x}{y^3},$$

et ob $PP = \frac{xx}{yy}$ aequatio proposita hanc induit formam:

$$0\left(\frac{\partial z}{\partial t}\right) + \frac{2x}{y^3}\left(\frac{\partial z}{\partial u}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{\partial\partial z}{\partial t \partial u}\right) = 0.$$

Nunc cum sit

$$ttu = 4xx, \text{ et } x = \tfrac{1}{2} t\sqrt{u},$$

atque $y = \frac{t}{2\sqrt{u}}$, habebimus

$$\frac{8uu}{tt}\left(\frac{\partial z}{\partial u}\right) - \frac{8uu}{t}\left(\frac{\partial\partial z}{\partial t \partial u}\right) = 0, \text{ seu } \left(\frac{\partial z}{\partial u}\right) = t\left(\frac{\partial\partial z}{\partial t \partial u}\right).$$

Fiat $\left(\frac{\partial z}{\partial u}\right) = v$, ut sit $v = t\left(\frac{\partial v}{\partial t}\right)$, et sumto u constante, $\frac{\partial t}{t} = \frac{\partial v}{v}$ ergo $v = \left(\frac{\partial z}{\partial u}\right) = t f' : u$. Sit jam t constans, fietque

$$z = t f : u + F : t = 2\sqrt{xy} \,.\, f : \frac{x}{y} + F : \sqrt{xy},$$

ut ante.

Corollarium.

308. Quemadmodum autem expressio inventa

$$z = x\Gamma : \frac{x}{y} + \Theta : xy$$

satisfaciat, differentialibus rite sumtis perspicietur

$$\left(\frac{\partial z}{\partial x}\right) = \Gamma : \frac{x}{y} + \frac{x}{y} F' : \frac{x}{y} + y\,\Theta' : xy, \quad \left(\frac{\partial z}{\partial y}\right) = \frac{-xx}{yy}\Gamma' : \frac{x}{y} + x\,\Theta' : xy,$$

unde porro fit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{2}{y}\Gamma' : \frac{x}{y} + \frac{x}{yy}\Gamma'' : \frac{x}{y} + yy\,\Theta'' : xy \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{2xx}{y^3}\Gamma' : \frac{x}{y} + \frac{x^2}{y^4}\Gamma'' : \frac{x}{y} + xx\,\Theta'' : xy.$$

Exemplum 2.

309. *Sumto* $f : P = \frac{PP}{2a}$, *ut sit*

$PP = 2aPy + 2ax$ *et* $P = ay \mp \sqrt{(aayy + 2ax)}$,

hujus aequationis:

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = [2aayy + 2ax + 2ay\sqrt{(aayy + 2ax)}]\left(\frac{\partial\partial z}{\partial x^2}\right),$$

integrale completum investigare.

Cum sit $f : P = \frac{PP}{2a}$, erit

$$f' : P = \frac{P}{a}, \text{ et } \int\frac{\partial P}{\sqrt{P}} f' : P = \int\frac{1}{a}\partial P\sqrt{P} = \frac{2}{3a}P\sqrt{P},$$

unde forma generalis supra inventa abit in

$$z = \int\partial P . \frac{2P}{3a} F : P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\int\frac{\partial P}{\sqrt{P}} F : P + \Phi : \left(\frac{2}{3a}P\sqrt{P} - y\sqrt{P}\right).$$

Statuatur $\int\frac{\partial P}{\sqrt{P}} . F : P = \Pi : P$, erit

$$\partial P . F : P = \partial P\sqrt{P} . \Pi' P,$$

atque

$$z = \frac{2}{3a}\int P^{\frac{3}{2}}\partial P . \Pi' : P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\Pi : P + \Phi : \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right).$$

Est autem

$$\frac{P}{3a} - y = \frac{-2}{3}y \mp \frac{1}{3}\sqrt{\left(yy + \frac{2x}{a}\right)};$$

quarum formularum evolutio deducit ad expressiones nimis perplexas. At substitutiones ad scopum perducentes sunt

$$t = \frac{2}{3a}P\sqrt{P} - 2y\sqrt{P} \text{ et } u = P.$$

Corollarium.

310. Si pro solutione magis restricta ponatur

$$\Pi : P = P^{n-\frac{1}{2}}, \text{ erit}$$

$$\Pi' : P = (n - \tfrac{1}{2})\, P^{n-\frac{3}{2}},$$

hincque colligitur

$$z = \tfrac{n}{(n+1)a} P^{n+1} - P^n y + \Phi : (\tfrac{P\surd P}{3a} - y\surd P).$$

Sit $n = 1$, et functio Φ evanescat, eritque

$$z = \tfrac{1}{2a} P P - P y = x;$$

at casus $n = 2$ dat

$$z = \tfrac{2}{3a} P^3 - P^2 y = \tfrac{4}{3} axy + \tfrac{2}{3} P\,(2x + ayy), \text{ seu}$$

$$z = aay^3 + 3axy + (ayy + 2x)\surd(aayy + 2ax).$$

Scholion.

311. Forma integralis inventa sequenti modo simplicior effici potest: Ponatur

$$\int \tfrac{\partial P}{\surd P} F : P = \Pi : P, \text{ fiet}$$

$$F : P = \surd P \,.\, \Pi' : P,$$

eritque (omittendo postremum membrum

$$\Phi\,(\int \tfrac{\partial P}{\surd P} f' : P - 2y\surd P),$$

quod nulla reductione indiget)

$$z = \int \partial P\,(\surd P \,.\, f' : P - \tfrac{1}{2}\int \tfrac{\partial P}{\surd P} f' : P)\ \Pi' : P$$

$$+ \tfrac{1}{2}\Pi : P \int \tfrac{\partial P}{\surd P} f' : P - y\surd P \,.\, \Pi : P; \text{ at}$$

$$\tfrac{1}{2}\Pi : P \,.\int \tfrac{\partial P}{\surd P} f' : P = \int (\tfrac{1}{2}\partial P\, \Pi' : P \,.\int \tfrac{\partial P}{\surd P} f' : P + \tfrac{1}{2}\tfrac{\partial P}{\surd P} \Pi : P \,.\, f' : P),$$

unde fit

$$z = \int \Pi' : P \,.\, \partial P \surd P\, f' : P + \tfrac{1}{2}\int \Pi : P \,.\, \tfrac{\partial P}{\surd P} f' : P - y\surd P \,.\, \Pi : P.$$

Porro est

$$\int \partial P . \Pi' : P . \sqrt{P} . f' : P = \Pi : P . \sqrt{P} . f' : P - \int \Pi : P \left(\frac{\partial P}{2\sqrt{P}} f' : P + \partial P \sqrt{P} . f'' : P \right),$$

ideoque

$$z = \Pi : P . \sqrt{P} . f' : P - \int \partial P . \Pi : P . \sqrt{P} . f'' : P - y \sqrt{P} . \Pi : P.$$

Statuatur porro

$$\int \partial P \, \Pi : P . \sqrt{P} . f'' : P = \Theta : P., \text{ erit}$$

$$\Pi : P = \frac{\Theta' : P}{\sqrt{P} . f'' : P} \text{ et}$$

$$z = \frac{\Theta' : P}{f'' : P} (f' : P - y) - \Theta : P + \Phi \left(\int \frac{\partial P}{\sqrt{P}} f' : P - 2 y \sqrt{P} \right),$$

quae forma sine dubio multo est simplicior quam primo inventa.

Problema 50.

312. Proposita aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - PP \left(\frac{\partial\partial z}{\partial x^2}\right) + Q \left(\frac{\partial z}{\partial y}\right) + R \left(\frac{\partial z}{\partial x}\right) = 0.$$

invenire valores quantitatum P, Q, R, quibus integratio ope reductionis ante adhibitae succedit.

Solutio.

Introductis binis novis variabilibus t et u, habebimus

$$0 = \left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial t}{\partial y}\right)^2 \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t \partial u}\right) + \left(\frac{\partial u}{\partial y}\right)^2 \left(\frac{\partial\partial z}{\partial u^2}\right)$$

$$- PP \left(\frac{\partial\partial t}{\partial x^2}\right) - PP \left(\frac{\partial\partial u}{\partial x^2}\right) - PP \left(\frac{\partial t}{\partial x}\right)^2 - 2 PP \left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) - PP \left(\frac{\partial u}{\partial x}\right)^2$$

$$+ Q \left(\frac{\partial t}{\partial y}\right) + Q \left(\frac{\partial u}{\partial y}\right)$$

$$+ R \left(\frac{\partial t}{\partial x}\right) + Q \left(\frac{\partial u}{\partial x}\right).$$

Statuamus ergo ut ante

$$\left(\frac{\partial t}{\partial y}\right) = P \left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = - P \left(\frac{\partial u}{\partial x}\right),$$

unde fit

$$\left(\frac{\partial\partial t}{\partial x \partial y}\right) = P \left(\frac{\partial\partial t}{\partial x^2}\right) + \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right), \text{ et}$$

$$\left(\frac{\partial\partial t}{\partial y^2}\right) = PP \left(\frac{\partial\partial t}{\partial x^2}\right) + P \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial t}{\partial x}\right),$$

atque

$$\left(\frac{\partial\partial u}{\partial y^2}\right) = PP\left(\frac{\partial\partial u}{\partial x^2}\right) + P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right),$$

et aequatio resolvenda erit

$$0 = \left[P\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right) + PQ + R\right]\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) - 4\,PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ \left[P\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right) - PQ + R\right]\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right).$$

Jam evidens est integrationem institui posse, si alterutra formula $\left(\frac{\partial z}{\partial t}\right)$ vel $\left(\frac{\partial z}{\partial u}\right)$ ex calculo abeat. Ponamus ergo esse

$$P\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right) - PQ + R = 0, \text{ seu}$$
$$R = PQ + \left(\frac{\partial P}{\partial y}\right) - P\left(\frac{\partial P}{\partial x}\right),$$

et aequatio resultans, per $\left(\frac{\partial t}{\partial x}\right)$ divisa, fit

$$0 = 2\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial z}{\partial t}\right) - 4\,PP\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right).$$

Fiat $\left(\frac{\partial z}{\partial t}\right) = v$, erit

$$\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right] v - 2\,PP\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial u}\right) = 0.$$

Sumatur t constans, ut fiat

$$\frac{\partial v}{v} = \frac{\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right]\partial u}{2\,PP\left(\frac{\partial u}{\partial x}\right)},$$

ut necesse est, ut quantitates P, Q, $\left(\frac{\partial P}{\partial y}\right)$ et $\left(\frac{\partial u}{\partial x}\right)$ per novas variabiles t et u exprimantur, quas ergo primum definiri convenit. Cum igitur sit

$$\left(\frac{\partial t}{\partial y}\right) = P\left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = -P\left(\frac{\partial u}{\partial x}\right), \text{ erit}$$
$$\partial t = \left(\frac{\partial t}{\partial x}\right)(\partial x + P\partial y) \text{ et } \partial u = \left(\frac{\partial u}{\partial x}\right)(\partial x - P\partial y).$$

Sunt ergo $\left(\frac{\partial t}{\partial x}\right)$ et $\left(\frac{\partial u}{\partial x}\right)$ factores integrabiles reddentes formulas $\partial x + P\partial y$ et $\partial x - P\partial y$: non enim opus est ut hinc valores t et u generalissime definiantur. Sint p et q tales multiplicatores, per x et y dati, eritque

$$t = \int p\,(\partial x + P\partial y) \text{ et } u = \int q\,(\partial x - P\partial y),$$

unde superior integratio fit

$$\frac{\partial v}{v} = \frac{[PQ + (\frac{\partial P}{\partial v})]\,\partial u}{2\,PPq},$$

in qua integratione quantitas $t = \int p\,(\partial x + P\partial y)$ constans est spectanda. Seu ob $\partial u = q\,(\partial x - P\partial y)$ erit

$$\frac{\partial v}{v} = \frac{[PQ + (\frac{\partial P}{\partial y})]\,(\partial x - P\partial y)}{2\,PP}.$$

Verum ob $\partial t = 0$ est $\partial x = -P\partial y$, ita ut prodeat

$$\frac{\partial v}{v} = -\frac{\partial y}{P}\,[PQ + (\frac{\partial P}{\partial y})],$$

ubi ob t constans, et datum per x et y, valor ipsius x per y et t expressus substitui potest, ut sola y variabilis insit, et invento integrali

$$-\int \frac{\partial y}{P}\,[PQ + (\frac{\partial P}{\partial y})] = lV,$$

erit $v = V\,f:t = (\frac{\partial z}{\partial t})$.

Nunc ponatur u constans eritque

$$z = \int V\partial t\, f:t + F:u.$$

Conditio autem, sub qua haec integratio locum habet, postulat ut sit

$$R = PQ + (\frac{\partial P}{\partial y}) - P\,(\frac{\partial P}{\partial x}).$$

Corollarium 1.

313. Eodem modo aequatio proposita resolutionem admittet, si fuerit

$$R = -PQ - (\frac{\partial P}{\partial y}) - P\,(\frac{\partial P}{\partial x});$$

manetque ut ante

$$t = \int p\,(\partial x + P\partial y) \text{ et } u = \int q\,(\partial x - P\partial y).$$

Tum vero fit

$$0 = -[PQ + (\frac{\partial P}{\partial x})] (\frac{\partial z}{\partial u}) - 2PP (\frac{\partial t}{\partial x}) (\frac{\partial\partial z}{\partial t \partial u}),$$

quae posito $(\frac{\partial z}{\partial u}) = v$, sumtoque u constante dat

$$\frac{\partial v}{v} = \frac{-[PQ + (\frac{\partial P}{\partial y})] \partial t}{2PP (\frac{\partial t}{\partial x})} = \frac{-[PQ + (\frac{\partial P}{\partial y})] (\partial x + P\partial y)}{2PP}$$

Corollarium 2.

314. Si porro habita ratione, quod

$$u = \int q (\partial x - P\partial y)$$

sit constans et $\partial x = P\partial y$, ponatur

$$\int - \frac{\partial y [PQ + (\frac{\partial P}{\partial y})]}{P} = lV, \text{ erit}$$

$$v = V f : u = (\frac{\partial z}{\partial u}),$$

unde tandem, sumendo jam

$$t = \int p (\partial x + P\partial y),$$

colligitur

$$z \int V\partial u f : u + F : t.$$

Exemplum 1.

315. *Si sumatur $P = a$ et $R = aQ$, quaecunque fuerit Q functio ipsarum x et y, integrare aequationem:*

$$(\frac{\partial\partial z}{\partial y^2}) - aa (\frac{\partial\partial z}{\partial x^2}) + Q (\frac{\partial z}{\partial y}) + aQ (\frac{\partial z}{\partial x}) = 0.$$

Cum hic sit $P = a$, erit $p = 1$, $q = 1$ et $t = x + ay$ atque $u = x - ay$, unde posito $(\frac{\partial z}{\partial t}) = v$ fit

$$\frac{\partial v}{v} = \frac{aQ\partial u}{2aa} = \frac{Q\partial u}{2a}.$$

Quoniam igitur est

$$x = \frac{t+u}{2} \text{ et } y = \frac{t-u}{2a},$$

his valoribus substitutis fit Q functio ipsarum t et u, ac spectata t ut constante erit

$$lv = \frac{1}{2a}\int Q\partial u + l\,f:t, \text{ seu}$$

$$\left(\frac{\partial z}{\partial t}\right) = e^{\frac{1}{2a}\int Q\partial u}\, f:t,$$

et sumta jam u constante

$$z = \int e^{\frac{1}{2a}\int Q\partial u}\,\partial t\, f:t + F:u.$$

Corollarium 1.

316. Si Q sit constans $= 2ab$, aequationis hujus

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - aa\left(\frac{\partial\partial z}{\partial x^2}\right) + 2ab\left(\frac{\partial z}{\partial y}\right) + 2aab\left(\frac{\partial z}{\partial x}\right) = 0,$$

integrale erit

$$z = e^{bu} f:t + F:u = e^{b(x-ay)} f:(x+ay) + F:(x-ay),$$

sive

$$z = e^{b(x-ay)}\,[f:(x+ay) + F:(x-ay)].$$

Corollarium 2.

317. Si $Q = \frac{a}{x}$, hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - aa\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{a}{x}\left(\frac{\partial z}{\partial y}\right) + \frac{aa}{x}\left(\frac{\partial z}{\partial x}\right) = 0$$

integrale ob

$$\int Q\partial u = \int \frac{a\partial u}{x} = \int \frac{2a\partial u}{t+u} = 2al\,(t+u), \text{ erit}$$

$$z = \int (t+u)\,\partial t\, f:t + F:u = \int t\partial t\, f:t + u\int \partial t\, f:t + F:u.$$

Vel sit $f:t = \Pi'':t$, erit

$$\int \partial t\, f:t = \Pi':t \text{ et}$$

$$\int t\partial t\, f:t = \int t\partial.\Pi':t = t\Pi':t - \int \partial t.\Pi':t = t\Pi':t - \Pi:t.$$

ergo

$$z = (t+u)\Pi' : t - \Pi : t + F : u, \text{ seu}$$
$$z = 2x\Pi' : (x+ay) - \Pi : (x+ay) + F : (x-ay).$$

Exemplum 2.

318. *Sit* $P = \frac{x}{y}$, *et* $R = \frac{-x}{y}Q + \frac{x}{yy} - \frac{x}{yy} = \frac{-x}{y}Q$, *sumaturque* $Q = \frac{1}{x}$, *ut sit* $R = \frac{-1}{y}$, *et haec aequatio integrari debeat*

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{xx}{yy}\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{1}{x}\left(\frac{\partial z}{\partial y}\right) - \frac{1}{y}\left(\frac{\partial z}{\partial x}\right) = 0.$$

Cum ergo sit

$$t = \int p\left(\partial x + \frac{x\partial y}{y}\right) \text{ et } u = \int q\left(\partial x - \frac{x\partial y}{y}\right),$$

sumatur $p = y$ et $q = \frac{1}{y}$, ut fiat $t = xy$ et $u = \frac{x}{y}$.
Posito nunc $\left(\frac{\partial z}{\partial u}\right) = v$ sumtoque u constante, ex Corollario 1. fit:

$$\frac{\partial v}{v} = \frac{-\left(\frac{1}{y} - \frac{x}{yy}\right)\partial t}{\frac{2xx}{yy}\cdot y} = \frac{-(y-x)\partial t}{2xxy}.$$

Est vero $tu = xx$, hincque $x = \sqrt{tu}$ et $y = \sqrt{\frac{t}{u}}$, atque

$$2xxy = 2t\sqrt{tu};$$

unde fit

$$\frac{\partial v}{v} = \frac{\left(\sqrt{tu} - \sqrt{\frac{t}{u}}\right)\partial t}{2t\sqrt{tu}} = \frac{\partial t}{2t} - \frac{\partial t}{2tu},$$

et ob u constans

$$lv = \tfrac{1}{2}lt - \frac{1}{2u}lt,$$

$$\left(\frac{\partial z}{\partial u}\right) = t^{\frac{1}{2} - \frac{1}{2u}} f : u.$$

Quare sumto jam t constante erit

$$z = t^{\frac{1}{2}} \int t^{-\frac{1}{2u}} \partial u\, f : u + F : t.$$

Vel ponatur $-\frac{1}{2u} = s$, ut sit $s = -\frac{y}{2x}$ eritque

$$z = t^{\frac{1}{2}} \int t^s \partial s \; f : s + F : t.$$

In hac integratione $\int t^s \partial s \; f : s$ sola s est variabilis, ac demum integrali sumto restitui debet $t = xy$ et $s = \frac{-y}{2x}$. Caeterum patet functionem quamcunque ipsius xy particulariter satisfacere.

Problema 51.

319. Proposita aequatione generali

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - 2P\left(\frac{\partial\partial z}{\partial x \partial y}\right) + (PP - QQ)\left(\frac{\partial\partial z}{\partial x^2}\right) + R\left(\frac{\partial z}{\partial y}\right) + S\left(\frac{\partial z}{\partial x}\right) + Tz + V = 0,$$

invenire conditiones quantitatum P, Q, R, S, T, ut integratio ope reductionis adhibitae succedat.

Solutio.

Facta eadem substitutione introducendis binis novis variabilibus t et u, aequatio nostra sequentem induet formam

$$\left.\begin{array}{lllll}
V + Tz + \left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right) & + \left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right) & + \left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right) & + 2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t \partial u}\right) & + \left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right) \\
- 2P\left(\frac{\partial\partial t}{\partial x \partial y}\right) & - 2P\left(\frac{\partial\partial u}{\partial x \partial y}\right) & - 2P\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right) & - 2P\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) & - 2P\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) \\
+ (PP - QQ)\left(\frac{\partial\partial t}{\partial x^2}\right) & + (PP - QQ)\left(\frac{\partial\partial u}{\partial x^2}\right) & + (PP - QQ)\left(\frac{\partial t}{\partial x}\right)^2 & - 2P\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right) & + (PP - QQ)\left(\frac{\partial u}{\partial x}\right)^2 \\
+ R\left(\frac{\partial t}{\partial y}\right) & + R\left(\frac{\partial u}{\partial y}\right) & & + 2(PP - QQ)\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) & \\
+ S\left(\frac{\partial t}{\partial x}\right) & + S\left(\frac{\partial u}{\partial x}\right) & & &
\end{array}\right\} = 0.$$

Determinentur jam hae duae novae variabiles t et u ita per x et y, ut formulae $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ evanescant: debebitque esse

$$\left(\frac{\partial t}{\partial y}\right) = (P + Q)\left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = (P - Q)\left(\frac{\partial u}{\partial x}\right),$$

unde patet has variabiles sequenti modo determinari

$$t = \int p\,[\partial x + (P + Q)\,\partial y] \text{ et } u = \int q\,[\partial x + (P - Q)\,\partial y],$$

sumendo p et q ita ut hae formulae integrationem admittant.

Cum nunc sit

$$\left(\frac{\partial\partial t}{\partial x\partial y}\right) = (P+Q)\left(\frac{\partial\partial t}{\partial x^2}\right) + \left[\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial t}{\partial x}\right),$$

$$\left(\frac{\partial\partial t}{\partial y^2}\right) = (P+Q)^2\left(\frac{\partial\partial t}{\partial x^2}\right) + (P+Q)\left[\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial t}{\partial x}\right)$$
$$+ \left[\left(\frac{\partial P}{\partial y}\right) + \left(\frac{\partial Q}{\partial y}\right)\right]\left(\frac{\partial t}{\partial x}\right),$$

$$\left(\frac{\partial\partial u}{\partial x\partial y}\right) = (P-Q)\left(\frac{\partial\partial u}{\partial x^2}\right) + \left[\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial u}{\partial x}\right),$$

$$\left(\frac{\partial\partial u}{\partial y^2}\right) = (P-Q)^2\left(\frac{\partial\partial u}{\partial x^2}\right) + (P-Q)\left[\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial u}{\partial x}\right)$$
$$+ \left[\left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x}\right).$$

Hinc reperitur formulae $2\left(\frac{\partial\partial z}{\partial t\partial u}\right)$ coëfficiens $= -2QQ\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)$, termini $\left(\frac{\partial z}{\partial t}\right)$ coëfficiens $=$

$$\left[-(P-Q)\left(\frac{\partial P+\partial Q}{\partial x}\right) + \left(\frac{\partial P+\partial Q}{\partial y}\right) + R(P+Q) + S\right]\left(\frac{\partial t}{\partial x}\right),$$

termini vero $\left(\frac{\partial z}{\partial u}\right)$ coëfficiens $=$

$$\left[-(P+Q)\left(\frac{\partial P-\partial Q}{\partial x}\right) + \left(\frac{\partial P-\partial Q}{\partial y}\right) + R(P-Q) + S\right]\left(\frac{\partial u}{\partial x}\right).$$

Est vero $\left(\frac{\partial t}{\partial x}\right) = p$ et $\left(\frac{\partial u}{\partial x}\right) = q$, unde si brevitatis gratia vocetur

$$S + R(P+Q) + \left(\frac{\partial P+\partial Q}{\partial y}\right) - (P-Q)\left(\frac{\partial P+\partial Q}{\partial x}\right) = M \text{ et}$$

$$S + R(P-Q) + \left(\frac{\partial P-\partial Q}{\partial y}\right) - (P+Q)\left(\frac{\partial P-\partial Q}{\partial x}\right) = N,$$

aequatio nostra resolvenda erit

$$0 = V + Tz + Mp\left(\frac{\partial z}{\partial t}\right) + Nq\left(\frac{\partial z}{\partial u}\right) - 4QQpq\left(\frac{\partial\partial z}{\partial t\partial u}\right),$$

seu ut cum formis supra §§. 294 et 295. exhibitis comparari queat

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{M}{4QQq}\left(\frac{\partial z}{\partial t}\right) - \frac{N}{4QQp}\left(\frac{\partial z}{\partial u}\right) - \frac{T}{4QQpq}z - \frac{V}{4QQpq} = 0,$$

quae si porro brevitatis gratia ponatur

$$\frac{M}{4QQq} = K \text{ et } \frac{N}{4QQp} = L,$$

duplici casu integrationem admittit: altero si fuerit

$$-\frac{T}{4QQpq} = +KL - \left(\frac{\partial L}{\partial u}\right), \text{ seu } T = 4QQpq\left(\frac{\partial L}{\partial u}\right) - \frac{MN}{4QQ},$$

altero vero si fuerit

$$-\frac{T}{4QQpq} = KL - \left(\frac{\partial K}{\partial t}\right), \text{ seu } T = 4QQpq\left(\frac{\partial K}{\partial t}\right) - \frac{MN}{4QQ}.$$

Quoniam vero K et L per x et y dantur, formulae illae $\left(\frac{\partial K}{\partial t}\right)$ et $\left(\frac{\partial L}{\partial u}\right)$ ita reduci possunt ut sit

$$\left(\frac{\partial K}{\partial t}\right) = \frac{Q-P}{2Qp}\left(\frac{\partial K}{\partial x}\right) + \frac{1}{2Qp}\left(\frac{\partial K}{\partial y}\right) \text{ et}$$
$$\left(\frac{\partial L}{\partial u}\right) = \frac{P+Q}{2Qp}\left(\frac{\partial L}{\partial x}\right) - \frac{1}{2Qq}\left(\frac{\partial L}{\partial y}\right).$$

Quemadmodum autem ipsa integralia his casibus inveniri debeant, id quidem supra est declaratum; unde superfluum foret calculos illos taediosos hic repetere: quovis enim casu oblato solutio inde peti poterit.

Scholion. 1.

320. Quod ad hanc reductionem formularum attinet, ea sequenti modo instituitur. Cum sit in genere

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right),$$

ex formulis

$$\partial t = p\partial x + p(P+Q)\partial y \text{ et } \partial u = q\partial x + q(P-Q)\partial y \text{ erit}$$
$$q\partial t - p\partial u = 2pqQ\partial y, \text{ seu } \partial y = \frac{q\partial t - p\partial u}{2Qpq} \text{ et}$$
$$q(P-Q)\partial t - p(P+Q)\partial u = -2Qpq\partial x, \text{ seu}$$
$$\partial x = \frac{p(P+Q)\partial u - q(P-Q)\partial t}{2Qpq}.$$

Quibus valoribus substitutis obtinebitur

$$\partial z = \left[\frac{(P+Q)\partial u}{2Qq} - \frac{(P-Q)\partial t}{2Qp}\right]\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial t}{2Qp} - \frac{\partial u}{2Qq}\right)\left(\frac{\partial z}{\partial y}\right),$$

ita ut ∂z per differentialia ∂t et ∂u exprimatur. Posito ergo u constante et $\partial u = 0$, erit

$$\left(\frac{\partial z}{\partial t}\right) = \frac{Q-P}{2Qp}\left(\frac{\partial z}{\partial x}\right) + \frac{1}{2Qp}\left(\frac{\partial z}{\partial y}\right);$$

at posito t constante et $\partial t = 0$, erit

$$\left(\frac{\partial z}{\partial u}\right) = \frac{P+Q}{2Qq}\left(\frac{\partial z}{\partial x}\right) - \frac{1}{2Qq}\left(\frac{\partial z}{\partial y}\right).$$

Scholion 2.

321. Methodus igitur hoc capite tradita in hoc consistit, ut hujusmodi aequationes ope introductionis binarum novarum variabilium t et u ad hanc formam reducantur

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) + P\left(\frac{\partial z}{\partial t}\right) + Q\left(\frac{\partial z}{\partial u}\right) + Rz + S = 0,$$

de qua in praecedente capite vidimus, quibusnam casibus ea integrari queat: Iisdem igitur quoque casibus omnes aequationes, quae ad talem formam se reduci patiuntur, integrationem admittent. Est vero ejusdem formae casus quidam maxime singularis, cujus integratio absolvi potest, unde denuo infinita multitudo aliarum aequationum, quae quidem eo reduci queant, oritur integrationem pariter admittentium. Quem propterea casum sequenti capite diligentius evoluamus.

CAPUT IV.

ALIA METHODUS PECULIARIS HUJUSMODI AEQUATIONES INTEGRANDI.

Problema 52.

322.

Si aequatio proposita hanc habuerit formam

$$(x+y)^2 \left(\frac{\partial\partial z}{\partial x \partial y}\right) + m(x+y)\left(\frac{\partial z}{\partial x}\right) + m(x+y)\left(\frac{\partial z}{\partial y}\right) + nz = 0,$$

ejus integrale completum investigare.

Solutio.

Cum hic binae variabiles x et y aequaliter insint, ponatur primo

$$z = A(x+y)^\lambda f:x + B(x+y)^{\lambda+1} f':x + C(x+y)^{\lambda+2} f'':x$$
$$+ D(x+y)^{\lambda+3} f''':x \text{ etc.}$$

ubi pro faciliori substitutione notetur, posito $v = (x+y)^\mu F:x$ fore

$$\left(\frac{\partial v}{\partial x}\right) = \mu(x+y)^{\mu-1} F:x + (x+y)^\mu F':x,$$
$$\left(\frac{\partial v}{\partial y}\right) = \mu(x+y)^{\mu-1} F:x,$$
$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) = \mu(\mu-1)(x+y)^{\mu-2} F:x + \mu(x+y)^{\mu-1} F':x.$$

Facta ergo substitutione obtinebimus hanc aequationem

$$\begin{array}{llll} 0 = nA(x+y)^\lambda f:x & + nB(x+y)^{\lambda+1} f':x & + nC(x+y)^{\lambda+2} f'':x & + \text{etc.} \\ \quad + 2m\lambda A & + mA & + mB & \\ \quad + \lambda(\lambda-1)A & + 2m(\lambda+1)B & + 2m(\lambda+2)C & \\ & + \lambda A & + (\lambda+1)B & \\ & + (\lambda+1)\lambda B & + (\lambda+2)(\lambda+1)C, & \end{array}$$

ubi totum negotium ad coëfficientium A, B, C, D, etc. determinationem revocatur; facile autem erat praevidere, forma superiori assumta potestates ipsius $(x+y)$ in singulis membris pares esse prodituras: Fieri igitur necesse est

$$n + 2m\lambda + \lambda\lambda - \lambda = 0,$$
$$(n + 2m\lambda + 2m + \lambda\lambda + \lambda)\,B + (m+\lambda)\,A = 0,$$
$$(n + 2m\lambda + 4m + \lambda\lambda + 3\lambda + 2)\,C + (m+\lambda+1)\,B = 0,$$
$$(n + 2m\lambda + 6m + \lambda\lambda + 5\lambda + 6)\,D + (m+\lambda+2)\,C = 0,$$
$$\text{etc.}$$

quae determinationes ope primae $n + 2m\lambda + \lambda\lambda - \lambda = 0$ ita commodius exprimuntur:

$$B = -\frac{(m+\lambda)\,A}{2(m+\lambda)}, \qquad F = -\frac{(m+\lambda+4)\,E}{5(2m+2\lambda+4)},$$
$$C = -\frac{(m+\lambda+1)\,B}{2(2m+2\lambda+1)}, \qquad G = -\frac{(m+\lambda+5)\,F}{6(2m+2\lambda+5)},$$
$$D = -\frac{(m+\lambda+2)\,C}{3(2m+2\lambda+2)}, \qquad H = -\frac{(m+\lambda+6)\,G}{7(2m+2\lambda+6)},$$
$$E = -\frac{(m+\lambda+3)\,D}{4(m+2\lambda+3)}, \qquad \text{etc.}$$

unde lex progressionis est manifesta. At pro exponente λ duplicem eruimus valorem

$$\lambda = \tfrac{1}{2} - m \pm \sqrt{(\tfrac{1}{4} - m - n + mm)},$$

quorum utrumque aeque pro λ accipere licet. Hic autem praecipue notandi sunt casus, quibus series assumta abrumpitur, quod fit, quoties $m + \lambda + i = 0$, denotante i numerum quemcunque integrum positivum cyphra non exclusa. Hoc ergo evenit quoties fuerit

$$\tfrac{1}{2} + i \pm \sqrt{(\tfrac{1}{4} - m - n + mm)} = 0,$$

id quod fieri nequit nisi $\frac{1}{4} - m - n + mm$ fuerit quadratum. Inventa autem hujusmodi serie sive finita sive in infinitum excurrente, alia similis pro functionibus ipsius y reperitur; unde valor ipsius z ita reperietur expressus

$$z = A(x+y)^{\lambda}(f:x+F:y) + B(x+y)^{\lambda+1}(f':x+F':y) + C(x+y)^{\lambda+2}(f'':x+F'':y) + D(x+y)^{\lambda+3}(f''':x+F''':y) + E(x+y)^{\lambda+4}(f^{IV}:x+F^{IV}:y) + F(x+y)^{\lambda+5}(f^{V}:x+F^{V}:y), + \text{etc.}$$

ubi cum binae functiones arbitrariae adsint, id certum est signum, hanc formam esse integrale completum aequationis propositae.

Corollarium 1.

323. Si fuerit $\lambda = -m$, hoc est $n - mm + m = 0$, seu $n = mm - m$, integrale ex unico membro constabit ob $B = 0$, eritque integrale

$$z = A(x+y)^{-m}(f:x+F:y).$$

Corollarium 2.

324. Integrale autem duo membra continebit, si

$$\lambda = -m-1 \text{ vel } n = mm - m - 2 = (m+1)(m-2);$$

tum erit $B = -\frac{1}{2}A$ et integrale erit

$$z = (x+y)^{-m-1}(f:x+F:y) - \tfrac{1}{2}(x+y)^{-m}(f':x+F':y).$$

Corollarium 3.

325. Integrale tribus terminis constabit, si $\lambda = -m-2$, vel $n = (m+2)(m-3)$; tum erit

$$B = -\tfrac{2}{2}A, \text{ et } C = -\tfrac{1}{6}B = +\tfrac{1}{12}A,$$

integrale vero

$$z = (x+y)^{-m-2}(f:x+F:y) - \tfrac{1}{2}(x+y)^{-m-1}(f':x+F':y) + \tfrac{1}{12}(x+y)^{-m}(f'':x+F'':y).$$

Corollarium 4.

326. Ex quatuor autem membris integrale constabit, si fuerit $\lambda = -m - 3$, seu $n = (m + 3)(m - 4)$; tum autem erit

$$B = -\tfrac{1}{2}A, \quad C = -\tfrac{1}{5}B = +\tfrac{1}{10}A, \quad D = -\tfrac{1}{12}C = -\tfrac{1}{120}A,$$

et integrale

$$z = (x+y)^{-m-3}(f:x + F:y) - \tfrac{1}{2}(x+y)^{-m-2}(f':x + F':y)$$
$$+ \tfrac{1}{10}(x+y)^{-m-1}(f'':x + F'':y) - \tfrac{1}{120}(x+y)^{-m}(f''':x + F''':y).$$

Scholion.

327. Quod si in genere ponamus $\lambda + m = -i$, erit $n = (m+i)(m-i-1)$, tum vero

$$B = -\tfrac{1}{2}A, \quad C = -\frac{(i-1)B}{2(2i-1)}, \quad D = -\frac{(i-2)C}{3(2i-2)}; \quad E = -\frac{(i-3)D}{4(2i-3)},$$

unde fit omnes ad primum reducendo

$$B = -\tfrac{1}{2}A, \quad C = \frac{(i-1)}{2.2(2i-1)}A, \quad D = \frac{-(i-2)}{2.2.2.3(i-1)}A,$$
$$E = \frac{+(i-2)(i-3)}{2.2.2.3.4(2i-1)(2i-3)}A, \quad F = \frac{-(i-3)(i-4)}{2^4.3.4.5(2i-1)(2i-3)}A, \text{ etc.}$$

qui ita se habent

	A	B	C	D	E	F
$i = 1$	1	$-\frac{1}{2}$	0	0	0	0
$i = 2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0
$i = 3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0
$i = 4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7.24}$	$\frac{2}{96.7.5}$	0
$i = 5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9.24}$	$\frac{3.2}{96.9.7}$	$\frac{2.1}{960.9.7}$
$i = 6$	1	$-\frac{1}{2}$	$\frac{4}{44}$	$-\frac{4}{11.24}$	$\frac{4.3}{96.11.9}$	$\frac{3.2}{960.11.9}$

ita hujus aequationis

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial y}\right) + \frac{(m+i)(m-i-1)}{(x+y)^2}z = 0,$$

integrale completum erit

$$\begin{aligned} z = &+ (x+y)^{-m-i}(\mathrm{f}:x + \mathrm{F}:y) \\ &- \frac{i}{2i}(x+y)^{-m-i+1}(\mathrm{f}':x + \mathrm{F}':y) \\ &+ \frac{i(i-1)}{2i\,.\,2(2i-1)}(x+y)^{-m-i+2}(\mathrm{f}'':x + \mathrm{F}'':y) \\ &- \frac{i(i-1)(i-2)}{2i\,.\,2(2i-1)\,.\,3(2i-2)}(x+y)^{-m-i+3}(\mathrm{f}''':x + \mathrm{F}''':y) \\ &+ \frac{i(i-1)(i-2)(i-3)}{2i\,.\,2(2i-1)\,.\,3(2i-2)\,.\,4(2i-3)}(x+y)^{-m-i+4}(\mathrm{f}^{\mathrm{IV}}:x + \mathrm{F}^{\mathrm{IV}}:y) \\ &- \frac{i(i-1)(i-2)(i-3)(i-4)}{2i\,.\,2(2i-1)\,.\,3(2i-2)\,.\,4(2i-3)\,.\,5(2i-4)}(x+y)^{-m-i+5}(\mathrm{f}^{\mathrm{V}}:x + \mathrm{F}^{\mathrm{V}}:y) \\ &+ \text{etc.} \end{aligned}$$

quae forma quoties i fuerit numerus integer positivus, finito constat terminorum numero: secus autem in infinitum excurrit. Imprimis autem ista integratio hoc habet singulare, quod non solum ipsas functiones arbitrarias $\mathrm{f}:x$ et $\mathrm{F}:y$ complectatur, sed etiam earum formulas differentiales.

Exemplum.

328. *Si occurrat ista aequatio*

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial y}\right) = 0,$$

definire casus, quibus ejus integrale per formam finitam exhiberi potest.

Cum hic sit $n = (m+i)(m-i-1) = 0$, sumendo pro i numeros integros positivos, duo ordines habebuntur casuum, quibus integratio succedit, alter quo est $m = -i$, alter quo $m = i+1$, ita ut in genere integratio finita locum habeat, quoties m fuerit numerus integer sive positivus sive negativus. Primo ergo si sit $m = -i$, erit

$$z = 1\,(f : x + F : y) - \frac{i}{2i}(x+y)(f' : x + F' : y)$$
$$+ \tfrac{1}{2} \cdot \frac{i(i-1)}{2i(2i-1)}(x+y)^2 (f'' : x + F'' : y)$$
$$- \tfrac{1}{6} \cdot \frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3 (f''' : x + F''' : y)$$
$$+ \tfrac{1}{24} \cdot \frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4 (f^{IV} : x + F^{IV} : y)$$
$$- \text{etc.}$$

Deinde si sit $m = i + 1$, erit

$$(x+y)^{2i+1} z = 1\,(f : x + F : y) - \frac{i}{2i}(x+y)(f' : x + F' : y)$$
$$+ \tfrac{1}{2} \cdot \frac{i(i-1)}{2i(2i-1)}(x+y)^2 (f'' : x + F'' : y)$$
$$- \tfrac{1}{6} \cdot \frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3 (f''' : x + F''' : y)$$
$$+ \tfrac{1}{24} \cdot \frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4 (f^{IV} : x + F^{IV} : y)$$
$$- \text{etc.}$$

utrinque scilicet eadem habetur expressio, cui casu priori ipsa quantitas z, posteriori quantitas $(x+y)^{2i+1} z$ aequatur. Ad singulos hos casus distinctius evolvendos ponamus

$$A = (f : x + F : y),$$
$$B = (f : x + F : y) - \tfrac{1}{2}(x+y)(f' : x + F' : y),$$
$$C = (f : x + F : y) - \tfrac{2}{4}(x+y)(f' : x + F' : y) + \frac{1}{4 \cdot 3}(x+y)^2 (f'' : x + F'' : y),$$
$$D = (f : x + F : y) - \tfrac{3}{6}(x+y)(f' : x + F' : y) + \frac{3}{6 \cdot 5}(x+y)^2 (f'' : x + F'' : y)$$
$$- \frac{1}{6 \cdot 5 \cdot 4}(x+y)^3 (f''' : x + F''' : y), \quad \text{etc.}$$

vel posito brevitatis gratia

$$\mathfrak{A} = f : x + F : y,$$
$$\mathfrak{B} = (x+y)\,(f' : x + F' : y),$$
$$\mathfrak{C} = (x+y)^2\,(f'' : x + F'' : y),$$
$$\mathfrak{D} = (x+y)^3\,(f''' : x + F''' : y),$$
$$\mathfrak{E} = (x+y)^4\,(f^{IV} : x + F^{IV} : y),$$

etc.

sit

$$A = \mathfrak{A},$$
$$B = \mathfrak{A} - \tfrac{1}{2}\mathfrak{B},$$
$$C = \mathfrak{A} - \tfrac{2}{4}\mathfrak{B} + \frac{1}{4 \cdot 3}\mathfrak{C},$$
$$D = \mathfrak{A} - \tfrac{3}{6}\mathfrak{B} + \frac{3}{6 \cdot 5}\mathfrak{C} - \frac{1}{6 \cdot 5 \cdot 4}\mathfrak{D},$$
$$E = \mathfrak{A} - \tfrac{4}{8}\mathfrak{B} + \frac{6}{8 \cdot 7}\mathfrak{C} - \frac{4}{8 \cdot 7 \cdot 6}\mathfrak{D} + \frac{1}{8 \cdot 7 \cdot 6 \cdot 5}\mathfrak{E},$$
$$F = \mathfrak{A} - \tfrac{5}{10}\mathfrak{B} + \frac{10}{10 \cdot 9}\mathfrak{C} - \frac{10}{10 \cdot 9 \cdot 8}\mathfrak{D} + \frac{5}{10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{E} - \frac{1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}\mathfrak{F},$$
$$G = \mathfrak{A} - \tfrac{6}{12}\mathfrak{B} + \frac{15}{12 \cdot 11}\mathfrak{C} - \frac{20}{12 \cdot 11 \cdot 10}\mathfrak{D} + \frac{15}{12 \cdot 11 \cdot 10 \cdot 9}\mathfrak{E} - \frac{6}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}\mathfrak{F}$$
$$+ \frac{1}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{G},$$
etc.

Quibus valoribus inventis, erit pro duplici ordine,

si	si
$m = 0, \quad z = A,$	$m = 1, \ (x+y)\ z = A;$
$m = -1, \quad z = B,$	$m = 2, \ (x+y)^3\ z = B,$
$m = -2, \quad z = C,$	$m = 3, \ (x+y)^5\ z = C,$
$m = -3, \quad z = D,$	$m = 4, \ (x+y)^7\ z = D,$
$m = -4, \quad z = E,$	$m = 5, \ (x+y)^9\ z = E,$
$m = -5, \quad z = F,$	$m = 6, \ (x+y)^{11}\ z = F,$
$m = -6, \quad z = G,$	$m = 7, \ (x+y)^{13}\ z = G,$
etc.	etc.

Scholion.

329. Si pro i sumatur numerus negativus, expressio in infinitum excurrit. Sit enim $i = -k$, et ex formula prima erit $m = k$, ideoque

$$z = \mathfrak{A} - \frac{k}{2k}\mathfrak{B} + \frac{1}{2} \cdot \frac{k(k+1)}{2k(2k+1)}\mathfrak{C} - \frac{1}{6} \cdot \frac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D} + \text{etc.}$$

Pro eodem autem casu $m = k$ altera forma ob $i = k - 1$ dat

$$(x+y)^{2k-1} z = \mathfrak{A} - \frac{(k-1)}{2k-2} \mathfrak{B} + \frac{1}{2} \cdot \frac{(k-1)(k-2)}{(2k-2)(2k-3)} \mathfrak{C}$$
$$- \frac{1}{6} \cdot \frac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)} \mathfrak{D} + \text{etc.}$$

quae autem formae non absolute aequales sunt censendae, sed in altera functiones $f:x$ et $F:y$ alias formas habebunt, ut nihilominus ambae aeque satisfaciant. Casu quidem $k = \frac{1}{2}$, ambae conveniunt perfecte: ponamus autem $k = 0$, ut prior det

$$z = \mathfrak{A} = f:x + F:y,$$

at posterior praebet

$$\frac{z}{x+y} = \mathfrak{A} - \frac{1}{2} \mathfrak{B} + \frac{1}{6} \mathfrak{C} - \frac{1}{24} \mathfrak{D} + \frac{1}{120} \mathfrak{E} - \text{etc.}$$

Quarum consensus ut appareat, sit in hac posteriori

$$f:x = ax^3 \text{ et } F:y = by^2, \text{ erit}$$
$$\mathfrak{A} = ax^3 + by^2, \quad \mathfrak{B} = (x+y)(3axx + 2by),$$
$$\mathfrak{C} = (x+y)^2 (6ax + 2b), \quad \mathfrak{D} = (x+y)^3\, 6a,$$

at reliquae partes evanescunt. Obtinebimus ergo ex posteriori

$$z = (x+y)(ax^3 + by^2) - \frac{1}{2}(x+y)^2 (3axx + 2by)$$
$$+ \frac{1}{3}(x+y)^3 (3ax + b) - \frac{1}{4}(x+y)^4 a,$$

quae evoluta praebet

$$\frac{1}{4} ax^4 - ay^4 + \frac{1}{3} bx^3 + \frac{1}{3} by^3 = z,$$

quae forma utique in priori $z = f:x + F:y$ continetur. Consensus ergo binarum illarum formarum generalium eo magis est notatu dignus.

Problema 53.

330. Invenire casus, quibus haec aequatio generalis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - QQ \left(\frac{\partial\partial z}{\partial x^2}\right) + R \left(\frac{\partial z}{\partial y}\right) + S \left(\frac{\partial z}{\partial x}\right) + Tz = 0.$$

ad formam praecedentem reduci, ideoque iisdem casibus integrari potest.

Solutio.

Introducendo binas novas variabiles t et u, ut sit quemadmodum reductio §. 319. adhibita, ubi $P = 0$ et $V = 0$, declarat

$$t = \int p\,(\partial x + Q\partial y) \text{ et } u = \int q\,(\partial x - Q\partial y),$$

si ponamus ad abbreviandum

$$M = S + QR + \left(\frac{\partial Q}{\partial y}\right) + Q\left(\frac{\partial Q}{\partial x}\right),$$

$$N = S - QR - \left(\frac{\partial Q}{\partial y}\right) + Q\left(\frac{\partial Q}{\partial x}\right),$$

prodibit haec aequatio

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{M}{4QQq}\left(\frac{\partial z}{\partial t}\right) - \frac{N}{4QQp}\left(\frac{\partial z}{\partial u}\right) - \frac{T}{4QQpq}z = 0,$$

quam ergo ad hanc formam revocari oportet

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) + \frac{n}{(t+u)^2}z = 0,$$

cujus casus integrabilitatis ante designavimus; scilicet quoties fuerit $n = (m+i)(m-i-1)$, denotante i numerum integrum quemcunque positivum cyphra non exclusa. Ad hoc ergo necesse est ut fiat

$$M = \frac{-4mQQq}{t+u},\quad N = \frac{-4mQQp}{t+u} \text{ et } T = \frac{-4nQQpq}{(t+u)^2}.$$

Quia autem hic integrabilitatis formularum t et u ratio haberi debet, sumamus $Q = \frac{\Phi':y}{\pi':x}$, sitque

$$p = a\pi':x \text{ et } q = b\pi':x,$$

eritque

$$t = a\pi:x + a\Phi:y \text{ et } u = b\pi:x - b\Phi:y.$$

Hinc si

$$M + N = 2S + 2Q\left(\frac{\partial Q}{\partial x}\right) = \frac{-4m(a+b)QQ\pi':x}{t+u} \text{ et}$$

$$M - N = 2QR + 2\left(\frac{\partial Q}{\partial y}\right) = \frac{4m(a-b)QQ\pi':x}{t+u},$$

ideoque

$$R = \frac{2m(a-b)Q\pi':x}{t+u} - \frac{1}{Q}\left(\frac{\partial Q}{\partial y}\right),$$
$$S = \frac{-2m(a+b)QQ\pi':x}{t+u} - Q\left(\frac{\partial Q}{\partial x}\right), \text{ et}$$
$$T = \frac{-4nabQQ\pi':x.\pi':x}{(t+u)^2} = \frac{-4nab\phi':y.\phi':y}{(t+u)^2},$$

ob $Q = \frac{\phi':y}{\pi':x}$; unde est

$$\left(\frac{\partial Q}{\partial y}\right) = \frac{\phi'':y}{\pi':x} \text{ et } \left(\frac{\partial Q}{\partial x}\right) = \frac{-\pi'':x.\phi':y}{\pi':x.\pi':x} \text{ et}$$
$$t+u = (a+b)\,\pi:x + (a-b)\,\phi:y.$$

Ideoque habebimus

$$R = \frac{2m(a-b)\phi':y}{t+u} - \frac{\phi'':y}{\phi':y} \text{ et}$$
$$\frac{S}{QQ} = \frac{-2m(a+b)\pi':x}{t+u} + \frac{\pi'':x}{\pi':x}.$$

Quo aequatio fiat simplicior, duo casus praecipue sunt considerandi, alter ubi $b = a$, alter ubi $b = -a$. Priori est $t+u = 2a\pi:x$ aequatio nostra erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\phi':y}{\pi':x}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{\phi'':y}{\phi':y}\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\phi':y}{\pi':x}\right)^2\left(\frac{\pi'':x}{\pi':x} - \frac{2m\pi':x}{\pi:x}\right)\left(\frac{\partial z}{\partial x}\right)$$
$$- \frac{n\phi':y.\phi':y}{\pi:x.\pi:x}\,z = 0.$$

Altero vero casu $b = -a$ fit $t+u = 2a\phi:y$ et

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\phi':y}{\pi':x}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(\frac{2m\phi':y}{\phi:y} - \frac{\phi'':y}{\phi':y}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\phi':y}{\pi':x}\right)^2 . \frac{\pi'':x}{\pi':x}\left(\frac{\partial z}{\partial x}\right)$$
$$+ \frac{n\phi':y.\phi':y}{\phi:y.\phi:y}\,z = 0,$$

quae ambae aequationes integrationem admittunt casibus

$$n = (m+i)\,(m-i-1).$$

Corollarium 1.

331. Aequationes postremo inventae a se invicem non differunt, nisi quod binae variabiles x et y invicem permutantur, unde sufficit alterutram solam considerasse. Prior autem transformatur ponendo

$$t = \pi:x + \phi:y \text{ et } u = \pi:x - \phi:y,$$

posterior vero ponendo

$$t = \pi : x + \Phi : y \text{ et } u = \Phi : y - \pi : x.$$

Corollarium 2.

332. Hae aequationes etiam sequenti forma magis perspicua repraesentari possunt, prior quidem

$$\frac{1}{(\Phi' : y)^2}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{(\pi' : x)^2}\left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{\Phi'' : y}{(\Phi' : y)^3}\left(\frac{\partial z}{\partial y}\right)$$
$$+ \left(\frac{\pi'' : y}{(\pi' : x)^3} - \frac{2m}{\pi x . \pi' : x}\right)\left(\frac{\partial z}{\partial x}\right) - \frac{n}{(\pi : x)^2} z = 0,$$

et posterior

$$\frac{1}{(\Phi' : y)^2}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{(\pi' : x)^2}\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(\frac{2m}{\Phi : y . \Phi' : y} - \frac{\Phi'' : y}{(\Phi' : y)^3}\right)\left(\frac{\partial z}{\partial y}\right)$$
$$+ \frac{\pi'' : x}{(\pi' : x)^3}\left(\frac{\partial z}{\partial x}\right) + \frac{n}{(\Phi : y)^2} z = 0.$$

Casus 1.

333. Ponamus $\pi' : x = a$, et $\Phi' : y = b$, erit $\pi : x = ax$ et $\Phi : y = by$ tum vero $\pi'' : x = 0$ et $\Phi'' : y = 0$; unde forma prior prodibit

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aa}\left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{aax}\left(\frac{\partial z}{\partial x}\right) - \frac{n}{aaxx} z = 0,$$

quae reducitur ad formam supra resolutam ponendo

$$t = ax + by \text{ et } u = ax - by.$$

Posterior vero forma est

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aa}\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2m}{bby}\left(\frac{\partial z}{\partial y}\right) + \frac{n}{bbyy} z = 0,$$

quae reducitur ad formam supra resolutam ponendo

$$t = ax + by \text{ et } u = by - ax,$$

utraque autem est integrabilis casu

$$n = (m + i)(m - i - 1),$$

Reductione enim ad variabiles t et u facta oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) + \frac{n}{(t+u)^2} z = 0.$$

Corollarium 1.

334. Si sumatur $n=0$, hae ambae aequationes

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right)-\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{2m}{x}\left(\frac{\partial z}{\partial x}\right)=0, \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{bb}{aa}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m}{y}\left(\frac{\partial z}{\partial y}\right)=0,$$

sunt integrabiles, quoties m fuerit numerus integer, ideoque $2m$ numerus par.

Corollarium 2.

335. En ergo aequationes ob simplicitatem notatu dignas, ex tribus tantum terminis constantes, quae infinitis casibus integrationem admittunt. Integrale autem quovis casu facile exhibetur ex §. 328, si modo ibi loco x et y scribatur t et u.

Casus 2.

336. Sit $\pi':x=ax^{\mu}$ et $\Phi':y=b$, erit

$$\pi:x=\frac{1}{\mu+1}ax^{\mu+1} \text{ et } \Phi:y=by,$$

tum vero

$$\pi'':x=\mu ax^{\mu-1} \text{ et } \Phi'':y=0.$$

Unde forma prior provenit

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial x}{\partial x^2}\right)+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ad formam supra resolutam ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+by \text{ et } u=\frac{1}{\mu+1}ax^{\mu+1}-by.$$

Posterior vero forma fit

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m}{bby}\left(\frac{\partial z}{\partial y}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)+\frac{n}{bbyy}z=0,$$

cujus reductio absolvitur ponendo

$$t = \frac{1}{\mu+1} ax^{\mu+1} + by \text{ et } u = by - \frac{1}{\mu+1} ax^{\mu+1}.$$

Haeque ambae aequationes integrationem admittunt, quoties fuerit $n = (m+i)(m-i-1)$.

Corollarium 1.

337. Ex priori forma casus maxime notabilis existit, si capiatur $m = \frac{\mu}{2\mu+2}$, et $n = 0$, tum enim erit

$$\frac{aa}{bb} x^{2\mu} \left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right),$$

quae est integrabilis, quoties $\frac{\mu}{2\mu+2}$ fuerit numerus integer m sive positivus sive negativus.

Corollarium 2.

338. Vel cum sit $\mu = \frac{-2m}{2m-1}$, haec aequatio

$$\frac{aa}{bb} x^{\frac{-4m}{2m-1}} \left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right), \text{ seu } \left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}} \left(\frac{\partial\partial z}{\partial x^2}\right),$$

erit integrabilis, quoties m fuerit numerus integer sive positivus sive negativus, reductio autem fit ponendo

$$t = -(2m-1)\, ax^{\frac{-1}{2m-1}} + by \text{ et}$$

$$u = -(2m-1)\, ax^{\frac{-1}{2m-1}} - by.$$

Casus 3.

339. Sit $\pi' : x = ax^{\mu}$ et $\Phi' : y = by^{\nu}$, erit

$$\pi : x = \frac{1}{\mu-1} ax^{\mu+1} \text{ et } \Phi : y = \frac{1}{\nu+1} by^{\nu+1},$$

tum vero

$$\pi'' : x = \mu ax^{\mu-1} \text{ et } \Phi'' : y = \nu by^{\nu-1}.$$

Hinc prior forma resultat

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ponendo

$$t=\tfrac{1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1} \text{ et}$$
$$u=\tfrac{1}{\mu+1}ax^{\mu+1}-\tfrac{1}{\nu+1}by^{\nu+1}.$$

Posterior vero forma evadit

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m\nu+2m-\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$+\frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)+\frac{n(\nu+1)^2}{bby^{2\nu+2}}z=0.$$

cujus reductio fit hac substitutione

$$t=\tfrac{1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1} \text{ et}$$
$$u=\tfrac{-1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}.$$

Vel cum hic tantum ratio inter a et b in computum ingrediatur, pro priori poni poterit

$$t=\tfrac{1}{2}x^{\mu+1}+\tfrac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1} \text{ et}$$
$$u=\tfrac{1}{2}x^{\mu+1}-\tfrac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1},$$

ut fiat $t+u=x^{\mu+1}$, quo expressio integralis fiat simplicior.

Corollarium 1.

340. Si ponatur in forma priori $\mu=\frac{-2m}{2m-1}$ minuetur ea uno termino, fietque

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aa}x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$

$$-\frac{n}{(2m-1)^2aa}x^{\frac{2}{2m-1}}z=0.$$

Statuatur $a=b$, et capiatur quoque $\nu=\frac{-2m}{2m-1}$, ut prodeat

$$y^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial y^2}\right)-x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m}{2m-1}y^{\frac{2m+1}{2m-1}}\left(\frac{\partial z}{\partial y}\right)$$

$$-\frac{n}{(2m-1)^2}x^{\frac{2}{2m-1}}z=0.$$

Corollarium 2.

341. Sumatur porro in priori forma $\nu=\mu$, ut fiat $\mu-2m\mu-2m=-\mu$, seu $m=\frac{\mu}{\mu+1}$, ut prodeat

$$\frac{1}{bby^{2\mu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\mu}{bby^{2\mu+1}}\left(\frac{\partial z}{\partial y}\right)$$

$$-\frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+1}}z=0,$$

quae integrabilis existit, quoties fuerit

$$n=-\frac{[\mu+(\mu+1)i]\,[(\mu+1)i+1]}{(\mu+1)^2},\text{ seu}$$

$$n=-\left(i+\frac{\mu}{\mu+1}\right)\left(i+\frac{1}{\mu+1}\right).$$

Scholion.

342. Largissima ergo hinc nobis suppeditatur copia aequationum satis concinnarum, quas ope methodi hic traditae integrare licet. Atque hic imprimis duo casus conspiciuntur, quorum alter

$$\left(\frac{\partial\partial z}{\partial y^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)$$

pro motu cordarum inaequali crassitie praeditarum determinando est inventus, alter autem hac aequatione

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{x}\left(\frac{\partial z}{\partial x}\right) = 0$$

contentus ideo est memorabilis, quod in analysi pro soni propagatione instituta ad talem formam pervenitur. Hae igitur binae aequationes prae caeteris merentur, ut pro casibus integrabilitatis integralia exhibeamus.

Problema 54.

343. Propositia aequatione differentiali

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{x}\left(\frac{\partial z}{\partial x}\right) = 0,$$

casibus quibus m est numerus integer sive positivus sive negativus, ejus integrale completum exhibere.

Solutio.

Facta substitutione $t = \frac{1}{2}x + \frac{b}{2a}y$ et $u = \frac{1}{2}x - \frac{b}{2a}y$, aequatio nostra hanc induit formam

$$\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) = 0.$$

Cum igitur sit $t + u = x$, si ponamus

$$\mathfrak{A} = \mathrm{f}:\frac{ax+by}{2a} + \mathrm{F}:\frac{ax-by}{2a},$$

$$\mathfrak{B} = x\left(\mathrm{f}':\frac{ax+by}{2a} + \mathrm{F}':\frac{ax-by}{2a}\right),$$

$$\mathfrak{C} = x^2\left(\mathrm{f}'':\frac{ax+by}{2a} + \mathrm{F}'':\frac{ax-by}{2a}\right),$$

$$\mathfrak{D} = x^3\left(\mathrm{f}''':\frac{ax+by}{2a} + \mathrm{F}''':\frac{ax-by}{2a}\right),$$

$$\mathfrak{E} = x^4\left(\mathrm{f}^{\mathrm{IV}}:\frac{ax+by}{2a} + \mathrm{F}^{\mathrm{IV}}:\frac{ax-by}{2a}\right),$$

$$\mathfrak{F} = x^5\left(\mathrm{f}^{\mathrm{V}}:\frac{ax+by}{2a} + \mathrm{F}^{\mathrm{V}}:\frac{ax-by}{2a}\right), \text{ etc.}$$

casus integrabiles ita se habebunt, primo negativi

si $m = 0$; $z = \mathfrak{A}$,

si $m = -1$; $z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$,

si $m = -2$; $z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4.3}\mathfrak{C}$,

si $m = -3$; $z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{1}{6.5}\mathfrak{C} - \frac{1}{6.5.4}\mathfrak{D}$,

si $m = -4$; $z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8.7}\mathfrak{C} - \frac{4}{8.7.6}\mathfrak{D} + \frac{1}{8.7.6.5}\mathfrak{E}$,

si $m = -5$; $z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10.9}\mathfrak{C} - \frac{10}{10.9.8}\mathfrak{D} + \frac{5}{10.9.8.7}\mathfrak{E} - \frac{1}{10.9.8.7.6}\mathfrak{F}$, etc.

Tum vero pro valoribus positivis ipsius m

si $m = 1$; $xz = \mathfrak{A}$,

si $m = 2$; $x^3 z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$,

si $m = 3$; $x^5 z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4.3}\mathfrak{C}$,

si $m = 4$; $x^7 z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6.5}\mathfrak{C} - \frac{1}{6.5.4}\mathfrak{D}$,

si $m = 5$; $x^9 z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8.7}\mathfrak{C} - \frac{1}{8.7.6}\mathfrak{D} + \frac{1}{8.7.6.5}\mathfrak{E}$,

si $m = 6$; $x^{11} z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10.9}\mathfrak{C} - \frac{10}{10.9.8}\mathfrak{D} + \frac{5}{10.9.8.7}\mathfrak{E} - \frac{1}{10.9.8.7.6}\mathfrak{F}$, etc.

Cui ergo expressioni casu $m = -i$ aequatur valor z, eidem aequatur casu $m = i + 1$ valor ipsius $x^{2i+1} z$.

Scholion.

344. Valores ipsarum t et u ita hic assumsi, ut fieret $t + u = x$, atque eosdem valores quoque in functionibus adhiberi oportet. Etsi enim $f : \frac{ax+by}{2a}$ etiam est functio ipsius $ax + by$, tamen functiones per differentiationem inde derivatae discrepant. Namque si ponamus

$$f : \frac{ax+by}{2a} = \phi : (ax + by),$$

erit differentiando

$$\frac{(a\partial x + b\partial y)}{2a} f' : \left(\frac{ax+by}{2a}\right) = (a\partial x + b\partial y)\, \Phi' : (ax + by),$$

unde erit

$$f' : \frac{ax+by}{2a} = 2a\Phi' : (ax + by),$$

neque ergo hae functiones differentiales sunt aequales etiamsi principales assumtae sint aequales, simili modo erit

$$f'' : \frac{ax+by}{2a} = 4aa\Phi'' : (ax + by), \text{ et}$$

$$f''' : \frac{ax+by}{2a} = 8a^3\Phi''' : (ax + by), \text{ etc.}$$

et ita porro.

Problema 55.

345. Proposita aequatione differentiali

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{4m}{2m-1}} \left(\frac{\partial\partial z}{\partial x^2}\right),$$

casibus quibus m est numerus integer sive positivus sive negativus, integrale completum exhibere.

Solutio.

Introductis novis variabilibus t et u, ita ut sit

$$t = \tfrac{1}{2}\, x^{\frac{-1}{2m-1}} - \frac{b}{2(2m-1)a}\, y \text{ et } u = \tfrac{1}{2}\, x^{\frac{-1}{2m-1}} + \frac{b}{2(2m-1)a}\, y,$$

aequatio nostra hanc induit formam

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) = 0,$$

ubi est

$$t + u = x^{\frac{-1}{2m-1}}.$$

Posito igitur

$$\mathfrak{A} = f : t + F : u, \quad \mathfrak{B} = x^{\frac{-1}{2m-1}} (f' : t + F' : u),$$

$$\mathfrak{C} = x^{\frac{-2}{2m-1}} (f'' : t + F'' : u), \quad \mathfrak{D} = x^{\frac{-3}{2m-1}} (f''' : t + F'' : u),$$

$$\mathfrak{E} = x^{\frac{-4}{2m-1}} (f^{IV} : t + F^{IV} : u), \quad \mathfrak{F} = x^{\frac{-5}{2m-1}} (f^{V} : t + F^{V} : u), \text{ etc.}$$

percurramus primo casus, quibus m a cyphra per numeros negativos decrescit.

I. Si $m = 0$, aequationis

$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} \left(\frac{\partial\partial z}{\partial x^2}\right)$ integrale

$$z = f : (\tfrac{1}{2} x + \tfrac{b}{2a} y) + F : (\tfrac{1}{2} x - \tfrac{b}{2a} y).$$

II. Si $m = -1$, ob

$$t = \tfrac{1}{2} x^{\frac{1}{3}} + \tfrac{b}{6a} y \text{ et } u = \tfrac{1}{2} x^{\frac{1}{3}} - \tfrac{b}{6a} y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{4}{3}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = f : t + F : u - \tfrac{1}{2} x^{\frac{1}{3}} (f' : t + F' : u).$$

III. Si $m = -2$, ob

$$t = \tfrac{1}{2} x^{\frac{1}{5}} + \tfrac{b}{10a} y \text{ et } u = \tfrac{1}{2} x^{\frac{1}{5}} - \tfrac{b}{10a} y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{8}{5}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = f : t + F : u - \tfrac{2}{4} x^{\frac{1}{5}} (f' : t + F' : u) + \tfrac{1}{4.3} x^{\frac{1}{5}} (f'' : t + F'' : u).$$

IV. Si $m = -3$, ob

$$t = \tfrac{1}{2} x^{\frac{1}{7}} + \tfrac{b}{14a} y \text{ et } u = \tfrac{1}{2} x^{\frac{1}{7}} - \tfrac{b}{14a} y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{12}{7}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = \mathrm{f}:t + \mathrm{F}:u - \tfrac{3}{6}\, x^{\frac{1}{7}} (\mathrm{f}':t + \mathrm{F}':u) + \frac{3}{6.5}\, x^{\frac{2}{7}} (\mathrm{f}'':t + \mathrm{F}'':u)$$

$$- \frac{1}{6.5.4}\, x^{\frac{3}{7}} (\mathrm{f}''':t + \mathrm{F}''':u),$$

V. Si $m = -4$, ob

$$t = \tfrac{1}{2} x^{\frac{1}{9}} + \frac{b}{18a} y \text{ et } u = \tfrac{1}{2} x^{\frac{1}{9}} - \frac{b}{18a} y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{16}{9}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = \mathrm{f}:t + \mathrm{F}:u - \tfrac{4}{8}\, x^{\frac{1}{9}} (\mathrm{f}':t + \mathrm{F}':u) + \frac{6}{8.7}\, x^{\frac{2}{9}} (\mathrm{f}'':t + \mathrm{F}'':u)$$

$$- \frac{4}{8.7.6}\, x^{\frac{3}{9}} (\mathrm{f}''':t + \mathrm{F}''':u) + \frac{1}{8.7.6.5}\, x^{\frac{4}{9}} (\mathrm{f}^{\mathrm{IV}}:t + \mathrm{F}^{\mathrm{IV}}:u),$$

et ita porro.

Pro altero vero casu ubi m habet valores positivos, integralia sequenti modo exprimentur

I. Si sit $m = 1$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^4 \left(\frac{\partial\partial z}{\partial x^2}\right)$,
ob $t = \tfrac{1}{2} x^{-1} - \frac{b}{2a} y$ et $u = \tfrac{1}{2} x^{-1} + \frac{b}{2a} y$,
erit integrale

$$x^{-1} z = \mathrm{f}:t + \mathrm{F}:u, \text{ seu } z = x(\mathrm{f}:t + \mathrm{F}:u).$$

II. Si sit $m = 2$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{8}{3}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \tfrac{1}{2} x^{-\frac{1}{3}} - \frac{b}{6a} y$ et $u = \tfrac{1}{2} x^{-\frac{1}{3}} + \frac{b}{6a} y$,

erit integrale

$$z = x(\mathrm{f}:t + \mathrm{F}:u) - \tfrac{1}{2} x^{\frac{2}{3}} (\mathrm{f}':t + \mathrm{F}':u).$$

III. Si sit $m = 3$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{12}{5}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{5}{5}} - \frac{b}{10a} y$ et $u = \frac{1}{2} x^{-\frac{5}{5}} + \frac{b}{10a} y$,

erit integrale

$$z = x(\text{f}:t + \text{F}:u) - \frac{2}{4} x^{\frac{4}{5}} (\text{f}':t + \text{F}':u) + \frac{1}{4.3} x^{\frac{3}{5}} (\text{f}'':t + \text{F}'':u).$$

IV. Si sit $m = 4$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{16}{7}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{7}{7}} - \frac{b}{14a} y$ et $u = \frac{1}{2} x^{-\frac{7}{7}} + \frac{b}{14a} y$,

erit integrale

$$z = x(\text{f}:t + \text{F}:u) - \frac{3}{6} x^{\frac{6}{7}} (\text{f}':t + \text{F}':u) + \frac{3}{6.5} x^{\frac{5}{7}} (\text{f}'':t + \text{F}'':u)$$
$$- \frac{1}{6.5.4} x^{\frac{4}{7}} (\text{f}''':t + \text{F}''':u).$$

V. Si sit $m = 5$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{20}{9}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{9}{9}} - \frac{b}{18a} y$ et $u = \frac{1}{2} x^{-\frac{9}{9}} + \frac{b}{18a} y$,

erit integrale

$$z = x(\text{f}:t + \text{F}:u) - \frac{4}{8} x^{\frac{8}{9}} (\text{f}':t + \text{F}':u) + \frac{6}{8.7} x^{\frac{7}{9}} (\text{f}'':t + \text{F}'':u)$$
$$- \frac{4}{8.7.6} x^{\frac{6}{9}} (\text{f}''':t + \text{F}''':u) + \frac{1}{8.7.6.5} x^{\frac{5}{9}} (\text{f}^{\text{IV}}:t + \text{F}^{\text{IV}}:u). \text{ etc.}$$

unde lex, qua has expressiones ulterius continuare licet, per se est manifesta.

Scholion 1.

346. Casus isti integrabilitatis congruunt cum iis, qui in aequatione *Riccatiana* dicta deprehenduntur, novimus scilicet aequa-

tionem hanc

$$\partial y + yy\partial x = ax^{\frac{-4m}{2m-1}}\partial x$$

integrari posse quoties m est numerus integer sive positivus sive negativus. Haec autem aequatio haud levi vinculo cum nostra forma est connexa, quod ita ostendi potest. Proposita forma generali

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = X\left(\frac{\partial\partial z}{\partial x^2}\right),$$

pro integralibus particularibus inveniendis statuatur $z = e^{\alpha y}v$, ut v sit functio ipsius x tantum, erit

$$\left(\frac{\partial z}{\partial x}\right) = e^{\alpha y}\cdot\frac{\partial v}{\partial x} \text{ et } \left(\frac{\partial\partial z}{\partial x^2}\right) = e^{\alpha y}\cdot\frac{\partial\partial v}{\partial x^2};$$

tum vero $\left(\frac{\partial\partial z}{\partial y^2}\right) = \alpha\alpha e^{\alpha y}v$; unde prodit haec aequatio $\alpha\alpha v = \frac{X\partial\partial v}{\partial x^2}$; in qua si porro statuatur $v = e^{\int p\partial x}$, oritur

$$\frac{\alpha\alpha\partial x}{X} = \partial p + pp\partial x,$$

ac si $X = Ax^{\frac{4m}{2m-1}}$, ut in nostro casu, haec aequatio sit

$$\partial p + pp\partial x = ax^{\frac{-4m}{2m-1}}\partial x.$$

Haud temere igitur evenire putandum est, quod utraque aequatio iisdem casibus integrationem admittat. Interim tamen notatu dignum occurit, quod casus $m = \infty$, qui in forma Riccatiana fit facillimus, idem in nostra aequatione neutiquam integrationem admittat. Habetur quippe haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}xx\left(\frac{\partial\partial z}{\partial x^2}\right),$$

cujus reductio modo supra §. 330. adhibito non succedit. Nam ob

$$Q = \frac{bx}{a},\ R = 0,\ S = 0 \text{ et } T = 0,$$

pro novis variabilibus ponitur

$$t = \int p\left(\partial x + \frac{bx\partial y}{a}\right) \text{ et } u = \int q\left(\partial x - \frac{bx\partial y}{a}\right);$$

unde ob $M = \frac{bbx}{aa} = N$, oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) - \frac{1}{4qx}\left(\frac{\partial z}{\partial t}\right) - \frac{1}{4px}\left(\frac{\partial z}{\partial u}\right) = 0,$$

quae sumendo

$$p = \frac{1}{x} \text{ et } q = \frac{1}{x},$$

ut sit

$$t = lx + \frac{by}{a} \text{ et } u = lx - \frac{by}{a},$$

transit in

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) - \frac{1}{4}\left(\frac{\partial z}{\partial t}\right) - \frac{1}{4}\left(\frac{\partial z}{\partial u}\right) = 0,$$

cujus integratio haud perspicitur.

Scholion 2.

347. Aequationis autem $\left(\frac{\partial\partial z}{\partial y^2}\right) = xx\left(\frac{\partial\partial z}{\partial x^2}\right)$ integralia particularia infinita exhibere licet, in hac forma $z = Ax^\lambda e^{\mu y}$ contenta. Cum enim hinc sit

$$\left(\frac{\partial z}{\partial y}\right) = \mu A x^\lambda e^{\mu y} \text{ et } \left(\frac{\partial z}{\partial x}\right) = \lambda A x^{\lambda-1} e^{\mu y}, \text{ erit}$$

$$\mu\mu A x^\lambda e^{\mu y} = \lambda(\lambda-1) A x^\lambda e^{\mu y}, \text{ ideoque}$$

$\mu = \sqrt{\lambda(\lambda-1)}$, unde ex quovis numero pro λ assumto bini valores pro μ oriuntur, ita ut habeatur

$$z = Ax^\lambda e^{y\sqrt{\lambda(\lambda-1)}} + Bx^\lambda e^{-y\sqrt{\lambda(\lambda-1)}},$$

et hujusmodi membrorum numerus variando λ in infinitum multiplicari potest Interim tamen singula haec membra adhuc generaliora reddi possunt. Posito enim $z = x^\lambda e^{\mu y} v$, videamus an v necessario constans esse debeat: hinc autem fit

$$\left(\frac{\partial z}{\partial y}\right) = \mu x^\lambda e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{\partial v}{\partial y}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial x}\right) = \lambda x^{\lambda-1} e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{\partial v}{\partial x}\right),$$

ideoque nostra aequatio praebet per $x^\lambda e^{\mu y}$ divisa

$$\mu\mu v + 2\mu \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial\partial v}{\partial y^2}\right)$$
$$= \lambda(\lambda-1)v + 2\lambda x\left(\frac{\partial v}{\partial x}\right) + xx\left(\frac{\partial\partial v}{\partial x^2}\right).$$

Statuatur ut ante $\mu\mu = \lambda(\lambda-1)$, sitque $v = \alpha lx + \beta y$, erit

$$2\beta\mu = 2\alpha\lambda - \alpha, \text{ seu } \frac{\alpha}{\beta} = \frac{2\mu}{2\lambda-1} = \frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1};$$

unde cujusque membri ex numero λ nati forma erit

$$z = x^\lambda \left\{ \begin{array}{l} e^{y\sqrt{\lambda(\lambda-1)}}\left(A + \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}} lx + \frac{2\lambda-1}{\mathfrak{A}} y\right) \\ + e^{-y\sqrt{\lambda(\lambda-1)}}\left(B - \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{B}} lx + \frac{2\lambda-1}{\mathfrak{B}} y\right) \end{array} \right\}.$$

Quomodocunque igitur non solum exponens λ sed etiam quantitates A, $\mathfrak{A}$, B, $\mathfrak{B}$ varientur, infinita hujusmodi membra formari possunt, quae omnia junctim sumta valorem completum functionis z praebere sunt censenda. Quin etiam pro λ imaginaria assumi possunt, posito enim

$$\lambda = a + b\sqrt{-1} \text{ fit } \mu = p + q\sqrt{-1},$$

existente

$$pp - qq = aa - a - bb \text{ et}$$
$$pp + qq = \sqrt{(aa+bb)(aa-2a+1+bb)},$$

tum vero est

$$x^\lambda = a^a (\cos. blx + \sqrt{-1} . \sin. blx) \text{ et}$$
$$e^{\mu y} = e^{py} (\cos. qy + \sqrt{-1} . \sin. qy),$$

unde colligitur forma realis

$$z = x^a e^{py} \left\{ \begin{array}{l} A\cos.(blx+qy) + B\,[2plx + (2a-1)y]\cos.(blx+qy) \\ \qquad - B(2qlx + 2by)\sin.(blx+qy) \\ \mathfrak{A}\sin.(blx+qy) + \mathfrak{B}\,[2plx + (2a-1)y]\sin.(blx+qy) \\ \qquad + \mathfrak{B}(2qlx + 2by)\cos.(blx+qy) \end{array} \right\},$$

ubi quantitates a et b pro lubitu assumere licet, unde simul p et q

definiuntur. Quodsi hic litteras b et q ut datas spectemus, binae reliquae a et p ex iis ita determinentur, ut sit

$$2a - 1 = q\sqrt{\left(\frac{1}{qq-bb} - 4\right)} \text{ et } p = \frac{b}{2}\sqrt{\left(\frac{1}{qq-bb} - 4\right)},$$

hic ergo necesse est sit $qq > bb$ et $qq < bb + \frac{1}{4}$, seu qq inter hos arctos limites bb et $bb + \frac{1}{4}$ contineri debet; statuatur $q = c$ et $\sqrt{\left(\frac{1}{qq-bb} - 4\right)} = 2f$, ut sit

$$\frac{1}{qq-bb} = 4(1+ff), \text{ seu } cc - bb = \frac{1}{4(1+ff)},$$

atque $2a - 1 = 2cf$ et $p = bf$,

ex quo forma integralium particularium erit

$$z = x^{cf+\frac{1}{2}} e^{bfy} \left\{ \begin{array}{r} \mathrm{A}\cos(blx+cy) + 2\mathrm{B}f(blx+cy)\cos.(blx+cy) \\ -2\mathrm{B}(clx+by)\sin.(blx+cy) \\ \mathfrak{A}\sin.(blx+cy) + 2\mathfrak{B}f(blx+cy)\sin.(blx+cy) \\ +2\mathfrak{B}(clx+by)\cos.(blx+cy) \end{array} \right\},$$

quae posito brevitatis gratia angulo $blx + cy = \Phi$ transformatur in hanc

$$z = x^{cf+\frac{1}{2}} e^{bfy} \left\{ \begin{array}{r} \mathrm{A}\cos.(\Phi+\alpha) + \mathrm{B}f(blx+cy)\sin.(\Phi+\beta) \\ + \mathrm{B}(clx+by)\cos.(\Phi+\beta) \end{array} \right\};$$

ubi quantitates b, c, A, B, α, β ab arbitrio nostro pendent.

Scholion 3.

343. Resolutio ergo aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = xx\left(\frac{\partial\partial z}{\partial x^2}\right),$$

ita institui potest, ut fingatur

$$z = x^\lambda e^{\mu y}(mlx + ny),$$

unde fit

$$\left(\frac{\partial z}{\partial x}\right) = \lambda x^{\lambda-1} e^{\mu y}(mlx+ny) + mx^{\lambda-1} e^{\mu y} \text{ et}$$
$$\left(\frac{\partial z}{\partial y}\right) = \mu x^\lambda e^{\mu y}(mlx+ny) + nx^\lambda e^{\mu y},$$

hincque ulterius differentiando

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = x^{\lambda-2}\, e^{\mu y}\,[m\,(2\lambda-1) + \lambda\,(\lambda-1)\,mlx + \lambda(\lambda-1)ny] \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = x^{\lambda}\, e^{\mu y}\,(2\mu n + \mu\mu\, mlx + \mu\mu\, ny).$$

Ex quo colligitur primo $\mu = \sqrt{\lambda\,(\lambda-1)}$, deinde

$$2n\sqrt{\lambda\,(\lambda-1)} = m\,(2\lambda-1),$$

ut sit

$$\frac{m}{n} = \frac{2\sqrt{\lambda\,(\lambda-1)}}{2\lambda-1},$$

sicque eadem prodit integratio quam modo ante dedimus.

CAPUT V.

TRANSFORMATIO SINGULARIS EARUNDEM AEQUATIONUM.

Problema 56.

349.

Proposita hac aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

in qua P, Q, R sint functiones ipsius x tantum, eam ope substitutionis

$$z = M\left(\frac{\partial v}{\partial x}\right) + Nv,$$

ubi quoque sint M et N functiones ipsius x tantum, in aliam ejusdem formae transmutare ut prodeat

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv,$$

existentibus F, G, H functionibus solius x.

Solutio.

Quia quantitates M et N ab y sunt immunes, erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = M\left(\frac{\partial^3 v}{\partial x\partial y^2}\right) + N\left(\frac{\partial\partial v}{\partial y^2}\right),$$

quae forma per aequationem, quam tandem resultare assumimus, abit in hanc

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = MF\left(\frac{\partial^3 v}{\partial x^3}\right) + \begin{matrix}\frac{M\partial F}{\partial x}\\ + MG\\ + NF\end{matrix}\left(\frac{\partial\partial v}{\partial x^2}\right) + \begin{matrix}\frac{M\partial G}{\partial x}\\ + MH\\ + NG.\end{matrix}\left(\frac{\partial v}{\partial x}\right) + \begin{matrix}\frac{M\partial H}{\partial x}\\ + NH\end{matrix}\, v$$

Deinde vero pro altero aequationis propositae membro nostra substitutio praebet

$$\left(\frac{\partial z}{\partial x}\right) = M\left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{\partial M}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial N}{\partial x}v,$$
$$+ N.$$

hincque porro

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = M\left(\frac{\partial^3 v}{\partial x^3}\right) + \left(\frac{2\partial M}{\partial x} + N\right)\left(\frac{\partial\partial v}{\partial x^2}\right)$$
$$+ \left(\frac{\partial\partial M}{\partial x^2} + \frac{2\partial N}{\partial x^2}\right)\left(\frac{\partial v}{\partial x}\right) + \frac{\partial\partial N}{\partial x^2}v.$$

Cum nunc sit per hypothesin

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

si hic valores modo inventi substituantur, singulaque membra $\left(\frac{\partial^3 v}{\partial x^3}\right)$, $\left(\frac{\partial\partial v}{\partial x^2}\right)$, $\left(\frac{\partial v}{\partial x}\right)$ et v seorsim ad nihilum redigantur, quatuor sequentes aequationes orientur, scilicet

ex	colligitur aequatio
$\left(\frac{\partial^3 v}{\partial x^3}\right)$	$MF = MP$
$\left(\frac{\partial\partial v}{\partial x^2}\right)$	$\frac{M\partial F}{\partial x} + MG + NF = \left(\frac{2\partial M}{\partial x} + N\right)P + MQ$
$\left(\frac{\partial v}{\partial x}\right)$	$\frac{M\partial G}{\partial x} + MH + NG = \left(\frac{\partial\partial M}{\partial x^2} + \frac{2\partial N}{\partial x}\right)P + \left(\frac{\partial M}{\partial x} + N\right)Q + MR$
v	$\frac{M\partial H}{\partial x} + NH = \frac{\partial\partial N}{\partial x^2}P + \left(\frac{\partial N}{\partial x}\right)Q + NR,$

ex quibus commodissime primo quaeruntur P, Q et R. Verum prima dat statim $P = F$, unde secunda fit

$$\frac{M\partial F - 2F\partial M}{M\partial x} + G = Q.$$

Ex binis ultimis autem eliminando R colligitur

$$\frac{M(N\partial G - M\partial H)}{\partial x} + NNG = \left(\frac{N\partial\partial M - M\partial\partial N}{\partial x^2} + \frac{2N\partial N}{\partial x}\right)F$$
$$+ \left(\frac{N\partial M - M\partial N}{\partial x} + NN\right)Q,$$

et illum valorem pro Q substituendo

$$0 = \frac{MM\partial H}{\partial x} - \frac{MN\partial G}{\partial x} + \frac{(N\partial\partial M - M\partial\partial N)}{\partial x^2} F + \frac{2NF\partial N}{\partial x}$$
$$+ \frac{N\partial M - M\partial N}{\partial x} G + \frac{(N\partial M - M\partial N)}{\partial x^2} \partial F + \frac{NN\partial F}{\partial x}$$
$$- \frac{2F\partial M\,(N\partial M - M\partial N)}{M\partial x^2} - \frac{2NNF\partial M}{M\partial x}$$

quae aequatio per $\frac{\partial x}{MM}$ multiplicata commode integrabilis redditur, inveniturque integrale

$$C = H - \frac{N}{M} G + \frac{N\partial M - M\partial N}{MM\partial x} F + \frac{NNF}{MM}.$$

Quod si ergo brevitatis gratia ponamus $N = Ms$, erit

$$C = H - Gs - F\frac{\partial s}{\partial x} + Fss, \text{ seu}$$
$$\partial s + \frac{G}{F} s\partial x - ss\partial x + \frac{(C - H)\partial x}{F} = 0.$$

Sive jam hinc definiatur quantitas $s = \frac{N}{M}$, sive una functionum F, G et H, pro ipsa aequatione proposita litterae P, Q et R, ita determinabuntur, ut sit

I. $P = F$

II. $Q = G + \frac{\partial F}{\partial x} - \frac{2F\partial M}{M\partial x}$,

et ex ultima aequatione derivatur

$$R = H + \frac{M\partial H}{N\partial x} - \frac{F\partial\partial N}{N\partial x^2} - \frac{\partial N}{N\partial x}\left(G + \frac{\partial F}{\partial x} - \frac{2F\partial M}{M\partial x}\right),$$

qui valor ob $N = Ms$ evadit

$$R = H + \frac{\partial H}{s\partial x} - \frac{G\partial s}{s\partial x} - \frac{G\partial M}{M\partial x} - \frac{F\partial\partial s}{s\partial x^2} - \frac{F\partial\partial M}{M\partial x^2}$$
$$+ \frac{2F\partial M^2}{MM\partial x^2} - \frac{\partial F\partial s}{s\partial x^2} - \frac{\partial F\partial M}{M\partial x^2},$$

et cum aequatio inventa, si differentietur, det

$$0 = \partial H - G\partial s - s\partial G - \frac{F\partial\partial s}{\partial x} - \frac{\partial F\partial s}{\partial x} + 2Fs\partial s + ss\partial F,$$

obtinebimus

III. $$R = H - \frac{G\partial M}{M\partial x} + \frac{\partial G}{\partial x} - \frac{F\partial\partial M}{M\partial x^2} - \frac{2F\partial s}{\partial x}$$
$$+ \frac{2F\partial M^2}{MM\partial x^2} - \frac{s\partial F}{\partial x} - \frac{\partial F\partial M}{M\partial x^2},$$

unde si aequatio

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv$$

resolutionem admittat, etiam resolutio succedet hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

cum sit

$$z = M\left(\frac{\partial v}{\partial x}\right) + Nv = M\left[sv + \left(\frac{\partial v}{\partial x}\right)\right].$$

Corollarium 1.

350. Si ponatur $M = 1$, ut fiat $z = sv + \left(\frac{\partial v}{\partial x}\right)$, erit

$$P = F,\ Q = G + \frac{\partial F}{\partial x},\ \text{et}\ R = H + \frac{\partial G}{\partial x} - \frac{2F\partial s - s\partial F}{\partial x},$$

neque hoc modo usus istius reductionis restringitur; quoniam si deinceps loco z ponatur Mz, etiam aequationis hinc ortae resolutio est in promtu.

Corollarium 2.

351. Quoties ergo aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv$$

resolutio est in potestate, toties etiam hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = F\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(G + \frac{\partial F}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) + \left(H + \frac{\partial G}{\partial x} - \frac{2F\partial s - s\partial F}{\partial x}\right)z$$

resolutio succedit, si modo capiatur s ex hac aequatione

$$F\partial s + Gs\partial x - Fss\partial x + (C - H)\partial x = 0,$$

tum enim erit $z = sv + \left(\frac{\partial v}{\partial x}\right)$. Sunt autem litterae F, G, H functiones ipsius x tantum.

Scholion.

352. Haec reductio methodum maxime naturalem suppeditare videtur ejusmodi integrationes perficiendi, quae simul functio-

num differentialia involvunt. Si enim aequationis pro v datae integrale sit $v = \Phi : t$, existente t functione ipsarum x et y, ob

$$\partial v = \partial t \Phi' : t, \text{ erit } \left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial t}{\partial x}\right) \Phi' : t$$

et aequationis inde derivatae pro z habebimus integrale

$$z = s\Phi : t + \left(\frac{\partial t}{\partial x}\right) \Phi' : t.$$

Deinde si fuerit generalius $v = u\Phi : t$, fiet

$$z = su\Phi : t + \left(\frac{\partial u}{\partial x}\right) \Phi : t + u \left(\frac{\partial t}{\partial x}\right) \Phi' : t,$$

unde ratio perspicitur ad ejusmodi aequationes perveniendi, quarum integralia praeter functionem $\Phi : t$ etiam functiones ex ejus differentiatione natas $\Phi' : t$, atque adeo etiam sequentes $\Phi'' : t$, $\Phi''' : t$, etc. complectantur. Quamobrem operae pretium erit hanc reductionem accuratius evolvere.

Problema 57.

353 Concessa resolutione hujus aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{n}{xx} v,$$

invenire aliam aequationem hujus formae

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

pro qua sit

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

Solutio.

Facta comparatione cum praecedente problemate habemus

$$F = 1, \quad G = \frac{m}{x} \quad \text{et} \quad H = \frac{n}{xx},$$

unde quantitatem s ex hac aequatione definiri oportet

$$\partial s + \frac{ms\partial x}{x} - ss\partial x + \left(f - \frac{n}{xx}\right)\partial x = 0,$$

qua inventa ob $\frac{\partial G}{\partial x} = -\frac{m}{xx}$, aequatio quaesita erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{n-m}{xx} - \frac{2\partial s}{\partial x}\right) z,$$

seu loco ∂s valore inde substituto

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \left(2f - \frac{m-n}{xx} + \frac{2ms}{x} - 2ss\right) z,$$

pro qua est

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

I. Ponamus primo quantitatem constantem $f = 0$, ut sit

$$\partial s + \frac{ms\partial x}{x} - ss\partial x - \frac{n\partial x}{xx} = 0,$$

cujus integrale particulare est $s = \frac{\alpha}{x}$, existente

$$-\alpha + m\alpha - \alpha\alpha - n = 0, \quad \text{seu} \quad \alpha\alpha - (m-1)\alpha + n = 0,$$

ex quo ob $\frac{\partial s}{\partial x} = \frac{-\alpha}{xx}$, oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{2\alpha - m + n}{xx} z,$$

pro qua est

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right),$$

seu exclusa $n = \alpha(m - 1 - \alpha)$, si constet resolutio hujus aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m-1-\alpha)}{xx} v,$$

pro hac

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha-1)(m-\alpha)}{xx} z,$$

erit

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right).$$

II. Maneat $f = 0$, et quaeramus pro s valorem completum ponendo $s = \frac{\alpha}{x} + \frac{1}{t}$, fietque ob

$$n = (m-1)\alpha - \alpha\alpha,\quad \partial t + \frac{(2\alpha - m)t\partial x}{x} + \partial x = 0,$$

quae per $x^{2\alpha - m}$ multiplicata et integrata praebet

$$t = \frac{cx^{m-2\alpha}}{2\alpha - m + 1} - \frac{x}{2\alpha - m + 1},$$

hincque

$$s = \frac{\alpha c x^{m-2\alpha-1} + \alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)} = \frac{\alpha}{x} + \frac{2\alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)},$$

unde fit

$$\frac{\partial s}{\partial x} = \frac{-\alpha}{xx} + \frac{(m-2\alpha-1)(m-2\alpha)}{xx(cx^{m-2\alpha-1} - 1)} + \frac{(m-2\alpha-1)^2}{xx(cx^{m-2\alpha-1} - 1)^2}.$$

Hic praecipue notetur casus $c = 0$, quo fit

$$s = \frac{m-\alpha-1}{x} \text{ et } \frac{\partial s}{\partial x} = \frac{-m+\alpha+1}{xx},$$

ita ut data aequatione

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m-1-\alpha)}{xx}v,$$

pro hac aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha+1)(m-2-\alpha)}{xx}z,$$

futurum sit

$$z = \frac{m-\alpha-1}{x}v + \left(\frac{\partial v}{\partial x}\right).$$

Pro generali autem valore sit $m - 2\alpha - 1 = \beta$, ut habeatur

$$s = \frac{\alpha}{x} - \frac{\beta}{x(cx^\beta - 1)} \text{ et}$$

$$\frac{\partial s}{\partial x} = \frac{-\alpha}{xx} + \frac{\beta(\beta+1)}{xx(cx^\beta - 1)} + \frac{\beta\beta}{xx(cx^\beta - 1)^2},$$

unde si detur haec aequatio

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2\alpha+\beta+1}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(\alpha+\beta)}{xx}v,$$

ejus ope resolvetur haec

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\beta+1}{x}\left(\frac{\partial z}{\partial x}\right)$$

$$+\left[(\alpha-1)(\alpha+\beta+1) - \frac{2\beta(\beta+1)}{cx^\beta-1} - \frac{2\beta\beta}{(cx^\beta-1)^2}\right]\frac{z}{xx},$$

cum sit

$$z = \left(\alpha - \frac{\beta}{cx^\beta-1}\right)\frac{v}{x} + \left(\frac{\partial v}{\partial x}\right).$$

III. Rationem quoque habeamus constantis f, ponamusque $f = \frac{1}{aa}$, ut facto $n = \alpha(m-1-\alpha)$ habeamus

$$\partial s + \frac{ms\partial x}{x} - ss\partial x - \frac{\alpha(m-1-\alpha)\partial x}{xx} + \frac{\partial x}{aa} = 0,$$

quae posito $s = \frac{\alpha}{x} + \frac{1}{t}$ abit in

$$\partial t - \frac{(m-2\alpha)t\partial x}{x} + \partial x = \frac{tt}{aa}\partial x.$$

Sit $m - 2\alpha = \gamma$, ut aequatio data sit

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(\alpha+\gamma-1)}{xx}v,$$

et inventa quantitate s prodeat haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\alpha\alpha-3\alpha+\alpha\gamma-\gamma}{xx} - \frac{2\partial s}{\partial x}\right)z,$$

seu

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{(\alpha-1)(\alpha+\gamma)}{xx} + \frac{2\partial t}{tt\,\partial x}\right)z,$$

pro qua est

$$z = \left(\frac{\alpha}{x} + \frac{1}{t}\right)v + \left(\frac{\partial v}{\partial x}\right),$$

ubi totum negotium ad inventionem quantitatis t redit ex aequatione

$$\partial t - \frac{\gamma t\partial x}{x} + \partial x = \frac{tt}{aa}\partial x.$$

Hunc in finem statuatur $t = a - \frac{aa\partial u}{u\partial x}$, ac reperitur

$$\frac{\partial\partial u}{\partial x^2} - \frac{\gamma\partial u}{x\partial x} - \frac{2\partial u}{a\partial x} + \frac{\gamma u}{ax} = 0,$$

cujus duplex resolutio datur, altera ponendo

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

existente

$$B = \frac{\gamma A}{\gamma a},\quad C = \frac{(\gamma-2)B}{2(\gamma-1)a},\quad D = \frac{(\gamma-4)C}{3(\gamma-2)a},\quad E = \frac{(\gamma-6)D}{4(\gamma-3)a},\quad \text{etc.}$$

altera vero ponendo

$$u = Ax^{\gamma+1} + Bx^{\gamma+2} + Cx^{\gamma+3} + Dx^{\gamma+4} + Ex^{\gamma+5} + \text{etc.}$$

ubi

$$B = \frac{(\gamma+2)A}{(\gamma+2)a},\quad C = \frac{(\gamma+4)B}{2(\gamma+3)a},\quad D = \frac{(\gamma+6)C}{3(\gamma+4)a},$$
$$E = \frac{(\gamma+8)D}{4(\gamma+5)a},\quad \text{etc.}$$

quarum illa abrumpitur, si sit γ numerus integer par positivus, haec vero si negativus. Qui valores etsi sunt particulares, tamen supra jam ostendimus, quomodo inde valores completi sint eliciendi.

Corollarium 1.

354. Supra autem vidimus, (§. 333.) hanc aequationem

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{(m+i)(m-i-1)}{xx}v,$$

esse integrabilem, si sit i numerus integer quicunque, unde colligimus hanc aequationem

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m-1-\alpha)}{xx}v$$

integrationem admittere, quoties fuerit vel $\alpha = \frac{1}{2}m + i$ vel $\alpha = \frac{1}{2}m - i - 1$, seu $m - 2\alpha$ numerus integer par sive positivus sive negativus, qui casus ob $m - 2\alpha = \gamma$ cum casibus integrabilitatis, pro valore generali ipsius s inveniendo, congruunt.

Corollarium 2.

355. Quando autem ex hac aequatione functionem v definire licet, tum etiam hae duae sequentes aequationes illi similes resolvi poterunt

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha - 1)(m - \alpha)}{xx} z \quad \text{et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha + 2)(m - \alpha - 1)}{xx} z,$$

cum pro illa sit

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right),$$

pro hac vero

$$z = \frac{m - \alpha - 1}{x} v + \left(\frac{\partial v}{\partial x}\right).$$

Corollarium 3.

356. Praeterea vero etiam aequationes alius generis, ub postremus terminus non est formae $\frac{n}{xx} z$, resolvi possunt, qui inveniuntur, si quantitatis s valor generalius investigatur, atque adeo constantis f ratio habetur.

Exemplum 1.

357. *Proposita aequatione* $\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right)$, *pro qua est*

$$v = \pi : (x + y) + \Phi : (x - y),$$

invenire aequationes magis complicatas, quae hujus ope integrari queant.

Cum hic sit $F = 1$, $G = 0$ et $H = 0$, resolvatur haec aequatio

$$\partial s - ss\,\partial x + C\,\partial x = 0,$$

et hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2\partial s}{\partial x} z$$

integrale erit

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

Sumta autem primo constante $C = 0$, fit $\frac{\partial s}{ss} = \partial x$ et $\frac{1}{s} = c - x$ seu $s = \frac{1}{c-x}$, atque $\frac{\partial s}{\partial x} = \frac{1}{(c-x)^2}$, ubi quidem sine ulla restrictione poni potest $c = 0$, ut hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2}{xx} z,$$

integrale sit

$$z = -\frac{1}{x}[\pi : (x+y) + \Phi : (x-y)] + \pi' : (x+y) + \Phi' : (x-y).$$

Sit deinde $C = aa$, et ob $\partial s = \partial x (ss - aa)$ fiet

$$x = \frac{1}{2a} l \frac{s-a}{s+a}, \text{ hincque}$$

$$\frac{s-a}{s+a} = Ae^{2ax} \text{ et } s = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}}, \text{ unde}$$

$$\frac{\partial s}{\partial x} = \frac{4Aaae^{2ax}}{(1-Ae^{2ax})^2},$$

et aequationes

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{8Aaae^{2ax}}{(1-Ae^{2ax})^2} z$$

integrale est

$$z = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}} v + \left(\frac{\partial v}{\partial x}\right).$$

Sit tandem $C = -aa$, et ob $\partial s = \partial x (aa + ss)$ fit

$$ax + b = \text{Ang. tang. } \frac{s}{a},$$

hincque

$$s = a \text{ tang. } (ax+b) \text{ et } \frac{\partial s}{\partial x} = \frac{aa}{\cos.(ax+b)^2},$$

quocirca hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2aa}{\cos.(ax+b)^2} z$$

integrale est

$$z = \frac{a \sin.(ax+b)}{\cos.(ax+b)} v + \left(\frac{\partial v}{\partial x}\right).$$

Exemplum 2.

358. *Proposita aequatione*

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) - \frac{2}{xx} v,$$

cujus integrale constat, invenire alias ejus ope integrabiles.

Pro hoc casu habemus

$$\partial s - ss\partial x + \left(C + \frac{2}{xx}\right)\partial x = 0,$$

qua resoluta erit hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - 2\left(\frac{1}{xx} + \frac{\partial s}{\partial x}\right) z$$

integrale

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

I. Sit primo $C = 0$, et ex aequatione

$$\partial s - ss\partial x - \frac{2\partial x}{xx} = 0$$

fit particulariter $s = \frac{1}{x}$ vel $s = -\frac{2}{x}$. Ponatur ergo $s = \frac{1}{x} + \frac{1}{t}$, eritque

$$\partial t + \frac{2t\partial x}{x} + \partial x = 0, \text{ hinc}$$

$$txx + \tfrac{1}{3}x^3 = \tfrac{1}{3}a^3.$$

Ergo

$$t = \frac{a^3 - x^3}{3xx} \quad \text{et} \quad s = \frac{a^3 + 2x^3}{x(a^3 - x^3)},$$

ideoque

$$\frac{\partial s}{\partial x} + \frac{1}{xx} = \frac{3x(2a^3 + x^3)}{(a^3 - x^3)^2},$$

unde hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{6x(2a^3 + x^3)}{(a^3 - x^3)^2} z$$

integrale est

$$z = \frac{a^3 + 2x^3}{x(a^3 - x^3)} v + \left(\frac{\partial v}{\partial x}\right).$$

II. Sit $C = \frac{1}{cc}$, et posito $s = \frac{1}{x} + \frac{1}{t}$ fit

$$\partial t + \frac{2t\partial x}{x} + \partial x = \frac{tt\partial x}{cc},$$

cui particulariter satisfacit $t = c + \frac{cc}{x}$, ut sit

$$s = \frac{cc + cx + xx}{cx(c+x)} \text{ et } \frac{\partial s}{\partial x} + \frac{1}{xx} = \frac{1}{(c+x)^2},$$

atque hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2}{(c+x)^2} z$$

integrale fit

$$z = \frac{cc + cx + xx}{cx(c+x)} v + \left(\frac{\partial v}{\partial x}\right).$$

Ad integrale autem pro t completum inveniendum statuatur

$$t = c + \frac{cc}{x} + \frac{1}{u},$$

fietque

$$\partial u + \frac{2u\partial x}{c} + \frac{\partial x}{cc} = 0, \text{ seu } \partial x = \frac{-cc\partial u}{1 + 2cu},$$

hinc $x = b - \frac{c}{2} l(1 + 2cu)$, ergo

$$u = \frac{e^{\frac{2(b-x)}{c}} - 1}{2c}, \text{ unde}$$

$$t = c + \frac{cc}{x} + \frac{2c}{e^{\frac{2(b-x)}{c}} - 1}, \text{ et}$$

$$s = \frac{1}{x} + \frac{x\left(e^{\frac{2(b-x)}{c}} - 1\right)}{c\left[(c+x)e^{\frac{2(b-x)}{c}} + c - x\right]},$$

atque

$$\frac{\partial s}{\partial x} + \frac{1}{xx} = \frac{-\partial t}{tt\partial x} = \frac{1}{tt}\left(1 + \frac{2t}{x} - \frac{tt}{cc}\right) = \frac{1}{tt}\left(\frac{cc}{xx} - \frac{4e^{\prime\prime}}{(e^{\prime\prime} - 1)^2}\right),$$

pro $e^{\prime\prime}$ legendo $e^{\frac{2(b-x)}{c}}$.

Scholion.

359. Quoniam supra invenimus hanc aequationem

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) - \frac{i(i+1)}{xx}\, v$$

integrationem admittere, quippe qui casus oritur ex generali forma (§. 354.) sumto $m = 0$, erit problemate huc translato

$$\partial s - ss\partial x + \left(f + \frac{i(i+1)}{xx}\right)\partial x = 0,$$

hincque inventa quantitate s, hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \left(2f + \frac{i(i+1)}{xx} - 2ss\right) z,$$

integrale erit

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

I. Quod si jam capiamus $f = 0$, erit particulariter $s = \frac{i}{x}$ vel $s = \frac{-i-1}{x}$, unde quidem aequationis integrabilis forma non mutatur. At facto $s = \frac{i}{x} + \frac{1}{t}$, oritur

$$\partial t + \frac{2it\partial x}{x} + \partial x = 0,$$

cujus integrale est

$$x^{2i} t + \frac{1}{2i+1} x^{2i+1} = \frac{g}{2i+1},$$

ideoque

$$s = \frac{ig + (i+1)x^{2i+1}}{x(g - x^{2i+1})},$$

et aequatio integrabilis fit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{[i(i-1)gg + 6i(i+1)gx^{2i+1} + (i+1)(i+2)x^{4i+2}]z}{xx(g - x^{2i+1})^2}.$$

II. At non rejecto f fit $s = \frac{i}{x} + u$, fietque

$$-\partial u + \frac{2iu\partial x}{x} + uu\partial x = f\partial x,$$

quae ut in aequationem differentialem secundi gradus facile per seriem resolubilem convertatur, ponatur

$$u = \sqrt{f} - \frac{i}{x} - \frac{\partial r}{r \partial x},$$

et prodit

$$\frac{\partial \partial r}{\partial x^2} - \frac{2 \partial r \sqrt{f}}{\partial x} - \frac{i(i+1)r}{xx} = 0:$$

sit $\sqrt{f} = a$ et statuatur

$$r = Ax^{i+1} + Bx^{i+2} + Cx^{i+3} + Dx^{i+4} + \text{etc.}$$

ac reperitur

$$B = \frac{2(i+1)a}{1(2i+2)} A, \quad C = \frac{2(i+2)a}{2(2i+3)} B, \quad D = \frac{2(i+3)a}{3(2i+4)} C, \quad E = \frac{2(i+4)a}{4(2i+5)} D, \text{ etc.}$$

quae abrumpitur quoties i est numerus integer negativus. Sin autem statuatur

$$r = Ax^{-i} + Bx^{1-i} + Cx^{2-i} + Dx^{3-i} + \text{etc.}$$

sequens relatio nascitur

$$B = \frac{2ia}{2i} A, \quad C = \frac{2(i-1)a}{2(2i-1)} B, \quad D = \frac{2(i-2)a}{3(2i-2)} C, \quad E = \frac{2(i-3)a}{4(2i-3)} D, \text{ etc.}$$

quae abrumpitur quoties i est numerus integer positivus.

Problema 58.

360. Proposita aequatione

$$\left(\frac{\partial \partial v}{\partial y^2}\right) = \left(\frac{\partial \partial v}{\partial x^2}\right) - \frac{2aa}{\cos.(ax+b)^2} v,$$

cujus integrale est

$$v = a \,\text{tang.}\,(ax+b) \,.\, [\pi : (x+y) + \Phi : (x-y)] \\ + \pi' : (x+y) + \Phi' : (x-y),$$

per transformationem hic traditam alias invenire aequationes ejus ope integrabiles.

Solutio.

Ponamus brevitatis gratia angulum $ax + b = \omega$, ut sit $\partial \omega = a \partial x$, et ex §. 351. cum sit $F = 1$, $G = 0$, $H = \frac{-2aa}{\cos.\omega^2}$, quaeratur quantitas s ex hac aequatione

$$\partial s - ss\partial x + (C + \frac{2aa}{\cos.\omega^2})\partial x = 0,$$

eritque hujus aequationis

$$(\frac{\partial\partial z}{\partial y^2}) = (\frac{\partial\partial z}{\partial x^2}) - (\frac{2aa}{\cos.\omega^2} + \frac{2\partial s}{\partial x})z$$

integrale

$$z = sv + (\frac{\partial v}{\partial x}), \text{ seu}$$

$$z = as\,\text{tang.}\,\omega[\pi:(x+y) + \Phi:(x-y)] + s[\pi':(x+y) + \Phi':(x-y)]$$
$$+ \frac{aa}{\cos.\omega^2}[\pi:(x+y) + \Phi:(x-y)] + a\,\text{tang.}\,\omega[\pi':(x+y) + \Phi':(x-y)]$$
$$+ \pi'':(x+y) + \Phi'':(x-y).$$

Totum ergo negotium ad inventionem quantitatis s reducitur, quem in finem ponamus

$$s = \alpha\ \text{tang.}\ \omega - \frac{\partial u}{u\partial x},$$

fietque

$$\frac{\partial s}{\partial x} = \frac{\alpha\alpha}{\cos.\ \omega^2} - \frac{\partial\partial u}{u\partial x^2} + \frac{\partial u^2}{uu\partial x^2},$$

et facta substitutione prodit

$$\frac{\alpha\alpha}{\cos.\ \omega^2} - \frac{\partial\partial u}{u\partial x^2} + \frac{2\alpha\partial u}{u\partial x}\ \text{tang.}\ \omega = 0,$$
$$- \frac{\alpha\alpha\sin.\omega^2}{\cos.\omega^2},$$
$$+ C + \frac{2aa}{\cos.\ \omega^2}.$$

Jam ob

$$- \frac{\alpha\alpha\sin.\omega^2}{\cos.\omega^2} = - \frac{\alpha\alpha}{\cos\ \omega^2} + \alpha\alpha,$$

sumatur α ita ut fiat

$$-\alpha\alpha + \alpha a + 2aa = 0.$$

Capiatur ergo $\alpha = -a$, ut sit

$$s = -a\,\text{tang.}\,\omega - \frac{\partial u}{u\partial x},$$

et pro quantitate u invenienda haec habetur aequatio

$$\frac{ddu}{udx^2} + \frac{2adu}{udx}\,\text{tang.}\,\omega + nau = 0,$$

posito $C = -aa - nau$

seu $\frac{\partial \partial u}{\partial \omega^2} + \frac{2 \partial u}{\partial \omega} \text{tang.}\, \omega + nu = 0$,

ob $\partial x = \frac{\partial \omega}{a}$,

cujus resolutio non parum ardua videtur, inter complures autem modos eam tractandi hic ad institutum maxime idoneus videtur. Fingatur

$$u = A \cos. \lambda\omega + B \cos. (\lambda + 2)\omega + C \cos. (\lambda + 4)\omega + \text{etc.}$$

eritque

$$\frac{\partial u}{\partial \omega} = -\lambda A \sin. \lambda\omega - (\lambda + 2) B \sin. (\lambda + 2)\omega$$
$$- (\lambda + 4) C \sin. (\lambda + 4)\omega - \text{etc.}$$
$$\frac{\partial \partial u}{\partial \omega^2} = -\lambda\lambda A \cos. \lambda\omega - (\lambda + 2)^2 B \cos. (\lambda + 2)\omega$$
$$- (\lambda + 4)^2 C \cos. (\lambda + 4)\omega - \text{etc.}$$

et aequatio hac forma repraesentata

$$\frac{2 \partial \partial u}{\partial \omega^2} \cos. \omega + \frac{4 \partial u}{\partial \omega} \sin. \omega + 2nu \cos. \omega = 0 \text{ dabit}$$

$$0 = -\lambda\lambda A \cos.(\lambda - 1)\omega - (\lambda + 2)^2 B \cos.(\lambda + 1)\omega - (\lambda + 4)^2 C \cos.(\lambda + 3)\omega - \text{etc.}$$

	$-\lambda\lambda A$	$-(\lambda+2)^2 B$
$-2\lambda A$	$-2(\lambda+2)B$	$-2(\lambda+4)C$
	$+2\lambda A$	$+2(\lambda+2)B$
$+nA$	$+nB$	$+nC$
	$+nA$	$+nB$

unde λ ita capi oportet ut sit

$$\lambda\lambda + 2\lambda = n, \text{ seu } \lambda = -1 \pm \sqrt{(n + 1)},$$

duplexque pro λ habeatur valor. Praeterea vero secundus terminus ob $n = \lambda\lambda + 2\lambda$ praebet $B = \frac{\lambda}{\lambda + 2} A$, tertius vero commode dat $C = 0$, unde et sequentes omnes evanescunt.

Sumamus $n = mm - 1$, ut sit

$$\lambda = -1 \pm m \text{ et } B = \frac{-1 \pm m}{1 \pm m} A;$$

atque integrale completum concludi videtur

$$u = \mathrm{A}\,[\cos.(m-1)\,\omega + \tfrac{m-1}{m+1}\cos.(m+1)\,\omega]$$
$$+\,\mathfrak{A}\,[\cos.(m+1)\,\omega + \tfrac{m+1}{m-1}\cos.(m-1)\,\omega],$$

sit

$$\mathrm{A} = (m+1)\,\mathrm{B} \text{ et } \mathfrak{A} = (m-1)\,\mathfrak{B},$$

fiet

$$u = (m+1)(\mathrm{B}+\mathfrak{B})\cos.(m-1)\,\omega + (m-1)(\mathrm{B}+\mathfrak{B})\cos.(m+1)\,\omega,$$

ubi cum binae constantes in unam coalescant, hoc integrale tantum est particulare, ex quo autem deinceps completum elici poterit. Cum ergo sit

$$\frac{\partial u}{u\,\partial\omega} = \frac{-(mm-1)\sin.(m-1)\,\omega - (mm-1)\sin.(m+1)\,\omega}{(m+1)\cos.(m-1)\,\omega + (m-1)\cos.(m+1\,\omega} \text{ erit}$$

$$\frac{s}{a} = -\,\mathrm{tang.}\,\omega + \frac{(mm-1)\,[\sin.(m-1)\,\omega + \sin.(m+1)\,\omega]}{(m+1)\cos.(m-1)\,\omega + (m-1)\cos.(m+1)\,\omega},$$

pro aequatione

$$\frac{\partial s}{a\,\partial\omega} - \frac{ss}{aa} - mm + \frac{2}{\cos.\omega^2} = 0,$$

ob $C = -(n+1)\,aa = -mmaa$.

Illud autem integrale inventum ad hanc formam reducitur

$$\frac{s}{a} = -\,\mathrm{tang.}\,\omega + \frac{(mm-1)\,\mathrm{tang.}\,m\omega}{m + \mathrm{tang.}\,m\omega\,\mathrm{tang.}\,\omega},$$

quae expressio substituta illi aequationi egregie satisfacere deprehenditur. Scribamus ejus loco Θ, ac ponamus $\frac{s}{a} = \Theta + \frac{1}{t}$ pro integrali completo eliciendo, prodibitque

$$-\frac{\partial t}{tt\,\partial\omega} - \frac{2\Theta}{t} - \frac{1}{tt} = 0, \text{ seu}$$

$$\partial t + 2\,\Theta t\,\partial\omega + \partial\omega = 0.$$

Erat autem modo ante

$$\Theta = \frac{s}{a} = -\,\mathrm{tang.}\,\omega - \frac{\partial u}{u\,\partial\omega}, \text{ unde}$$

$$\int\Theta\,\partial\omega = l\cos.\omega - l\,u \text{ et } e^{2\int\Theta\partial\omega} = \frac{\cos.\omega^2}{uu},$$

qui est multiplicator pro illa aequatione, sicque fit

$$\frac{t\cos.\omega^2}{uu} = C - \int\frac{\partial\omega\cos.\omega^2}{uu};$$

at est

$$u = 2\, m \cos. m\,\omega \cos.\omega + 2 \sin. m\,\omega \sin.\omega,$$

ideoque

$$\frac{t}{(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega)^2} = A - \int \frac{d\,\omega}{(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}\,\omega)^2},$$

cujus postremi membri integrale deprehenditur

$$\frac{-m \operatorname{tang}. m\,\omega + \operatorname{tang}.\omega}{m(mm-1)(m + \operatorname{tang}. m\,\omega \operatorname{tang}.\omega)} = \frac{-m \sin. m\,\omega + \operatorname{tang}.\omega \cos. m\,\omega}{m(mm-1)(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega)},$$

ita ut sit

$$\frac{t}{(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega)^2} = A + \frac{\cos. m\,\omega \operatorname{tang}.\omega - m \sin. m\,\omega}{m(mm-1)(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega)},$$

seu

$$\frac{1}{t} = \frac{m(mm-1)}{[C(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega) + \cos. m\,\omega \operatorname{tang}.\omega - m \sin. m\,\omega\,(m \cos\, m\,\omega + \sin. m\,\omega \operatorname{tang}\,\omega)]},$$

cui addatur

$$\Theta = -\operatorname{tang}.\omega + \frac{(mm-1)\sin. m\,\omega}{m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega},$$

ut prodeat $\frac{s}{a}$, eritque

$$\frac{s}{a} = -\operatorname{tang}.\omega + \frac{(mm-1)(C \sin. m\,\omega + \cos\, m\,\omega)}{C(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega) + \cos. m\,\omega \operatorname{tang}.\omega - m \sin. m\,\omega},$$

seu

$$\frac{s}{a} = \frac{(mm - 1 - \operatorname{tang}.\omega^2)(C \sin. m\,\omega + \cos. m\,\omega) - m \operatorname{tang}.\omega\,(C \cos\, m\,\omega - \sin. m\,\omega)}{C(m \cos. m\,\omega + \sin. m\,\omega \operatorname{tang}.\omega) + \cos. m\,\omega \operatorname{tang}.\omega - m \sin. m\,\omega}.$$

Corollarium 1.

361. Hic praecipue notandum est, hujus aequationis

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2\,\partial u}{\partial\omega} \operatorname{tang}.\omega + (m\,m - 1)\,u = 0.$$

integrale particulare esse

$$u = m \cos. m\,\omega \cos.\omega + \sin. m\,\omega \sin.\omega,$$

aliud vero integrale particulare reperitur simili modo

$$u = m \sin. m\,\omega \cos.\omega - \cos. m\,\omega \sin.\omega,$$

unde concluditur completum

$$u = A(m \cos. m\,\omega \cos.\omega + \sin. m\,\omega \sin.\omega)$$
$$+ B(m \sin. m\,\omega \cos.\omega - \cos. m\,\omega \sin.\omega).$$

Corollarium 2.

362. Si hic ponatur

$$A = C \cos. \alpha \text{ et } B = - C \sin. \alpha,$$

hoc integrale completum ad hanc formam redigitur

$$u = C [m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega],$$

quod quidem ex integrali particulari primum invento statim concludi potuisset, cum ibi loco anguli $m \omega$ scribere liceat $m \omega + \alpha$.

Corollarium 3.

363. Hinc multo facilius reperitur valor

$\frac{s}{a} = - \text{tang. } \omega - \frac{\partial u}{u \partial \omega}$, cum enim sit

$\frac{\partial u}{\partial \omega} = - C (m m - 1) \sin. (m \omega + \alpha) \cos. \omega$, erit

$$\frac{s}{a} = - \text{tang. } \omega + \frac{(mm-1) \sin. (m\omega + \alpha) \cos. \omega}{m \cos. (m\omega + \alpha) \cos. \omega + \sin. (m\omega + \alpha) \sin. \omega},$$

hincque

$$\frac{\partial s}{a \partial \omega} = \frac{\partial s}{a a \partial x} = \frac{-1}{\cos. \omega^2} + \frac{(mm-1)[m^2 \cos. \omega^2 - \sin. (m\omega + \alpha)^2]}{[m \cos. (m\omega + \alpha) \cos. \omega + \sin. (m\omega + \alpha) \sin \omega]^2},$$

et aequatio, cujus integrationem invenimus, erit

$$\left(\frac{\partial \partial z}{\partial y^2}\right) = \left(\frac{\partial \partial z}{\partial x^2}\right) - \frac{2(mm-1) a a [m^2 \cos. \omega^2 - \sin. (m\omega + \alpha)^2]}{[m \cos. (m\omega + \alpha) \cos. \omega + \sin. (m\omega + \alpha) \sin. \omega]^2},$$

ejusque integrale colligitur

$$z = \frac{m a a [m \sin. (m\omega + \alpha) \sin. \omega + \cos (m\omega + \alpha) \cos. \omega]}{m \cos. (m\omega + \alpha) \cos \omega + \sin. (m\omega + \alpha) \sin \omega} [\pi : (x + y) + \Phi : (x - y)]$$
$$+ \frac{(mm-1) a \sin. (m\omega + \alpha) \cos. \omega}{m \cos (m\omega + \alpha) \cos. \omega + \sin. (m\omega + \alpha) \sin. \omega} [\pi' : (x + y) + \Phi' : (x - y)]$$
$$+ \pi'' : (x + y) + \Phi'' : (x - y)],$$

existente $\omega = a x + b$.

Scholion 1.

364. Omnino memoratu digna est integratio hujus aequationis

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2\partial u}{\partial\omega}\,\text{tang.}\,\omega + (mm - 1)\,u = 0;$$

unde occasionem carpo, hanc aequationem generaliorem tractandi

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\,\text{tang.}\,\omega + g\,u = 0,$$

quam primum observo posito

$$\frac{\partial u}{u} = -(2f+1)\,\partial\omega\,\text{tang.}\,\omega + \frac{\partial v}{v}, \text{ ut sit}$$

$$u = \cos.\,\omega^{2f+1}\,v,$$

abire in hanc formam

$$\frac{\partial\partial v}{\partial\omega^2} - \frac{2(f+1)\partial v}{\partial\omega}\,\text{tang.}\,\omega + (g - 2f - 1)\,v = 0,$$

ita ut si illa integrabilis existat casu $f = n$, integrabilis quoque sit casu $f = -n - 1$. Jam pro illa aequatione ponatur

$$u = \text{A}\sin.\,\lambda\,\omega + \text{B}\sin.\,(\lambda+2)\,\omega + \text{C}\sin.\,(\lambda+4)\,\omega$$
$$+ \text{D}\sin.\,(\lambda+6)\,\omega + \text{etc.}$$

et facta substitutione in aequatione

$$\frac{2\partial\partial u}{\partial\omega^2}\cos.\,\omega + \frac{4f\partial u}{\partial\omega}\sin.\,\omega + 2\,g\,u\cos.\,\omega = 0,$$

reperitur

$$\begin{array}{llll}
-\lambda\lambda\text{A}\,\text{si.}(\lambda-1)\omega & -(\lambda+2)^2\text{B}\,\text{si.}(\lambda+1)\omega & -(\lambda+4)^2\text{C}\,\text{si.}(\lambda+3)\omega & -(\lambda+6)^2\text{D}\,\text{si.}(\lambda+5)\omega \\
-2\lambda\text{A}f & -\lambda\lambda\text{A} & -(\lambda+2)^2\text{B} & -(\lambda+4)^2\text{C} \\
+\text{A}g & +2\lambda\text{A}f & +2(\lambda+2)\text{B}f & +2(\lambda+4)\text{C}f \\
 & -2(\lambda+2)\text{B}f & -2(\lambda+4)\text{C}f & -2(\lambda+6)\text{D}f \\
 & +\text{A}g & +\text{B}g & +\text{C}g \\
 & +\text{B}g & +\text{C}g & +\text{D}g
\end{array}$$

Oportet ergo sit $g = \lambda\lambda + 2\lambda f$, tum vero coëfficientes assumti ita determinantur

$$\text{B} = \frac{\lambda f \text{A}}{\lambda+f+1},\quad \text{C} = \frac{(\lambda+1)(f-1)\text{B}}{2(\lambda+f+2)},\quad \text{D} = \frac{(\lambda+2)(f-2)\text{C}}{3(\lambda+f+3)},\ \text{etc.}$$

Statuamus ergo $g = mm - ff$, ut fiat $\lambda = m - f$, et aequationes nostrae sint

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\,\text{tang.}\,\omega + (mm - ff)\,u = 0 \text{ et}$$

$$\frac{\partial \partial v}{\partial \omega^2} - \frac{2(f+1)\partial v}{\partial \omega} \text{tang.}\, \omega + [m m - (f+1)^2]\, v = 0,$$

existente

$$u = v \cos.\, \omega^{2f+1} \text{ seu } v = \frac{u}{\cos.\, \omega^{2f+1}}.$$

Quoniam nunc series nostra abrumpitur, quoties est f numerus integer, percurramus casûs simpliciores.

I. Sit $f = 0$, erit

$$\lambda = m \text{ et } B = 0,\ C = 0, \text{ etc.}$$

ideoque

$$u = A \sin.\, m\, \omega \text{ et } v = \frac{A \sin.\, m\, \omega}{\cos.\, \omega}.$$

II. Sit $f = 1$, erit

$$\lambda = m - 1 \text{ et } B = \frac{(m-1)A}{m+1},\ C = 0, \text{ etc.}$$

ergo

$$\frac{u}{a} = (m + 1) \sin.\, (m-1)\omega + (m-1) \sin.\, (m+1)\,\omega, \text{ et } v = \frac{u}{\cos.\, \omega^3},$$

seu $\frac{u}{a} = m \sin.\, m\, \omega \cos.\, \omega - \cos.\, m\, \omega \sin.\, \omega$.

III. Sit $f = 2$, erit $\lambda = m - 2$, et

$$B = \frac{2(m-2)A}{m+1},\ C = \frac{(m-1)B}{2(m+2)} = \frac{(m-1)(m-2)A}{(m+1)(m+2)},\ D = 0, \text{ etc.}$$

hinc

$$\frac{u}{a} = (m+1)(m+2) \sin.\, (m-2)\omega + 2(m-2)(m+2) \sin.\, m\, \omega$$
$$- (m-1)(m-2) \sin.\, (m+2)\, \omega,$$

indeque $v = \frac{u}{\cos.\, \omega^6}$ seu

$$\frac{u}{2a} = (m m - 2) \sin.\, m\, \omega \cos.\, 2\, \omega + 2(m m - 4) \sin.\, m\, \omega$$
$$- 3 m \cos.\, m\, \omega \sin.\, 2\, \omega.$$

IV. Sit $f = 3$, erit $\lambda = m - 3$, et

$$B = \frac{3(m-3)A}{m+1},\ C = \frac{2(m-2)B}{2(m+2)},\ D = \frac{(m+1)C}{3(m+3)},\ E = 0, \text{ etc.}$$

Ergo

$$\frac{u}{a} = +(m+1)(m+2)(m+3)\sin.(m-3)\omega + 3(m+2)(mm-9)\sin.(m-1)\omega$$
$$+(m-1)(m-2)(m-3)\sin.(m+3)\omega + 3(m-2)(mm-9)\sin.(m+1)\omega$$

existente $u = \frac{u}{\cos.\omega^7}$.

V. Sit $f = 4$, erit $\lambda = m - 4$, ac reperitur

$$\frac{u}{a} = +(m+1)(m+2)(m+3)(m+4)\sin.(m-4)\omega + 4(m+2)(m+3)(mm-16)\sin.(m-2)\omega$$
$$+(m-1)(m-2)(m-3)(m-4)\sin.(m+4)\omega + 4(m-2)(m-3)(mm-16)\sin.(m+2)\omega$$
$$+6(mm-9)(mm-16)\sin.m\omega$$

existente $v = \frac{u}{\cos.\omega^9}$,

unde ratio progressionis per se est manifesta. Notari autem convenit si posuissemus

$$0 = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

easdem coëfficientium determinationes prodituras fuisse, ex qua hi duo valores conjuncti integrale completum exhibebunt: quod etiam ex forma inventa colligitur, si modo loco anguli $m\omega$ generalius scribatur $m\omega + \alpha$.

Scholion 2.

365. Pluribus autem aliis modis eadem aequatio

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\,\text{tang.}\,\omega + gu = 0$$

tractari, et ejus integrale pér series exprimi potest, undè alii casus integrabilitatis obtinentur. Ad hoc primum notetur, posito $u = \sin.\omega^\lambda$ fore

$$\frac{\partial u}{\partial\omega} = \lambda\sin\,\omega^{\lambda-1}\cos.\omega, \text{ hincque}$$
$$\frac{\partial u}{\partial\omega}\,\text{tang.}\,\omega = \lambda\sin.\omega^\lambda, \text{ et}$$
$$\frac{\partial\partial u}{\partial\omega^2} = \lambda(\lambda-1)\sin.\omega^{\lambda-2}\cos.\omega^2 - \lambda\sin.\omega^\lambda$$
$$= \lambda(\lambda-1)\sin.\omega^{\lambda-2} - \lambda\lambda\sin.\omega^\lambda.$$

Hinc si ponamus

$$u = A\sin.\omega^{\lambda} + B\sin.\omega^{\lambda+2} + C\sin.\omega^{\lambda+4} + D\sin.\omega^{\lambda+6} + \text{etc.}$$

facta substitutione adipiscimur

$$\begin{array}{llll} 0 = \lambda(\lambda-1)A\,si.\omega^{\lambda-2} & + (\lambda+2)(\lambda+1)B\,si.\omega^{\lambda} & + (\lambda+4)(\lambda+3)C\,si.\omega^{\lambda+2} & + \text{etc} \\ & -\lambda\lambda A & -(\lambda+2)^2 B & \\ & +2\lambda f A & +2(\lambda+2)fB & \\ & +gA & +gB & \end{array}$$

unde sumi oportet vel $\lambda = 0$ vel $\lambda = 1$, tum vero erit

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+1)(\lambda+2)}A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+3)(\lambda+4)}B, \text{ etc.}$$

hinc duo casus evolvi convenit

$\lambda = 0$,	$\lambda = 1$,
$B = \frac{-g}{1.2}A$,	$B = \frac{1-2f-g}{2.3}A$,
$C = \frac{4-4f-g}{3.4}B$,	$C = \frac{9-6f-g}{4.5}B$,
$D = \frac{16-8f-g}{5.6}C$,	$D = \frac{25-10f-g}{6.7}C$,
$E = \frac{36-12f-g}{7.8}D$,	$E = \frac{49-14f-g}{8.9}D$,
etc.	etc.

Integratio ergo succedit, quoties fuerit $g = ii - 2if$ denotante i numerum integrum positivum. Quare cum posito $u = v\cos.\omega^{2f+1}$ aequatio transformata sit

$$\frac{\partial\partial v}{\partial\omega^2} - \frac{2(f+1)\partial v}{\partial\omega}\operatorname{tang.}\omega + (g - 2f - 1)v = 0,$$

haec ideoque et illa erit integrabilis, quoties fuerit

$$g = (i+1)^2 + 2(i+1)f,$$

quos binos casus ita uno complecti licet, ut integratio succedat, dum sit $g = ii \pm 2if$.

Scholion 3.

366. Eidem aequationi adhuc inhaerens, cum posito $u = \cos.\omega^{\lambda}$, sit

$$\frac{\partial u}{\partial\omega} = -\lambda\cos.\omega^{\lambda-1}\sin.\omega, \text{ ideoque}$$

$$\frac{\partial u}{\partial\omega}\operatorname{tang.}\omega = -\lambda\cos.\omega^{\lambda-2} + \lambda\cos.\omega^{\lambda}, \text{ et}$$

$$\frac{\partial\partial u}{\partial\omega^2} = \lambda(\lambda-1)\cos.\omega^{\lambda-2} - \lambda\lambda\cos.\omega^{\lambda},$$

statuo

$$u = A\cos.\omega^{\lambda} + B\cos.\omega^{\lambda+2} + C\cos.\omega^{\lambda+4} + D\cos.\omega^{\lambda+6} + \text{etc.}$$

et facta substitutione orietur

$$0 = \begin{array}{lll} \lambda(\lambda-1)A\cos.\omega^{\lambda-2} & +(\lambda+2)(\lambda+1)B\cos.\omega^{\lambda} & +(\lambda+4)(\lambda+3)C\cos.\omega^{\lambda+2} + \text{etc.} \\ -2\lambda fA & -\lambda\lambda A & -(\lambda+2)^2 B \\ & -2(\lambda+2)fB & -2(\lambda+4)fC \\ & -2\lambda fA & +2(\lambda+2)fB \\ & +gA & +gB \end{array}$$

Oportet ergo sit vel $\lambda = 0$ vel $\lambda = 2f+1$, tum vero

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+2)(\lambda+1+2f)} A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+4)(\lambda+3-2f)} B, \text{ etc.}$$

et ambo casus ita se habebunt

$\lambda = 0,$	$\lambda = 2f+1$
$B = \frac{-g}{2(1+2f)} A,$	$B = \frac{1+2f-g}{2(2f+3)} A,$
$C = \frac{4-4f-g}{4(3-2f)} B,$	$C = \frac{9+6f-g}{4(2f+5)} B,$
$D = \frac{16-8f-g}{6(5-2f)} C,$	$D = \frac{25+10f-g}{6(2f+7)} C,$
etc.	etc.

Ex priori integratio succedit si $g = 4ii - 4if$, ex posteriori si $g = (2i+1)^2 + 2(2i+1)f$, qui casus cum iis, qui ex transformata nascuntur juncti, eodem redeunt ac in §. praec. inventi. Omnes ergo hactenus inventi integrabilitatis casus huc revocantur, ut posito $g = mm - ff$, sit vel $f = \pm i$, vel $m = i \pm f$, hoc est vel $f = \pm i$, vel $f = \pm i \pm m$. Caeterum hi posteriores casus etiam ex prima resolutione (§. 364) sequuntur, ubi series quoque abrumpitur si $\lambda = -i$, ideoque $g = mm - ff = ii - 2if$, ergo $i - f = \pm m$, et transformatione in subsidium vocata $f = \pm i \pm m$. Contra vero casus primo inventi in resolutionibus posterioribus non occurrunt.

Problema 59.

367. Concessa hujus aequationis integratione $\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv,$

invenire **aequationem hujus formae**

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + R\,z,$$

pro qua sit

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v,$$

ubi F, G, H; P, Q, R; et r, s sunt functiones ipsius x tantum.

Solutio.

Cum sit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial^4 v}{\partial x^2\partial y^2}\right) + r\left(\frac{\partial^3 v}{\partial x\partial y^2}\right) + s\left(\frac{\partial\partial v}{\partial y^2}\right), \text{ ob}$$

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + H\,v, \text{ erit}$$

$$\left(\frac{\partial^3 v}{\partial x\partial y^2}\right) = F\left(\frac{\partial^3 v}{\partial x^3}\right) + \begin{matrix}\frac{\partial F}{\partial x}\\ +G\end{matrix}\left(\frac{\partial\partial v}{\partial x^2}\right) + \begin{matrix}\frac{\partial G}{\partial x}\\ +H\end{matrix}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial H}{\partial x}\,v, \text{ et}$$

$$\left(\frac{\partial^4 v}{\partial x^2\partial y^2}\right) = F\left(\frac{\partial^4 v}{\partial x^4}\right) + \begin{matrix}\frac{2\partial F}{\partial x}\\ +G\end{matrix}\left(\frac{\partial^3 v}{\partial x^3}\right) + \begin{matrix}\frac{\partial\partial F}{\partial x^2}\\ +\frac{2\partial G}{\partial x}\\ +H\end{matrix}\left(\frac{\partial\partial v}{\partial x^2}\right) + \begin{matrix}\frac{\partial\partial G}{\partial x^2}\\ +\frac{2\partial H}{\partial x}\end{matrix}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial\partial H}{\partial x^2}\,v.$$

Deinde vero ob

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v, \text{ fit}$$

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial^3 v}{\partial x^3}\right) + r\left(\frac{\partial\partial v}{\partial x^2}\right) + \begin{matrix}\frac{\partial r}{\partial x}\\ +s\end{matrix}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial s}{\partial x}\,v, \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial^4 v}{\partial x^4}\right) + r\left(\frac{\partial^3 v}{\partial x^3}\right) + \begin{matrix}\frac{2\partial r}{\partial x}\\ +s\end{matrix}\left(\frac{\partial\partial v}{\partial x^2}\right) + \begin{matrix}\frac{\partial\partial r}{\partial x^2}\\ +\frac{2\partial s}{\partial x}\end{matrix}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial\partial s}{\partial x^2}\,v.$$

His jam substitutis necesse est, ut omnes termini affecti per

$$\left(\frac{\partial^4 v}{\partial x^4}\right),\ \left(\frac{\partial^3 v}{\partial x^3}\right),\ \left(\frac{\partial\partial v}{\partial x^2}\right),\ \left(\frac{\partial v}{\partial x}\right), \text{ et } v$$

seorsim evanescant unde sequentes resultant aequationes

ex	
$\left(\frac{\partial^4 v}{\partial x^4}\right)$	I. $F = P,$
$\left(\frac{\partial^3 v}{\partial x^3}\right)$	II. $G + \frac{2\partial F}{\partial x} + F r = P r + Q,$
$\left(\frac{\partial\partial v}{\partial x^2}\right)$	III. $H + \frac{2\partial G}{\partial x} + \frac{\partial\partial F}{\partial x^2} + G r + \frac{r\partial F}{\partial x} + F s = P\left(s + \frac{2\partial r}{\partial x}\right) + Q r + R,$
$\left(\frac{\partial v}{\partial x}\right)$	IV. $\frac{2\partial H}{\partial x} + \frac{\partial\partial G}{\partial x^2} + H r + \frac{r\partial G}{\partial x} + G s = P\left(\frac{2\partial s}{\partial x} + \frac{\partial\partial r}{\partial x^2}\right) + Q\left(s + \frac{\partial r}{\partial x}\right) + R r,$
v	V. $\frac{\partial\partial H}{\partial x^2} + \frac{r\partial H}{\partial x} + H s = P\frac{\partial\partial s}{\partial x^2} + Q\frac{\partial s}{\partial x} + R s.$

Ex prima fit $P = F$, ex secunda $Q = G + \frac{2\partial F}{\partial x}$, et tertia

$$R = H + \frac{2\partial G}{\partial x} + \frac{\partial\partial F}{\partial x^2} - \frac{r\partial F - 2F\partial r}{\partial x},$$

qui valores in binis ultimis substituti praebent

$$\frac{2\partial H}{\partial x} + \frac{\partial\partial G}{\partial x^2} - \frac{r\partial G - G\partial r}{\partial x} - \frac{r\partial\partial F}{\partial x^2} - \frac{2\partial F\partial r}{\partial x^2} - \frac{2s\partial F - 2F\partial s}{\partial x}$$
$$+ \frac{rr\partial F + 2Fr\partial r}{\partial x} - \frac{F\partial\partial r}{\partial x^2} = 0 \text{ et}$$

$$\frac{\partial\partial H}{\partial x^2} + \frac{r\partial H}{\partial x} - \frac{s\partial\partial F - 2\partial F\partial s - F\partial\partial s}{\partial x^2} - \frac{2s\partial G - G\partial s}{\partial x}$$
$$+ \frac{s(r\partial F + 2F\partial r)}{\partial x} = 0,$$

quarum illa sponte est integrabilis, praebens

$$2H + \frac{\partial G}{\partial x} - G r - \frac{r\partial F - F\partial r}{\partial x} - 2 F s + F r r = A;$$

deinde binis illis aequationibus ita repraesentatis

$$-\frac{\partial\partial.Fr}{\partial x^2} - \frac{2\partial.Fs}{\partial x} + \frac{\partial.Frr}{\partial x} + \frac{\partial\partial G}{\partial x^2} - \frac{\partial.Gr}{\partial x} + \frac{2\partial H}{\partial x} = 0,$$

$$-\frac{\partial\partial.Fs}{\partial x^2} + \frac{s}{r}\cdot\frac{\partial.Frr}{\partial x} - \frac{2s\partial G - G\partial s}{\partial x} + \frac{r\partial H}{\partial x} + \frac{\partial\partial H}{\partial x^2} = 0,$$

vel adeo hoc modo

$$\frac{\partial\partial.(G - Fr)}{\partial x} - \partial.r(G - Fr) + 2\,\partial.(H - Fs) = 0,$$

$$\frac{\partial\partial.(H - Fs)}{\partial x} + 2 F s\, dr + r s\partial F - G\partial s - 2 s\partial G + r\partial H = 0,$$

ultima vero ita repraesentari potest

$$\frac{\partial\partial.(H - Fs)}{\partial x} - 2s\partial.(G - Fr) - \partial s(G - Fr) + r\partial.(H - Fs) = 0.$$

Quod si jam prior per $H - Fs$ haec vero per $-(G - Fr)$ multiplicetur, summa fit

$$\begin{aligned}&\frac{(H-Fs)\partial\partial.(G-Fr)-(G-Fr)\partial\partial.(H-Fs)}{\partial x} && -(G-Fr)(H-Fs)\partial r = 0.\\ &+2(H-Fs)\partial.(H-Fs) && -r(H-Fs)\partial.(G-Fr)\\ &+2s(G-Fr)\partial.(G-Fr)+(G-Fr)^2\partial s && -r(G-Fr)\partial.(H-Fs)\end{aligned}$$

cujus integrale manifesto est

$$\frac{(H-Fs)\partial.(G-Fr)-(G-Fr)\partial.(H-Fs)}{\partial x} + (H+Fs)^2 + (G-Fr)^2 s - (G-Fr)(H-Fs)r = B;$$

integrale autem prius inventum est

$$\frac{\partial.(G-Fr)}{\partial x} - (G-Fr)r + 2(H-Fs) = A,$$

quae per $H - Fs$ multiplicata et ab illa subtracta relinquit

$$-\frac{(G-Fr)\partial.(H-Fs)}{\partial x} - (H-Fs)^2 + (G-Fr)^2 s = B - A(H-Fs),$$

sicque habentur duae aequationes simpliciter differentiales, ex quibus binas quantitates r et s definiri oportet, quibus cognitis etiam functiones P, Q et R innotescunt.

Corollarium 1.

368. Si sit $F = 1$, $G = 0$ et $H = 0$, aequationes inventae erunt

$$-\frac{\partial r}{\partial x} + rr - 2s = a \text{ et } \frac{s\partial r - r\partial s}{\partial r} + ss = b,$$

unde ∂x eliminando fit

$$\frac{r\partial s - s\partial r}{\partial r} = \frac{b - ss}{a - 2s - rr}, \text{ seu } \frac{r\partial s}{\partial r} = \frac{b + as + ss - rrs}{a + 2s - rr},$$

cujus resolutio in genere vix suscipienda videtur. Sumtis autem constantibus $a = 0$ et $b = 0$, aequatio $\frac{r\partial s}{\partial r} = \frac{ss - rrs}{2s - rr}$, posito $s = rrt$, transit in

$$\frac{r\partial t + 2t\partial r}{\partial r} = \frac{tt - t}{2t - 1}, \text{ seu } \frac{r\partial t}{\partial r} = \frac{-3tt + t}{2t - 1},$$

unde fit

$$\frac{\partial r}{r} = \frac{\partial t(1-2t)}{t(3t-1)} = \frac{-\partial t}{t} + \frac{\partial t}{3t-1}, \text{ et}$$

$$r = \frac{\alpha \sqrt[3]{(3t-1)}}{t}, \text{ hinc}$$

$$s = \frac{\alpha\alpha \sqrt[3]{(3t-1)^2}}{t}.$$

Corollarium 2.

369. Pro eodem casu singulari ponamus $3t - 1 = u^3$, ut fiat

$$r = \frac{3\alpha u}{1+u^3} \text{ et } s = \frac{3\alpha\alpha uu}{1+u^3}.$$

Jam ob $\alpha = 0$ est

$$\partial x = \frac{\partial r}{rr-2s} = \frac{\partial r}{rr(1-2t)} = \frac{3\partial r}{rr(1-2u^3)} \text{ et}$$

$$\frac{\partial r}{rr} = \frac{\partial u}{3\alpha uu} - \frac{2u\partial u}{3\alpha} = \frac{\partial u(1-2u^3)}{3\alpha uu},$$

ita ut sit

$$\partial x = \frac{\partial u}{\alpha uu}, \text{ hincque}$$

$$\frac{1}{u} = \beta - \alpha x \text{ et } u = \frac{1}{\beta - \alpha x};$$

ubi quidem salva generalitate sumi potest

$$\beta = 0 \text{ et } u = \frac{-1}{\alpha x},$$

unde fit

$$r = \frac{-3xx}{x^3+c^3} \text{ facto}$$

$$\alpha = -\frac{1}{c} \text{ et } s = \frac{3x}{x^3+c^3}.$$

Tandem ergo colligitur

$$P = 1, \; Q = 0 \text{ et } R = -\frac{2\partial r}{\partial x} = -\frac{6x(2c^3-x^3)}{(c^3+x^3)^2}.$$

Corollarium 3.

370. Proposita ergo aequatione $\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right)$, cujus integrale est

$$v = \Gamma:(x+y) + \Delta:(x-y),$$

hujus aequationis integrale assignari poterit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{6x(2c^3 - x^3)}{(c^3+x^3)^2}\, z,$$

est enim

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) - \frac{3xx}{c^3+x^3}\left(\frac{\partial v}{\partial x}\right) + \frac{3x}{c^3+x^3}\, v.$$

Scholion 1.

371. Haec pro casu

$$F = 1,\ G = 0 \text{ et } H = 0,$$

multo facilius atque generalius computari possunt pro quocunque valore quantitatis a, dum sit $b = 0$, tum enim altera aequatio statim dat

$$\partial x = \frac{r\partial s - s\partial r}{ss}, \text{ hineque}$$

$$x = \frac{-r}{s} \text{ et } s = \frac{-r}{x},$$

ex quo prima aequatio hanc induit formam

$$\frac{\partial r}{\partial x} - rr - \frac{2r}{x} + a = 0.$$

Ponamus $r = \frac{a}{t}$, fiet

$$dt + \frac{2t\partial x}{x} - tt\partial x + a\partial x = 0,$$

cui particulariter satisfacit

$$t = \sqrt{a} + \frac{1}{x}.$$

Statuatur ergo

$$t = \sqrt{a} + \tfrac{1}{x} + \tfrac{1}{u},$$

ac prodit

$$\partial u + \partial x + 2u\partial x\sqrt{a} = 0,$$

quae per $e^{2x\sqrt{a}}$ multiplicata et integrata praebet

$$e^{2x\sqrt{a}}u + \tfrac{1}{2\sqrt{a}}e^{2x\sqrt{a}} = \tfrac{n}{2\sqrt{a}},$$

ideoque

$$\frac{1}{u} = \frac{2e^{2x\sqrt{a}}\sqrt{a}}{n - e^{2x\sqrt{a}}} = \frac{2\sqrt{a}}{ne^{-2x\sqrt{a}} - 1},$$

$$t = \frac{1}{x} + \frac{ne^{-2x\sqrt{a}} + 1}{ne^{-2x\sqrt{a}} - 1}\sqrt{a} = \frac{1}{x} + \frac{n + e^{2x\sqrt{a}}}{n - e^{2x\sqrt{a}}}\sqrt{a} \text{ et}$$

$$r = \frac{ax(n - e^{2x\sqrt{a}})}{n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)},$$

ac propterea

$$s = \frac{-a(n - e^{2x\sqrt{a}})}{n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)}.$$

tum vero postremo

$$P = -1,\ Q = 0 \text{ et } R = -\tfrac{2\partial r}{\partial x} = -2rr - \tfrac{4r}{x} + 2a,$$

seu

$$R = \frac{-2a(nn - 4naxxe^{2x\sqrt{a}} - 2ne^{2x\sqrt{a}} + e^{4x\sqrt{a}})}{[n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)]^2}$$

$$= \frac{-2a(n - e^{2x\sqrt{a}})^2 + 8naxxe^{2x\sqrt{a}}}{[n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)]^2}.$$

Si jam sumatur a evanescens et $n = 1 + \tfrac{2}{3}ac^3\sqrt{a}$, formulae ante inventae resultant. At si a sit quantitas negativa puta $a = -m^2$, capiaturque $n = \frac{\alpha\sqrt{-1} + \beta}{\alpha\sqrt{-1} - \beta}$, reperitur

$$r = \frac{-mmx(\beta\cos.mx + \alpha\sin.mx)}{\beta\cos.mx + \alpha\sin.mx - mx(\alpha\cos.mx - \beta\sin.mx)} = \frac{-mmx\cos.(mx + \gamma)}{\cos.(mx + \gamma) - mx\sin.(mx + \gamma)}$$

et

$$s = \frac{mm\cos.(mx+\gamma)}{\cos.(mx+\gamma) - mx\sin.(mx+\gamma)},$$

indeque

$$R = \frac{2mm[\cos.(mx+\gamma)^2 + mmxx]}{[\cos.(mx+\gamma) - mx\sin.(mx+\gamma)]^2}.$$

Quantitas R reducitur ad hanc

$$R = \frac{8naaxx - 2a(ne^{-x\sqrt{a}} - e^{x\sqrt{a}})^2}{[n(1+x\sqrt{a})e^{-x\sqrt{a}} - (1-x\sqrt{a})e^{x\sqrt{a}}]^2},$$

quae forma sumto a valde parvo abit in

$$R = \frac{8naaxx - 2a[n-1-(n+1)x\sqrt{a} + \frac{(n-1)}{2}axx - \frac{(n+1)}{6}ax^3\sqrt{a} + \text{etc.}]^2}{[n-1-\frac{1}{2}(n-1)axx + \frac{1}{3}(n+1)ax^3\sqrt{a}]^2}.$$

Statuatur $n = 1 + \beta a\sqrt{a}$, ut sit

$$n - 1 = \beta a\sqrt{a} \text{ et } n + 1 = 2 = \beta a\sqrt{a}, \text{ erit}$$

$$R = \frac{8naaxx - 2a(\beta a\sqrt{a} - 2x\sqrt{a} - \beta aax + \frac{\beta aaxx\sqrt{a}}{2} - \frac{1}{3}ax^3\sqrt{a})^2}{(\beta a\sqrt{a} - \frac{1}{2}\beta aaxx\sqrt{a} + \frac{2}{3}ax^3\sqrt{a})^2},$$

ubi numerator fit

$$8aaxx + 8\beta a^3xx\sqrt{a} - 2a(\beta\beta a^3 - 4\beta aax - 2\beta\beta a^3x\sqrt{a})$$
$$+ 4axx + \frac{4}{3}aax^4,$$

ubi cum termini per aa affecti se destruant, retineantur ii soli qui per a^3 sunt affecti, erit idem in denominatore observato

$$R = \frac{8\beta a^3x - \frac{8}{3}a^3x^4}{a^3(\beta + \frac{2}{3}x^3)^2} = \frac{8x(\beta - \frac{1}{3}x^3)}{(\beta + \frac{2}{3}x^3)^2},$$

quae jam facile ad formam

$$R = \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2}$$

reducitur, sumendo

$$3\beta = 2c^3, \text{ ut sit } \beta = \frac{2}{3}c^3.$$

Quare hic casus oritur, sumendo a evanescens et

$$n = 1 + \frac{2}{3}c^3a\sqrt{a}.$$

Scholion 2.

372. Cum evolutio solutionis inventae sit difficillima, neque ulla via pateat, quomodo ambae quantitates incognitae r et s ex binis aequationibus erutis definiri queant, in scientiae incrementum haud parum juvabit observasse, idem problema per repetitionem transformationis in primo problemate hujus capitis quoque solvi posse, neque proinde usu carebit has duas solutiones inter se comparasse. Proposita ergo aequatione

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + H v,$$

ponamus primo

$$u = \left(\frac{\partial v}{\partial x}\right) + p v,$$

ac p ex hac aequatione determinetur

$$F\partial p + G p\partial x - F p p\partial x + (C - H)\partial x = 0,$$

ac tum ista resultabit aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = F\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(G + \frac{\partial F}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \left(H + \frac{\partial G}{\partial x} - \frac{2F\partial p - p\partial F}{\partial x}\right) u.$$

Nunc pro hac aequatione porro transformando, statuamus simili modo

$$z = \left(\frac{\partial u}{\partial x}\right) + q u,$$

ita ut sit quoque

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + (p + q)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial p}{\partial x} + p q\right) v,$$

et quantitate q ex hac aequatione definita

$$F\partial q + \left(G + \frac{\partial F}{\partial x}\right) q\partial x - F q q\partial x + \left(D - H - \frac{\partial G}{\partial x} + \frac{2F\partial p + p\partial F}{\partial x}\right)\partial x = 0,$$

orietur haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + R z,$$

cujus quantitates P, Q, R ita se habent

$$P = F,\ Q = G + \frac{2\partial F}{\partial x} \text{ et}$$

$$R = H + \frac{2\partial G}{\partial x} - \frac{2F\partial p - p\partial F}{\partial x} + \frac{\partial\partial F}{\partial x^2} + \frac{2F\partial q - q\partial F}{\partial x}.$$

Cum hac ergo solutione convenire debet ea, quam postremum problema suppeditavit, in quo cum statim posuerimus

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v,$$

erit utique

$$r = p + q \text{ et } s = \frac{dp}{\partial x} + p\,q,$$

unde quidem statim valores pro P, Q et R manifesto prodeunt iidem. Verum multo minus apparet, si pro r et s isti valores per p et q substituantur, tum istas binas aequationes

$$\frac{\partial.(G - Fr)}{\partial x} - (G - F\,r)\,r + 2\,(H - F\,s) = A \text{ et}$$

$$\frac{(G - Fr)\partial.(H - Fs)}{\partial x} + (H - F\,s)^2 - (G - F\,r)^2\,s - A\,(H - F\,s) = B,$$

ad eas quas ante invenimus reduci

$$\frac{F\partial p}{\partial x} + G\,p - F\,p\,p - H + C = 0 \text{ et}$$

$$\frac{F\partial q}{\partial x} + \left(G + \frac{\partial F}{\partial x}\right) q - F\,q\,q - H - \frac{\partial G}{\partial x} + \frac{2F\partial p + p\partial F}{\partial x} + D = 0,$$

ita ut hae constantes C et D ad illas A et B certam teneant relationem. Interim patet has postremas aequationes multo esse simpliciores, dum prior duas tantum variabiles p et x complectitur, indeque p per x, cujus F, G et H sunt functiones datae, determinari debet, qua inventa quantitatem q simili modo ex altera aequatione elici oportet. Verum in ambabus superioribus aequationibus binae variabiles r et s ita inter se sunt permixtae, ut nulla methodus eas resolvendi, vel adeo ad aequationem inter duas tantum variabiles perveniendi, habeatur. Cum igitur certum sit priores solutu difficillimas ad posteriores multo faciliores ope substitutionum assignatarum perduci posse, sine dubio methodus hanc reductionem efficiendi haud contemnenda subsidia in Analysin esse allatura videtur.

Scholion 3.

373. Cum adeo consensus harum duarum solutionum maxime sit absconditus, casum specialem accuratius perpendi expediet. Sit igitur

$$F = 1, \quad G = 0 \text{ et } H = 0,$$

ac binae priores aequationes inter r et s has induent formas

$$\text{I.} \quad \frac{-\partial r}{\partial x} + rr - 2s = A \text{ et}$$

$$\text{II.} \quad \frac{r \partial s}{\partial x} + ss - rrs + As = B,$$

posteriores vero istas

$$\text{III.} \quad \frac{\partial p}{\partial x} - pp + C = 0 \text{ et}$$

$$\text{IV.} \quad \frac{\partial q}{\partial x} - qq + \frac{2 \partial p}{\partial x} + D = 0,$$

quas cum illis certum est ita cohaerere, ut sit

$$r = p + q \text{ et } s = \frac{\partial p}{\partial x} + pq.$$

Ut saltem consensum a posteriori agnoscamus, sit $C = -mm$ et tertia dat

$$\partial x = \frac{\partial p}{mm + pp}, \text{ hinc}$$

$$x = \frac{1}{m} \text{ Ang. tang. } \frac{p}{m} \text{ et } pm = \text{tang. } mx.$$

Hinc cum sit

$$\frac{\partial p}{\partial x} = mm + pp, \text{ erit}$$

$$s = mm + pp + pq = mm + pr = m(m + r \text{ tang. } mx),$$

qui valor in I. substitutus dat

$$\frac{-\partial r}{\partial x} + rr - 2mr \text{ tang. } mx - 2mm = A, \text{ seu}$$

$$\frac{\partial r}{\partial x} = rr - 2mr \text{ tang. } mx - 2mm - A,$$

secunda vero ob

$$\frac{\partial s}{\partial x} = \frac{m \partial r}{\partial x} \text{ tang. } mx + \frac{mmr}{\cos. mx^2}$$

abit in

$$\frac{m r \partial r}{\partial x} \text{tang.}\, m x = m r^3 \text{tang.}\, m x - 2 m m r r \text{tang.}\, m x^2$$
$$- m(A + 2mm) r \text{tang.}\, m x - m^4 - A m m + B,$$

ex quibus ∂r eliminando fit

$$B = A m m + m^4.$$

Pro quarta vero ob

$$q = r - p = r - m \text{tang.}\, m x,$$

resultat

$$\frac{\partial r}{\partial x} = r r - 2 m r \text{tang.}\, m x - m m - D,$$

ita ut sit

$$D = m m + A.$$

Consensus ergo nostrarum aequationum in hac constantium relatione consistit, ut ob $m m = - C$ sit

$$D = A - C \text{ et } B = - C(A - C) = - C D.$$

In genere vero etiam eaedem relationes locum habent, nam si III et IV. in unam summam colligantur, ob

$$C + D = A \text{ et } p + q = r, \text{ erit}$$
$$\frac{F\partial r}{\partial x} + Gr + \frac{r\partial F}{\partial x} - Fpp - Fqq - 2H - \frac{\partial G}{\partial x} + \frac{2F\partial p}{\partial x} + A = 0,$$

cum vero sit $\frac{\partial p}{\partial x} = s - p q$, fit

$$\frac{F\partial r + r\partial F - \partial G}{\partial x} + G r - F r r - 2 H + 2 F s + A = 0, \text{ seu}$$
$$\frac{\partial.(G - Fr)}{\partial x} - (G - F r) r + 2 (H - F s) = A,$$

quae est ipsa aequatio prima. Porro aequatio III. ob $\frac{\partial p}{\partial x} = s - p q$ dat

$$Fs - Fpr + Gp - H + C = 0, \text{ seu } C = H - Fs - p(G - Fr),$$

quarta vero reducitur ad hanc formam

$$\frac{F\partial r}{\partial x} + G q + \frac{q\partial F}{\partial x} - F q q - H - \frac{\partial G}{\partial x} + F s - F p q + \frac{p\partial F}{\partial x} + D = 0,$$

seu

$$\frac{\partial.(Fr-G)}{\partial x} + q(G - Fr) - H + Fs + D = 0,$$

hincque

$$D = \frac{\partial.(G-Fr)}{\partial x} - q(G - Fr) + H - Fs,$$

ex quibus concluditur

$$CD = \frac{(H-Fs)\partial.(G-Fr)}{\partial x} - q(G-Fr)(H-Fs) + (H-Fs)^2$$
$$- \frac{p(G-Fr)\partial.(G-Fr)}{\partial x} + pq(G-Fr)^2 - p(G-Fr)(H-Fs).$$

Ex secunda vero habemus

$$B = \frac{(G-Fr)\partial.(H-Fs)}{\partial x} - \frac{(H-Fs)\partial.(G-Fr)}{\partial x} - (H - Fs)^2$$
$$+ (G-Fr)(H-Fs)r - (G-Fr)^2 s,$$

quibus expressionibus conjunctis fit

$$\frac{CD+B}{G-Fr} = \frac{\partial.(H-Fs)}{\partial x} - \frac{p\partial.(G-Fr)}{\partial x} - \frac{\partial p(G-Fr)}{\partial x}$$
$$= \frac{\partial.(H-Fs) - \partial.p(G-Fr)}{\partial x} = 0,$$

siquidem est

$$C = H - Fs - p(G - Fr),$$

ex quo etiam in genere est

$$B = -CD \text{ et } A = C + D.$$

Interim tamen hinc non perspicitur, quomodo ex aequationibus I. et II. binae reliquae III. et IV. derivari queant.

Scholion 4.

374. Omnibus his diligenter pensitatis manifestum fiet, totum negotium ope substitutionis satis simplicis confici posse. Quod quo facilius ostendatur, ponamus brevitatis causa

$$G - Fr = R \text{ et } H - Fs = S,$$

ut habeantur hae duae aequationes

$$\text{I.}\quad A = \frac{\partial R}{\partial x} - \frac{GR}{F} + \frac{RR}{F} + 2S,$$

$$\text{II.}\quad B = \frac{R\partial S - S\partial R}{\partial x} - \frac{HRR}{F} + \frac{GRS}{F} - SS,$$

ex quibus duas quantitates R et S erui oporteat, dum F, G, H sunt functiones quaecunque ipsius x, at A et B quantitates constantes. Ad hoc adhibeatur ista substitutio $S = C + Rp$ ita adornanda, ut binae illae aequationes coalescant in unam, in qua praeter x unica insit nova variabilis p, deinceps per methodos cognitas investiganda. Hinc ob

$\partial S = R\partial p + p\partial R$ habebitur

$$\text{I.}\quad A = \frac{\partial R}{\partial x} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp,$$

$$\text{II.}\quad B = \frac{RR\partial p}{\partial x} - \frac{C\partial R}{\partial x} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F}$$
$$- CC + 2CRp - RRpp,$$

unde primo eliminando ∂R, concluditur

$$B + AC = \frac{RR\partial p}{\partial x} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp,$$

dummodo ergo constantem C ita assumamus, ut sit $CC = B + AC$, per divisionem etiam ipsa quantitas R tolletur, resultabitque haec aequatio

$$0 = \frac{\partial p}{\partial x} + \frac{C}{F} - \frac{H}{F} - pp,$$

cujus resolutio ad methodos magis cognitas pertinet. Cum igitur ista methodus maximi sit momenti, sequens problema, etiamsi ad primam partem calculi integralis sit referendum, hic adjicere operae pretium videtur.

Problema 60.

375. Propositis hujusmodi duabus aequationibus differentialibus

$$\text{I.}\quad 0 = \frac{\partial y}{\partial x} + F + Gy + Hz + Iyy + Kyz + Lzz,$$

$$\text{II.}\quad 0 = \frac{y\partial z - z\partial y}{\partial x} + P + Qy + Rz + Syy + Tyz + Vzz,$$

ubi F, G, H, etc. P, Q, R, etc. sint functiones ipsius x, methodum exponere has aequationes, siquidem fieri licet, resolvendi.

Solutio.

Methodus indicata in hoc consistit, ut ope substitutionis $z = a + yv$ ex illis aequationibus una elici queat duas tantum variabiles x et v implicans. Quoniam igitur est

$$y \partial z - z \partial y = yy \partial v - a \partial y,$$

ex $\mathrm{I} \times a + \mathrm{II}$. nascitur haec aequatio

$$0 = \frac{yy\partial v}{\partial x} + P + Qy + Rz + Syy + Tyz + Vzz$$
$$+ aF + aGy + aHz + aIyy + aKyz + aLzz,$$

quae, loco z substituto valore $a + yv$, ita exhibeatur secundum potestates ipsius y

$$0 = \frac{yy\partial v}{\partial x} + y^0 [P + aF + a(R + aH) + aa(V + aL)]$$
$$+ y^2 [Q + aG + v(R + aH) + a(T + aK) + 2av(V + aL)]$$
$$+ y^2 [S + aI + v(T + aK) + vv(V + aL)],$$

nuncque efficiendum est, ut tota aequatio per yy dividi queat. ideoque partes per y^0 et y^1 affectae evanescant. Ex parte ergo y^0 fieri oportet

$$P + aF + a(R + aH) + aa(V + aL) = 0,$$

ex parte autem y^1, quia v est nova variabilis in calculum inducta, hae duae conditiones nascuntur

$$Q + aG + a(T + aK) = 0 \text{ et}$$

$$R + aH + 2a(V + aL) = 0,$$

unde prima dabit

$$P + aF - aa(V + aL) = 0.$$

Conditiones ad istam reductionem requisitae sunt hae tres

$$\text{I. } P + aF - aa(V + aL) = 0,$$
$$\text{II. } Q + aG + a(T + aK) = 0,$$
$$\text{III. } R + aH + 2a(V + aL) = 0,$$

unde vel P, Q et R, vel F, G et H commode definiuntur.

His autem conditionibus stabilitis, totum negotium ad resolutionem hujus aequationis revocatur

$$0 = \frac{\partial v}{\partial x} + S + aI + v(T + aK) + vv(V + aL),$$

quae duas tantum continet variabiles x et v, ex qua v per x determinari oportet. Cum deinde posito $z = a + yv$ prima aequatio induat hanc formam

$$0 = \frac{\partial y}{\partial x} + F + aH + aaL + y(G + Hv + aK + 2aLv)$$
$$+ yy(I + Kv + Lvv),$$

secunda vero istam

$$0 = \frac{yy\partial v}{\partial x} - \frac{a\partial y}{\partial x} + P + aR + aaV + y(Q + Rv + aT + 2aVv)$$
$$+ yy(S + Tv + Vvv),$$

seu hinc superiorem per yy multiplicatam subtrahendo

$$0 = \frac{-a\partial y}{\partial x} + P + aR + aaV + y(Q + Rv + aT + 2aVv)$$
$$- yy(Ia + aKv + aLvv),$$

quae quidem cum illa congruit, ut natura rei postulat.

Corollarium 1.

376. Si ergo hujusmodi binae aequationes fuerint propositae

$$0 = \frac{\partial y}{\partial x} + F + Gy + Hz + Iyy + Kyz + Lzz,$$
$$0 = \frac{y\partial z - z\partial y}{\partial x} - aF - aGy - aHz + Syy + Tyz + Vzz$$
$$+ a^3L - aaKy - 2aaLz$$
$$+ aaV - aTy - 2aVz,$$

facto $z = a + yv$, primo resolvi debet haec aequatio

$$0 = \frac{\partial v}{\partial x} + S + aI + v(T + aK) + vv(V + aL),$$

unde definita v per x, hanc aequationem tractari oportet

$$0 = \frac{\partial y}{\partial x} + F + aH + aaL + y(G + aK) + yy(I + Kv + Lvv) + vy(H + 2aL),$$

quo facto habebitur quoque $z = a + vy$.

Corollarium 2.

377. Si $F = A$, $K = 0$, $L = 0$, $H = -2b$, $V = b$ et $T = -G$, casus supra §. 374. tractatus resultat harum aequationum

$$0 = \frac{\partial y}{\partial x} + A + Gy - 2bz + Iyy,$$

$$0 = \frac{y\partial z - z\partial y}{\partial x} - aA + Syy - Gyz + bzz, + aab,$$

ubi G, I et S sunt functiones quaecunque ipsius x, et resolutio ita se habet, ut posito $x = a + y.v$, hae aequationes successive debeant expediri

$$0 = \frac{\partial v}{\partial x} + S + aI - Gv + bvv \text{ et}$$

$$0 = \frac{\partial y}{\partial x} + A - 2ab + y(G - 2bv) + Iyy.$$

Corollarium 3.

378. Evidens est postremam aequationem nulla laborare difficultate, etiam in genere dum sit

$$F + aH + aaL = 0,$$

prioris autem solutio in promtu est, si sit vel $S + aT = 0$, vel $V = aL = 0$.

CALCULI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA,

SEU

INVESTIGATIO FUNCTIONUM DUARUM VARIABILIUM EX DATA DIFFERENTIALIUM CUJUSVIS GRADUS RELATIONE.

SECTIO TERTIA,

INVESTIGATIO DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM TERTII ALTIORUMQUE GRADUUM RELATIONE.

CAPUT I.

DE RESOLUTIONE AEQUATIONUM SIMPLICISSIMARUM UNICAM FORMULAM DIFFERENTIALEM INVOLVENTIUM.

Problema 61.

379.

Indolem functionis binarum variabilium x et y indagare, si ejus quaepiam formula differentialis tertii gradus evanescat.

Solutio.

Sit z functio illa quaesita, et cum ejus sint quatuor formulae differentiales tertii gradus

$$\left(\frac{\partial^3 z}{\partial x^3}\right),\ \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right),\ \left(\frac{\partial^3 z}{\partial x \partial y^2}\right) \text{ et } \left(\frac{\partial^3 z}{\partial y^3}\right),$$

prout quaelibet harum nihilo aequalis statuitur, totidem habemus casus evolvendos.

I. Sit igitur primo $\left(\frac{\partial^3 z}{\partial x^3}\right) = 0$, et sumta y constante prima integratio praebet

$$\left(\frac{\partial \partial z}{\partial x^2}\right) = \Gamma : y,$$

tum simili modo secunda integratio dat

$$\left(\frac{\partial z}{\partial x}\right) = x\Gamma : y + \Delta : y,$$

unde tandem fit

$$z = \tfrac{1}{2} x x \Gamma : y + x \Delta : y + \Sigma : y,$$

ubi $\Gamma : y$, $\Delta : y$ et $\Sigma : y$ denotant functiones quascunque ipsius y, ita ut ob triplicem integrationem tres functiones arbitrariae in calculum sint ingressae, ut rei natura postulat.

II. Sit $\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = 0$, ac primo bis integrando per solius x variabilitatem reperitur ut ante

$$\left(\frac{\partial z}{\partial y}\right) = x\, \Gamma' : y + \Delta' : y,$$

nunc autem sola y pro variabili habita, adipiscimur

$$z = x\, \Gamma : y + \Delta : y + \Sigma : x,$$

quandoquidem apices signis functionum inscripti hic semper hunc habent significatum, ut sit

$$\int \partial y\, \Gamma' : y = \Gamma : y \text{ et } \int \partial y\, \Delta' : y = \Delta : y.$$

III. Sit $\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = 0$, et quia hic casus a praecedente non differt, nisi quod binae variabiles x et y inter se sint permutatae, integrale quaesitum est

$$z = y\, \Gamma : x + \Delta : x + \Sigma : y.$$

IV. Sit $\left(\frac{\partial^3 z}{\partial y^3}\right) = 0$, et ob similem permutationem ex casu primo intelligitur fore

$$z = \tfrac{1}{2} y^2\, \Gamma : x + y\, \Delta : x + \Sigma : x.$$

Corollarium 1.

380. Tres functiones arbitrariae, hic per triplicem integrationem ingressae, sunt vel ipsius x, vel ipsius y tantum; omnes tres sunt ipsius y tantum casu primo $\left(\frac{\partial^3 z}{\partial x^3}\right) = 0$, ipsius x vero tantum casu quarto $\left(\frac{\partial^3 z}{\partial y^3}\right) = 0$; duae vero sunt ipsius y et una ipsius x casu secundo $\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = 0$; contra autem duae ipsius x et una ipsius y casu tertio $\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = 0$,

Corollarium 2.

381. Porro observasse juvabit, si ejusdem variabilis puta x duae pluresve occurrant functiones arbitrariae, unam quidem absolute poni, alteram per y multiplicari, tertiam vero si adsit per $\frac{1}{2}yy$, seu quod eodem redit, per yy multiplicatam accedere.

Corollarium 3.

382. Perpetuo autem tenendum est has functiones ita arbitrio nostro relinqui, et etiam functiones discontinuae seu nulla continuitatis lege contentae non excludantur. Scilicet si libero manus tractu linea quaecunque describatur, applicata respondens abscissae x hujusmodi functionem $\Gamma : x$ referet.

Scholion 1.

383. Minus hic immorandum arbitror transformationi formularum differentialium altioris gradus, dum loco binarum variabilium x et y aliae quaecunque in calculum introducuntur, quoniam in genere expressiones nimis fierent complicatae vixque ullum usum habiturae, tum vero imprimis quod methodus has transformationes inveniendi jam supra (§. 229) satis luculenter est tradita. Casum tantum simpliciorem, quo binae novae variabiles t et u loco x et y introducendae ita accipiuntur, ut sit

$$t = \alpha x + \beta y \text{ et } u = \gamma x + \delta y,$$

hic quoque ad formulas differentiales altiores accommodabo. Cum igitur viderimus esse,

pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = \alpha\left(\frac{\partial z}{\partial t}\right) + \gamma\left(\frac{\partial z}{\partial u}\right),$$

$$\left(\frac{\partial z}{\partial y}\right) = \beta\left(\frac{\partial z}{\partial t}\right) + \delta\left(\frac{\partial z}{\partial u}\right),$$

et pro formulis secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \alpha^2 \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\,\alpha\,\gamma \left(\frac{\partial\partial z}{\partial t \partial u}\right) + \gamma^2 \left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \alpha\,\beta \left(\frac{\partial\partial z}{\partial t^2}\right) + (\alpha\,\delta + \beta\,\gamma) \left(\frac{\partial\partial z}{\partial t \partial u}\right) + \gamma\,\delta \left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \beta^2 \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\,\beta\,\delta \left(\frac{\partial\partial z}{\partial t \partial u}\right) + \delta^2 \left(\frac{\partial\partial z}{\partial u^2}\right),$$

erit pro formulis tertii gradus

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = \alpha^3 \left(\frac{\partial^3 z}{\partial t^3}\right) + 3\alpha^2\gamma \left(\frac{\partial^3 z}{\partial t^2 \partial u}\right) + 3\alpha\gamma^2 \left(\frac{\partial^3 z}{\partial t \partial u^2}\right) + \gamma^3 \left(\frac{\partial^3 z}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = \alpha^2\beta \left(\frac{\partial^3 z}{\partial t^3}\right) + (\alpha^2\delta + 2\alpha\beta\gamma) \left(\frac{\partial^3 z}{\partial t^2 \partial u}\right) + (\beta\gamma^2 + 2\alpha\gamma\delta) \left(\frac{\partial^3 z}{\partial t \partial u^2}\right) + \gamma^2\delta \left(\frac{\partial^3 z}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = \alpha\beta^2 \left(\frac{\partial^3 z}{\partial t^3}\right) + (\beta\beta\gamma + 2\alpha\beta\delta) \left(\frac{\partial^3 z}{\partial t^2 \partial u}\right) + (\alpha\delta^2 + 2\beta\gamma\delta) \left(\frac{\partial^3 z}{\partial t \partial u^2}\right) + \gamma\delta^2 \left(\frac{\partial^3 z}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 z}{\partial y^3}\right) = \beta^3 \left(\frac{\partial^3 z}{\partial t^3}\right) + 3\beta^2\delta \left(\frac{\partial^3 z}{\partial t^2 \partial u}\right) + 3\beta\delta^2 \left(\frac{\partial^3 z}{\partial t \partial u^2}\right) + \delta^3 \left(\frac{\partial^3 z}{\partial u^3}\right),$$

et pro formulis quarti gradus

	$\left(\frac{\partial^4 z}{\partial t^4}\right)$	$\left(\frac{\partial^4 z}{\partial t^3 \partial u}\right)$	$\left(\frac{\partial^4 z}{\partial t^2 \partial u^2}\right)$	$\left(\frac{\partial^4 z}{\partial t \partial u^3}\right)$	$\left(\frac{\partial^4 z}{\partial u^4}\right)$
$\left(\frac{\partial^4 z}{\partial x^4}\right)$	$=\alpha^4$	$+4\alpha^3\gamma$	$+6\alpha^2\gamma^2$	$+4\alpha\gamma^3$	$+\gamma^4$
$\left(\frac{\partial^4 z}{\partial x^3 \partial y}\right)$	$=\alpha^3\beta$	$\alpha^3\delta+3\alpha^2\beta\gamma$	$3\alpha^2\gamma\delta+3\alpha\beta\gamma^2$	$+3\alpha\gamma^2\delta+\beta\gamma^3$	$+\gamma^3\delta$
$\left(\frac{\partial^4 z}{\partial x^2 \partial y^2}\right)$	$=\alpha^2\beta^2$	$2\alpha^2\beta\delta+2\alpha\beta^2\gamma$	$\alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2$	$2\alpha\gamma\delta^2+2\beta\gamma^2\delta$	$+\gamma^2\delta^2$
$\left(\frac{\partial^4}{\partial x \partial y^3}\right)$	$=\alpha\beta^3$	$3\alpha\beta^2\delta+\beta^3\gamma$	$3\alpha\beta\delta^2+3\beta^2\gamma\delta$	$\alpha\delta^3+3\beta\gamma\delta^2$	$+\gamma\delta^3$
$\left(\frac{\partial^4 z}{\partial y^4}\right)$	$=\beta^4$	$+4\beta^3\delta$	$+6\beta^2\delta^2$	$+4\beta\delta^3$	$+\delta^4$

unde simul lex pro altioribus gradibus elucet: pro formula scilicet generali $\left(\frac{\partial^{m+n} z}{\partial z^m \partial y^n}\right)$ hi coefficientes iidem sunt qui oriuntur ex evolutione hujus formae

$$(\alpha + \gamma\,v)^m\,(\beta + \delta\,v)^n,$$

siquidem termini secundum potestates ipsius v disponantur.

Scholion 2.

384. Haud alienum fore arbitror evolutionem istius formulae ex principiis ante stabilitis accuratius docere. Sit igitur

$$s = (\alpha + \gamma v)^m (\beta + \delta v)^n,$$

ac ponatur

$$s = A + Bv + Cv^2 + Dv^3 + Ev^4 + Fv^5 + \text{etc.}$$

ubi quidem primo patet esse $A = \alpha^m \beta^n$; pro reliquis vero coefficientibus inveniendis, sumtis differentialibus logarithmorum, habebimus

$$\frac{\partial s}{s \partial v} = \frac{m\gamma}{\alpha + \gamma v} + \frac{n\delta}{\beta + \delta v}, \text{ ideoque}$$

$$\frac{\partial s}{\partial v}[\alpha\beta + (\alpha\delta + \beta\gamma)v + \gamma\delta vv]$$
$$- s[m\beta\gamma + n\alpha\delta + (m+n)\gamma\delta v] = 0,$$

ubi si loco s series assumta substituatur, orietur haec aequatio

$$\begin{array}{lllll}
0 = \alpha\beta B + 2\alpha\beta C v & + 3\alpha\beta D v^2 & + 4\alpha\beta E v^3 & + 5A\beta F v^4 + \text{etc.} \\
\quad\quad + \alpha\delta B & + 2\alpha\delta C & + 3\alpha\delta D & + 4\alpha\delta E \\
\quad\quad + \beta\gamma B & + 2\beta\gamma C & + 3\beta\gamma D & + 4\beta\gamma E \\
 & + \gamma\delta B & + 2\gamma\delta C & + 3\gamma\delta D \\
- m\beta\gamma A - m\beta\gamma B & - m\beta\gamma C & - m\beta\gamma D & - m\beta\gamma E \\
- n\alpha\delta A \;\; - n\alpha\delta B & - n\alpha\delta C & - n\alpha\delta D & - n\alpha\delta E \\
\quad\quad - (m+n)\gamma\delta A & - (m+n)\gamma\delta B & - (m+n)\gamma\delta C & - (m+n)\gamma\delta D
\end{array}$$

unde quilibet coefficiens ex praecedentibus ita definitur

$$A = \alpha^m \beta^n,$$

$$B = \frac{m\beta\gamma + n\alpha\delta}{\alpha\beta} A,$$

$$C = \frac{(m-1)\beta\gamma + (n-1)\alpha\delta}{2\alpha\beta} B + \frac{(m+n)\gamma\delta}{2\alpha\beta} A,$$

$$D = \frac{(m-2)\beta\gamma + (n-2)\alpha\delta}{3\alpha\beta} C + \frac{(m+n-1)\gamma\delta}{3\alpha\beta} B,$$

$$E = \frac{(m-3)\beta\gamma + (n-3)\alpha\delta}{4\alpha\beta} D + \frac{(m+n-2)\gamma\delta}{4\alpha\beta} C,$$

etc.

His igitur coefficientibus inventis, si ponatur

$$t = \alpha x + \beta y \text{ et } \gamma x + \delta y,$$

transformatio formulae differentialis cujuscunque ita se habebit, ut sit

$$\left(\frac{\partial^{m+n} z}{\partial x^m \partial y^n}\right) = A \left(\frac{\partial^{m+n} z}{\partial t^{m+n}}\right) + B \left(\frac{\partial^{m+n} z}{\partial t^{m+n-1} \partial u}\right)$$

$$+ C \left(\frac{\partial t^{m+n} z}{\partial t^{m+n-2} \partial u^2}\right) + \text{etc.}$$

Problema 62.

385. Indolem functionis binarum variabilium x et y investigare, si ejus formula differentialis cujuscunque gradus evanescat.

Solutio.

Ex iis quae de formulis differentialibus tertii gradus nihilo aequatis ostendimus in praecedente problemate, satis perspicuum est solutionem hujus problematis pro formulis differentialibus quarti gradus ita se habere.

I. Si sit $\left(\frac{\partial^4 z}{\partial x^4}\right) = 0$, erit

$$z = x^3 \Gamma : y + x^2 \Delta : y + x \Sigma : y + \Theta : y.$$

II. Si sit $\left(\frac{\partial^4 z}{\partial x^3 \partial y}\right) = 0$, erit

$$z = x^2 \Gamma : y + x \Delta : y + \Sigma : y + \Theta : x.$$

III. Si sit $\left(\frac{\partial^4 z}{\partial x^2 \partial y^2}\right) = 0$, erit

$$z = x \Gamma : y + \Delta : y + y \Sigma : x + \Theta : x.$$

IV. Si sit $\left(\frac{\partial^4 z}{\partial x \partial y^3}\right) = 0$, erit

$$z = \Gamma : y + y^2 \Delta : x + y \Sigma : x + \Theta : x.$$

V. Si sit $\left(\frac{\partial^4 z}{\partial y^4}\right) = 0$, erit

$$z = y^3 \Gamma : x + y^2 \Delta : x + y \Sigma : x + \Theta : x,$$

unde simul progressus ad altiores gradus est manifestus.

Corollarium 1.

386. Cum hic quatuor functiones arbitrariae occurrunt, totidem scilicet quot integrationes institui oportet, in hoc ipso criterium integrationis completae continetur.

Corollarium 2.

387. Quin etiam vicissim facile ostenditur, formas inventas aequationi propositae satisfacere. Sic cum pro casu tertio invenerimus:

$$z = x \Gamma : y + \Delta : y + y \Sigma : x + \Theta : x,$$

differentiando hinc colligimus

Primo $\left(\frac{\partial z}{\partial x}\right) = \Gamma : y + y \Sigma' : x + \Theta' : x$,

deinde $\left(\frac{\partial \partial z}{\partial x^2}\right) = y \Sigma'' : x + \Theta'' : x$,

tertio $\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = \Sigma'' : x$ et

quarto $\left(\frac{\partial^4 z}{\partial x^2 \partial y^2}\right) = 0$,

eodemque pervenitur, quocunque ordine differentiationes, vel solam x vel solam y variabilem sumendo, instituantur.

Scholion 1.

388. Hactenus unam formulam differentialem nihilo esse aequalem assumsimus, calculus autem perinde succedit, si hujusmodi formula functioni cuicunque ipsarum x et y aequalis statuatur: quemadmodum in sequentibus problematibus sum ostensurus. Hoc tantum inculcandum censeo, si V fuerit functio quaecunque binarum variabilium x et y, tum $\int V \partial x$ id denotare integrale, quod obti-

netur si sola x pro variabili habeatur, in hac vero formula $\int V \partial y$ solam y pro variabili haberi: quod idem tenendum est de integrationibus repetitis veluti $\int \partial x \int V \partial x$, ubi in utraque sola x variabilis assumitur, in hac vero $\int \partial y \int V \partial x$, postquam integrale $\int V \partial x$ ex sola ipsius x variabilitate fuerit erutum, tum in altera integratione $\int \partial y \int V \partial x$ solam y variabilem accipiendam esse. Et cum perinde utra integratio prior instituatur, etiam hoc discrimen e modo signandi tolli potest, hocque integrale geminatum ita $\int\int V \partial x \partial y$ exhiberi: hincque intelligitur, quomodo has formulas

$$\int\int\int V \partial x^2 \partial y, \text{ seu } \int^3 V \partial x^2 \partial y \text{ et } \int^{m+n} V \partial x^m \partial y^n,$$

interpretari oporteat; hic scilicet signo integrationis $\int$ indices suffigimus, prorsus uti signo differentiationis ∂ suffigi solent, quippe qui indicant, quoties integratio sit repetenda.

Scholion 2.

389. Singulas has integrationes repetendas ita institui hic assumimus, ut nulla relatio inter binas variabiles x et y in subsidium vocetur, quae circumstantia eo diligentius est animadvertenda, cum vulgo, ubi talibus integrationibus opus est, calculus prorsus diverso modo institui debeat. Quodsi enim proposito quopiam corpore geometrico, ejus soliditas seu superficies sit investiganda per duplicem integrationem hujusmodi formula $\int\int V \partial x \partial y$ evolvi debet, existente V certa functione ipsarum x et y; ubi quidem primo quaeritur integrale $\int V \partial y$ spectata x ut constante; at absoluta integratione ad terminos integrationi praescriptos respici oportet, dum scilicet altero praescribitur, ut hoc integrale $\int V \partial y$ evanescat posito $y = 0$, altero vero id eo usque extendendum est, donec y datae cuipiam functioni ipsius x aequetur. Tum vero postquam hoc integrale $\int V \partial y$ isto modo fuerit determinatum, altera demum integratio formulae $\partial x \int V \partial y$ suscipitur, in qua quantitas y non amplius inest, dum ejus loco certa quaepiam functio ipsius x est sub-

stituta, eaque formula jam revera unicam variabilem x complectitur. Hic ergo prima integratione absoluta, variabilis y in functionem ipsius x abire est censenda, quam propterea in altera integratione, ubi x est variabilis, minime ut constantem spectare licebit. Ex quo patet hunc casum toto coelo esse diversum ab iis integrationibus repetendis, quas hic contemplamur, ad quem propterea hic eo minus respicimus, cum ista peculiaris ratio tantum in formula $\int\int V \partial x \partial y$ locum habere possit; reliquis vero ubi alterum differentiale ∂x vel ∂y saepius repetitur, adeo adversetur. Quam ob causam hinc omnem relationem, quae forte peracta una integratione inter binas variabiles x et y statui posset, merito removemus.

Problema 63.

390. Si formula quaepiam differentialis tertii altiorisve gradus aequetur functioni cuicunque binarum variabilium x et y, indolem functionis z definire.

Solutio.

Sit V functio quaecunque binarum variabilium x et y, et incipientes a formulis tertii ordinis sit primo $\left(\frac{\partial^3 z}{\partial x^3}\right) = V$, et posita sola x variabili erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \int V \partial x + \Gamma : y:$$

tum vero porro

$$\left(\frac{\partial z}{\partial x}\right) = \int \partial x \int V \partial x + x \Gamma : y + \Delta : y = \int\int V \partial x^2 + x \Gamma : y + \Delta : y,$$

ac denique

$$z = \int^3 V \partial x^3 + \tfrac{1}{2} x^2 \Gamma : y + x \Delta : y + \Sigma : y.$$

Simili modo patet, si fuerit $\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = V$ fore

$$z = \int^3 V \partial x^2 \partial y + x \Gamma : y + \Delta : y + \Sigma : x;$$

ac si sit $\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = V$, erit

$$z = \int^3 V \partial x \partial y^2 + \Gamma : y + y \Delta : x + \Sigma : x;$$ denique

si sit $\left(\frac{\partial^3 z}{\partial y^3}\right) = V$, erit

$$z = \int^3 V \partial y^3 + y^2 \Gamma : x + y \Delta : x + \Sigma : x.$$

Eodem modo ad formulas altiorum graduum progredientes, reperiemus ut sequitur:

si sit $\left(\frac{\partial^4 z}{\partial x^4}\right) = V$, fore

$$z = \int^4 V \partial x^4 + x^3 \Gamma : y + x^2 \Delta : y + x \Sigma : y + \Theta : y,$$

si sit $\left(\frac{\partial^4 z}{\partial x^3 \partial y}\right) = V$, fore

$$z = \int^4 V \partial x^3 \partial y + x^2 \Gamma : y + x \Delta : y + \Sigma : y + \Theta : x,$$

si sit $\left(\frac{\partial^4 z}{\partial x^2 \partial y^3}\right) = V$, fore

$$z = \int^4 V \partial x^2 \partial y^2 + x \Gamma : y + \Delta : y + y \Sigma : x + \Theta : x,$$

si sit $\left(\frac{\partial^4 z}{\partial x \partial y^3}\right) = V$, fore

$$z = \int^4 V \partial x \partial y^3 + \Gamma : y + y^2 \Delta : x + y \Sigma : x + \Theta : x,$$

si sit $\left(\frac{\partial^4 z}{\partial y^4}\right) = V$, fore

$$z = \int^4 V \partial y^4 + y^3 \Gamma : x + y^2 \Delta : x + y \Sigma : x + \Theta : x,$$

neque pro altioribus gradibus res eget ulteriori explicatione.

Corollarium 1.

391. Quemadmodum signum integrationis in primo libro usitatum jam per se involvit constantem per integrationem ingredientem, ita quoque hic functiones arbitrariae per integrationem ingressae jam in formula integrali involvi sunt censendae, ita ut non sit opus eas exprimere.

Corollarium 2.

392. Sufficit ergo pro aequatione $\left(\frac{\partial^3 z}{\partial x^3}\right) = V$ integrale tri-

plicatum hoc modo dedisse $z = \int^3 V \partial x^3$, quae forma jam potestate complectitur partes supra adjectas

$$x x \Gamma : y + x \Delta : y + \Sigma : y,$$

quod idem de reliquis est tenendum.

Corollarium 3.

393. Si ergo in genere haec habeatur aequatio

$$\left(\frac{\partial^{m+n} z}{\partial x^m \partial y^n}\right) = V,$$

ejus integrale statim hoc modo exhibetur

$$z = \int^{m+n} V \partial x^m \partial y^n,$$

quae potestate jam involvit omnes illas functiones arbitrarias numero $m+n$ per totidem integrationes invectas.

Scholion.

394. Hi casus utique sunt simplicissimi, qui ad hoc reverendi videntur, pro magis autem complicatis vix certa praecepta tradere licet, cum ista calculi integralis pars vix adhuc coli sit coepta. Interim tamen jam intelligitur, si aequationes magis complicatae ope cujusdam transformationis ad has simplicissimas revocare liceat, etiam earum integrationem in promtu esse futuram, quod quidem negotium hic non copiosius persequendum videtur. Progredior igitur ad casus magis reconditos, eosque ita comparatos, ut ope aequationum inferiorum ordinum expediri queant, unde quidem insignis methodus satis late patens colligi poterit, qua saepius haud sine successu uti licebit. Neque tamen in hac pertractatione nimis diffusum esse convenit, sed sufficiet praecipuos fontes adhuc quidem cognitos patefecisse.

CAPUT II.

DE

INTEGRATIONE AEQUATIONUM ALTIORUM PER REDUCTIONEM AD INFERIORES.

Problema 64.

395.

Proposita hac aequatione tertii gradus $\left(\frac{\partial^3 z}{\partial x^3}\right) = a^3 z$, indolem functionis z investigare.

Solutio.

Fingatur huic aequationi sasisfacere haec simplicior primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = n z,$$

et cum hinc differentiando obtineatur

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = n \left(\frac{\partial z}{\partial x}\right) = n n z,$$

hincque porro

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = n n \left(\frac{\partial z}{\partial x}\right) = n^3 z,$$

evidens est quaesito satisfieri, dum sit $n^3 = a^3$, id quod triplici modo evenire potest

I. $n = a$,

II. $n = \frac{-1+\sqrt{-3}}{2} a$,

III. $n = \frac{-1-\sqrt{-3}}{2} a$.

Pro quolibet ergo valore quaeratur integrale completum aequationis

$\left(\frac{\partial z}{\partial x}\right) = nz$, et tria haec integralia conjuncta praebebunt integrale completum aequationis propositae. Cum autem in aequatione $\left(\frac{\partial z}{\partial x}\right) = nz$ quantitas y constans sumatur, erit

$$\partial z = nz\partial x, \text{ seu } \frac{\partial z}{z} = n\partial x,$$

unde fit

$$lz = nx + l\Gamma : y, \text{ seu } z = e^{nx}\Gamma : y.$$

Tribuantur jam ipsi n terni valores, eritque pro aequatione proposita

$$z = e^{ax}\Gamma : y + e^{\frac{-1+\sqrt{-3}}{2}}\Delta : y + e^{\frac{-1-\sqrt{-3}}{2}}\Sigma : y.$$

Cum autem sit

$$e^{m\sqrt{-1}} = \cos. m + \sqrt{-1}.\sin. m,$$

erit functionum arbitrarium formam mutando

$$z = e^{ax}\Gamma : y + e^{-\frac{1}{2}ax}\cos.\frac{ax\sqrt{3}}{2}.\Delta : y + e^{-\frac{1}{2}ax}\sin.\frac{ax\sqrt{3}}{2}\Sigma : y.$$

Corollarium 1.

§. 396. Integrale hoc etiam ita repraesentari potest

$$z = e^{ax}\Gamma : y + e^{-\frac{1}{2}ax}\Delta : y.\cos.\left(\frac{ax\sqrt{3}}{2} + Y\right),$$

denotante Y functionem quamcunque ipsius y.

Corollarium 2.

397. Quia tribus integrationibus est opus, et in singulis quantitas y ut constans tractatur; secundum praecepta libri primi haec aequatio $\partial^3 z = a^3 z\partial x^3$ resolvatur, et loco trium constantium functiones quaecunque ipsius y introducantur; unde eadem solutio elicitur.

Problema 65.

398. Proposita hac aequatione cujuscunque gradus

$$Pz + Q\left(\frac{\partial z}{\partial x}\right) + R\left(\frac{\partial\partial z}{\partial x^2}\right) + S\left(\frac{\partial^3 z}{\partial x^3}\right) + T\left(\frac{\partial^4 z}{\partial x^4}\right) + \text{etc.} = 0,$$

ubi litterae P, Q, R, S, T, etc. functiones denotant quascunque binarum variabilium x et y, indolem functionis z definire.

Solutio.

Cum in omnibus integrationibus instituendis quantitas y perpetuo ut constans spectetur, haec aequatio inter duas tantum variabiles x et z consistere est censenda. Quare per praecepta libri primi haec tractanda erit aequatio

$$Pz + \frac{Q\partial z}{\partial x} + \frac{R\partial\partial z}{\partial x^2} + \frac{S\partial^3 z}{\partial x^3} + \frac{T\partial^4 z}{\partial x^4} + \text{etc.} = 0,$$

cujus resolutio si succedat, tantum opus est, ut loco constantium per singulas integrationes invectarum functiones quaecunque ipsius y scribantur; sicque habebitur integrale desideratum, idque completum siquidem hanc aequationem complete integrare licuerit.

Corollarium 1.

399. Si ergo litterae P, Q, R, S, etc. sint constantes, vel solam variabilem y involvant, integratio semper succedit, quoniam in primo libro hujusmodi aequationes in genere integrare docuimus.

Corollarium 2.

400. Deinde etiam resolutio succedit hujus aequationis

$$Az + Bx\left(\frac{\partial z}{\partial x}\right) + Cx^2\left(\frac{\partial\partial z}{\partial x^2}\right) + Dx^3\left(\frac{\partial^3 z}{\partial x^3}\right) + \text{etc.} = 0,$$

sive litterae A, B, C, etc. sint constantes sive functiones ipsius y tantum.

Corollarium 3.

401. Tum vero etiam si hae formae non sint aequales nihilo, sed functioni cuicunque ipsarum x et y aequentur, resolutio

nihilo minus succedit, per ea, quae in postremis capitibus libri primi sunt exposita.

Scholion.

402. Haec etiam multo latius extendi possunt ad omnes plane aequationes, in quibus nullae aliae formulae differentiales praeter has

$$\left(\frac{\partial z}{\partial x}\right),\ \left(\frac{\partial\partial z}{\partial x^2}\right),\ \left(\frac{\partial^3 z}{\partial x^3}\right),\ \text{etc.}$$

quae solam x ut variabilem implicant occurrunt. Quomodocunque enim istae formulae cum quantitatibus finitis x, y et z fuerint complicatae, aequatio semper ad librum primum pertinere est censenda; quoniam in omnibus integrationibus instituendis quantitas y perpetuo ut constans tractatur. Confectis demum integrationibus discrimen in hoc consistit, ut loco constantium arbitrariarum functiones arbitrariae ipsius y in calculum introducantur. Superfluum foret hic monere, quae de altera variabilium y sunt dicta, etiam de altera x esse intelligenda.

Problema 66.

403. Proposita hac aequatione

$$\left(\frac{\partial\partial z}{\partial x^2}\right)+b\left(\frac{\partial\partial z}{\partial x\partial y}\right)-2a\left(\frac{\partial z}{\partial x}\right)-ab\left(\frac{\partial z}{\partial y}\right)+aaz=0,$$

investigare indolem functionis z.

Solutio.

Facile patet huic aequationi satisfacere hanc aequationem simplicem $\left(\frac{\partial z}{\partial x}\right)=az$, unde fit $z=e^{ax}$; statuamus ergo $z=e^{ax}v$ eritque

$$\left(\frac{\partial z}{\partial x}\right)=e^{ax}\left[av+\left(\frac{\partial v}{\partial x}\right)\right],\ \left(\frac{\partial z}{\partial y}\right)=e^{ax}\left(\frac{\partial v}{\partial y}\right),$$

hincque

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = e^{ax}\left[a a v + 2 a \left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial\partial v}{\partial x^2}\right)\right] \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = e^{ax}\left[a\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial\partial v}{\partial x\partial y}\right)\right],$$

quibus valoribus substitutis et divisa aequatione per e^{ax}, habebimus

$$\left(\frac{\partial\partial v}{\partial x^2}\right) + b\left(\frac{\partial\partial v}{\partial x\partial y}\right) = 0.$$

Quia nunc hic ubique occurrit $\left(\frac{\partial v}{\partial x}\right)$, faciamus $\left(\frac{\partial v}{\partial x}\right) = u$, erit

$$\left(\frac{\partial u}{\partial x}\right) + b\left(\frac{\partial u}{\partial y}\right) = 0,$$

cujus integrale est

$$f : (y - b x) = u.$$

Scribamus ergo

$$u = \left(\frac{\partial v}{\partial x}\right) = - b \Gamma' : (y - b x);$$

ut prodeat

$$v = \Gamma : (y - bx) + \Delta : y,$$

ideoque integrale quaesitum erit

$$z = e^{ax}[\Gamma : (y - bx) + \Delta : y],$$

quae forma ob duas functiones arbitrarias utique est integrale completum.

Problema 67.

404. Proposita hac aequatione

$$0 = (a + 2b) z - (2a + 3b)\left(\frac{\partial z}{\partial x}\right) + c\left(\frac{\partial z}{\partial y}\right) + a\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$- 2c\left(\frac{\partial\partial z}{\partial x\partial y}\right) + b\left(\frac{\partial^3 z}{\partial x^3}\right) + c\left(\frac{\partial^3 z}{\partial x^2\partial y}\right),$$

indolem functionis z investigare.

Solutio.

Aequatio haec ita est comparata ut ei manifesto satisfaciat $z = e^x$, statuamus ergo $z = e^x v$, eritque

$$\left(\frac{\partial z}{\partial x}\right) = e^x \left[v + \left(\frac{\partial v}{\partial x}\right)\right], \quad \left(\frac{\partial z}{\partial y}\right) = e^x \left(\frac{\partial v}{\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = e^x \left[v + 2\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial\partial v}{\partial x^2}\right)\right], \quad \left(\frac{\partial\partial z}{\partial x \partial y}\right) = e^x \left[\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial\partial v}{\partial x \partial y}\right)\right],$$

$$\left(\frac{\partial^3 v}{\partial x^3}\right) = e^x \left[v + 3\left(\frac{\partial v}{\partial x}\right) + 3\left(\frac{\partial\partial v}{\partial x^2}\right) + \left(\frac{\partial^3 v}{\partial x^3}\right)\right],$$

$$\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = e^x \left[\left(\frac{\partial v}{\partial y}\right) + 2\left(\frac{\partial\partial v}{\partial x \partial y}\right) + \left(\frac{\partial^3 v}{\partial x^2 \partial y}\right)\right],$$

quibus valoribus substitutis emergit haec satis simplex aequatio

$$0 = (a + 3b)\left(\frac{\partial\partial v}{\partial x^2}\right) + b\left(\frac{\partial^3 v}{\partial x^3}\right) + c\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right),$$

in qua commode evenit ut in singulis terminis formula $\left(\frac{\partial\partial v}{\partial x^2}\right)$, contineatur, quare posito $\left(\frac{\partial\partial v}{\partial x^2}\right) = u$, prodit haec aequatio primi gradus

$$0 = (a + 3b)\, u + b\left(\frac{\partial u}{\partial x}\right) + c\left(\frac{\partial u}{\partial y}\right),$$

ex qua patet si ponatur

$$\partial u = p\partial x + q\partial y,$$

esse debere

$$(a + 3b)\, u + bp + cq = 0,$$

quae ita resolvitur.

Cum posito $a + 3b = f$ sit

$$q = -\frac{bp}{c} - \frac{fu}{c}, \quad \text{erit}$$

$$\partial u = p\,\partial x - \frac{bp\,\partial y}{c} - \frac{fu\,\partial y}{c}, \quad \text{seu}$$

$$\partial x - \frac{b\,\partial y}{c} = \frac{1}{p}\left(\partial u + \frac{fu\,\partial y}{c}\right) = \frac{u}{p}\left(\frac{\partial u}{u} + \frac{f\,\partial y}{c}\right),$$

sicque necesse est ut sit $\frac{u}{p}$ functio ipsius $x - \frac{by}{c}$, unde fit

$$lu + \frac{fy}{c} = \text{f} : (cx - by) \quad \text{et}$$

$$u = e^{\frac{-fy}{c}}\, \Gamma'' : \left(x - \frac{by}{c}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right).$$

Jam ob y constans spectandum, prima integratio dat

$$\left(\frac{\partial v}{\partial x}\right) = e^{\frac{-fy}{c}}\, \Gamma' : \left(x - \frac{by}{c}\right) + \Delta : y,$$

et altera

$$v = e^{\frac{-fy}{c}} \Gamma : (x - \tfrac{by}{c}) + x\Delta : y + \Sigma : y.$$

Quare posito $a + 3b = f$ aequationis propositae integrale completum est

$$z = e^{x - \frac{fy}{c}} \Gamma : (x - \tfrac{by}{c}) + e^x x \Delta : y + e^x \Sigma : y.$$

Problema 68.

405. Proposita hac aequatione differentiali tertii gradus

$$0 = Pz - 3P\left(\frac{\partial z}{\partial x}\right) + 3P\left(\frac{\partial\partial z}{\partial x^2}\right) - P\left(\frac{\partial^3 z}{\partial x^3}\right) + Q\left(\frac{\partial z}{\partial y}\right)$$
$$- 2Q\left(\frac{\partial\partial z}{\partial x\partial y}\right) + Q\left(\frac{\partial^3 z}{\partial x^2\partial y}\right),$$

ubi P et Q sint functiones quaecunque ipsarum x et y, investigare indolem functionis z.

Solutio.

Facta substitutione $z = e^x v$, quandoquidem ex data forma facile perspicitur valorem e^x loco z positum satisfacere, pervenitur ad hanc aequationem

$$-P\left(\frac{\partial^3 v}{\partial x^3}\right) + Q\left(\frac{\partial^3 v}{\partial x^2\partial y}\right) = 0,$$

quae porro posito $\left(\frac{\partial\partial v}{\partial x^2}\right) = u$, ut sit $v = \int\int u\partial x^2$, abit in hanc

$$-P\left(\frac{\partial u}{\partial x}\right) + Q\left(\frac{\partial u}{\partial y}\right) = 0.$$

Statuamus $\partial u = p\partial x + q\partial y$, erit $Qq = Pp$, hinc $q = \frac{Pp}{Q}$, ideoque

$$\partial u = p\left(\partial x + \frac{P}{Q}\partial y\right);$$

ex quo intelligitur, quantitatem p ita comparatam esse debere, ut formula

$$\partial x + \frac{P}{Q}\partial y$$

per eam multiplicata integrabilis evadat. Quaeratnr ergo multiplicator M formulam

$$Q\partial x + P\partial y,$$

integrabilem reddens, ita ut sit

$$\int M\,(Q\partial x + P\partial y) = s,$$

quam ergo functionem s ipsarum x et y inveniri posse assumo, et ob

$$Q\partial x + P\partial y = \frac{\partial s}{M},$$

habebimus $\partial u = \frac{p\partial s}{MQ}$, unde patet, $\frac{p}{MQ}$ functionem denotare quantitatis s. Posito ergo $\frac{p}{MQ} = \Gamma' = s$, statim erit $u = \Gamma : s$, hincque $v = \int\partial x \int\partial x\,\Gamma : s$, in qua utraque integratione quantitas y ut constans spectatur. Quocirca resolutio problematis ita se habebit.

Pro formula differentiali $Q\partial x + P\partial y$ quaeratur multiplicator M eam reddens integrabilem, ut sit

$$M\,(Q\partial x + P\partial y) = \partial s,$$

et inventa hac ipsarum x et y functione s, erit

$$z = e^x \int\partial x \int\partial x\,\Gamma : s + e^x\,x\Delta : y + e^x\,\Sigma : y.$$

Scholion.

406. In istis aequationibus hoc commodi usu venit, ut facta substitutione $z = e^x v$ ejusmodi induant formam, quae facile porro ad speciem simplicem in prima sectione consideratam revocari queat, etiamsi enim differentialia tertii gradus non sint destructa, tamen reliqua membra ista e calculo excesserunt, ut deinceps nova substitutione $\left(\frac{\partial\partial v}{\partial x^2}\right) = u$ uti, ejusque ope ad aequationem differentialem primi gradus perveniri licuerit. Unica igitur substitutio hoc praestitura fuisset, si statim posuissemus $z = e^x \int\int u\,\partial x^2$. Utinam prae-

cepta haberentur, quorum ope hujusmodi substitutiones facile dignosci possunt! Interim postremo problemate, multo latius patente, in subsidium vocato §. 209. resolvi poterit.

Problema 69.

407. Proposita hac aequatione differentiali tertii gradus

$$0 = (P + Q)z - (2P + 3Q)\left(\frac{\partial z}{\partial x}\right) + (P + 3Q)\left(\frac{\partial\partial z}{\partial x^2}\right) - Q\left(\frac{\partial^3 z}{\partial x^3}\right)$$
$$- R\left(\frac{\partial z}{\partial y}\right) + 2R\left(\frac{\partial\partial z}{\partial x \partial y}\right) - R\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right),$$

ubi P, Q et R sint functiones quaecunque datae ipsarum x et y, investigare indolem functionis z.

Solutio.

Eadem adhibita substitutione $z = e^x v$, qua hactenus sumus usi, aequatio proposita transmutatur in sequentem

$$0 = P\left(\frac{\partial\partial v}{\partial x^2}\right) - Q\left(\frac{\partial^3 v}{\partial x^3}\right) - R\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right).$$

ubi commode evenit, ut posito $\left(\frac{\partial\partial v}{\partial x^2}\right) = u$, ista resultet aequatio differentialis primi gradus

$$0 = Pu - Q\left(\frac{\partial u}{\partial x}\right) - R\left(\frac{\partial u}{\partial y}\right),$$

unde qualis ipsarum x et y functio sit u est inquirendum. Ponamus esse

$$\partial u = p\partial x + q\partial y,$$

et quia jam illa conditio praebet

$$Pu = Qp + Rq,$$

secundum artificium supra §. 209. usurpatum formemus hinc tres sequentes aequationes

$$L\partial u = Lp\partial x + Lq\partial y,$$
$$MPu\partial x = MQp\partial x + MRq\partial x;$$
$$NPu\partial y = NQp\partial y + NRq\partial y,$$

quae in unam summam collectae dabunt

$$L\partial u + Pu(M\partial x + N\partial y) = p[(L + MQ)\partial x + NQ\partial y] + q[(L + NR)\partial y + MR\partial x],$$

ubi cum tres quantitates L, M et N ab arbitrio nostro pendeant, inter eas statuatur primo ejusmodi relatio, ut binae partes posterioris membri communem obtineant factorem scilicet

$$L + MQ : NQ = MR : L + NR, \text{ seu } L = -MQ - NR,$$

et habebimus

$$-\partial u(MQ + NR) + Pu(M\partial x + N\partial y) = (Mq - Np)(R\partial x - Q\partial y).$$

Quaeratur multiplicator T formulam $R\partial x - Q\partial y$ reddens integrabilem, ut sit

$$T(R\partial x - Q\partial y) = \partial s,$$

ex quo tam functio T quam s ut cognita spectari poterit, et quia nunc habemus

$$-\partial u(MQ + NR) + Pu(M\partial x + N\partial y) = (Mq - Np)\frac{\partial s}{T}, \text{ seu}$$
$$\frac{\partial u}{u} - \frac{P(M\partial x + N\partial y)}{MQ + NR} = \frac{Np - Mq}{u(MQ + NR)} \cdot \frac{\partial s}{T}.$$

Nunc cum P, Q, R sint functiones datae ipsarum x et y, probe notandum est inter binas nondum definitas M et N semper ejusmodi relationem statui posse, ut formula $\frac{P(M\partial x + N\partial y)}{MQ + NR}$ integrationem admittat; sit ergo ejus integrale $= lw$, ita ut sit

$$M\partial x + N\partial y = \frac{MQ + NR}{P} \cdot \frac{\partial w}{w}, \text{ et}$$
$$\frac{\partial u}{u} = \frac{\partial w}{w} + \frac{Np - Mq}{Tu(MQ + NR)} \cdot \partial s.$$

Necesse ergo est quantitates p et q ita sint comparatae, ut fiat

$$\frac{Np - Mq}{u(MQ + NR)} = f' : s;$$

hincque

$$l u = l w + f : s.$$

Loco $f : s$ scribamus $l\,\Gamma : s$, ut prodeat

$$u = w\,\Gamma : s,$$

ac propterea

$$v = \int \partial x \int w\, \partial x\, \Gamma : s + x\,\Delta : y + \Sigma : y.$$

Consequenter

$$z = e^x \int \partial x \int w\, \partial x\, \Gamma : s + e^x\, x\,\Delta : y + e^x\, \Sigma : y.$$

Corollarium 1.

408. Ad hanc ergo solutionem ex forma proposita statim eruendam, primo quaeratur ejusmodi functio ipsarum x et y, quae vocetur s, ut sit

$$\partial s = T\,(R\,\partial x - Q\,\partial y),$$

id quod expedietur multiplicatorem T investigando, quo formula differentialis $R\partial x - Q\partial y$ integrabilis reddatur.

Corollarium 2.

409. Praeterea vero quoque quantitatem w investigari oportet. In hunc finem inter quantitates M et N ejusmodi rationem indagari convenit, ut fiat

$$\int \frac{P\,(M\partial x + N\partial y)}{MQ + NR} = l w,$$

quae quidem investigatio semper est concedenda.

Scholion.

410. Cum statim totum negotium eo sit perductum, ut functio u ex hac aequatione definiri debeat

$$Pu = Q\left(\frac{\partial u}{\partial x}\right) + R\left(\frac{\partial u}{\partial y}\right),$$

sine ambagibus, quibus in solutione sum usus, solutio sequenti modo multo facilius absolvi poterit, id quod insigne supplementum in sectionem primam suppeditat. Statuatur

$$\left(\frac{\partial u}{\partial x}\right) = LMu \quad \text{et} \quad \left(\frac{\partial u}{\partial y}\right) = LNu,$$

erit primo

$$P = L\,(MQ + NR), \text{ hinc}$$

$$L = \frac{P}{MQ + NR}, \text{ deinde ob}$$

$$\partial u = \partial x\left(\frac{\partial u}{\partial x}\right) + \partial y\left(\frac{\partial u}{\partial y}\right), \text{ habebimus}$$

$$\frac{\partial u}{u} = L\,(M\partial x + N\partial y) = \frac{P\,(M\partial x + N\partial y)}{MQ + NR},$$

ubi M et N ita accipi oportet, ut integratio succedat, quod cum innumeris modis fieri possit, solutio hinc completa obtineri est aestimanda. Verum dum casus integrationis particularis constet, multo commodius inde solutio completa sequenti ratione elicietur. Posito scilicet

$$\frac{\partial w}{w} = \frac{P\,(M\partial x + N\partial y)}{MQ + NR},$$

ita ut valor ipsius w pro u sumtus jam particulariter satisfaciat, sitque

$$Pw = Q\left(\frac{\partial w}{\partial y}\right) + R\left(\frac{\partial w}{\partial y}\right).$$

Statuamus pro valore completo $u = w\Gamma : s$, et facta substitutione consequimur

$$Pw\Gamma : s = Q\left(\frac{\partial w}{\partial x}\right)\Gamma : s + R\left(\frac{\partial w}{\partial y}\right)\Gamma : s$$
$$+ Qw\left(\frac{\partial s}{\partial x}\right)\Gamma' : s + Rw\left(\frac{\partial s}{\partial y}\right)\Gamma' : s,$$

quae aequatio subito in hanc contrahitur

$$Q\left(\frac{\partial s}{\partial x}\right) + R\left(\frac{\partial s}{\partial y}\right) = 0,$$

ex qua concludimus

$$\left(\frac{\partial s}{\partial x}\right) = TR \text{ et } \left(\frac{\partial s}{\partial y}\right) = -TQ,$$

ac propterea

$$\partial s = T(R\partial x - Q\partial y),$$

unde patet hanc quantitatem s inveniri ex formula $R\partial x - Q\partial y$, pro qua primo factor T eam reddens integrabilem quaeri, tum vero ejus integrale pro s sumi debet. Imprimis igitur hic attendatur, quam concinne eandem solutionem elicere liceat, ad quam per tantas ambages perveneramus.

Problema 68.

411. Proposita hac aequatione differentiali quarti gradus

$$\left(\frac{\partial^4 z}{\partial y^4}\right) = aa\left(\frac{\partial\partial z}{\partial x^2}\right),$$

functionis z inventionem saltem ad resolutionem aequationis simplicioris reducere.

Solutio.

Hanc aequationem attentius contemplanti mox patebit, ei satisfacere hujusmodi simpliciorem

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = b\left(\frac{\partial z}{\partial x}\right),$$

hinc enim per y differentiando fit

$$\left(\frac{\partial^3 z}{\partial y^3}\right) = b\left(\frac{\partial\partial z}{\partial x\partial y}\right),$$

ac denuo eodem modo

$$\left(\frac{\partial^4 z}{\partial y^4}\right) = b\left(\frac{\partial^3 z}{\partial x\partial y^3}\right),$$

at ex ipsa assumta per x differentiata prodit

$$\left(\frac{\partial^3 z}{\partial x\partial y^2}\right) = b\left(\frac{\partial\partial z}{\partial x^2}\right),$$

quo valore ibi inducto colligitur

$$\left(\frac{\partial^4 z}{\partial y^4}\right) = bb\left(\frac{\partial\partial z}{\partial x^2}\right),$$

quae forma cum proposita congruit, dum sit $bb = aa$, quod cum duplici modo evenire queat

$b = + a$ et $b = - a$,

postquam has aequationes simpliciores resolverimus

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - a\left(\frac{\partial z}{\partial x}\right) = 0, \text{ quae praebeat } z = P,$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) + a\left(\frac{\partial z}{\partial x}\right) = 0, \text{ quae praebeat } z = Q,$$

erit pro aequatione proposita

$$z = P + Q,$$

et quia tam P quam Q binas functiones arbitrarias involvit, integrale hoc modo inventum quatuor ejusmodi functiones complectetur, ideoque erit completum.

Corollarium 1.

412. Solutiones particulares infinitae facile eliciuntur, ponendo

$$z = e^{\mu x + \nu y},$$

facta enim substitutione fieri necesse est

$$\nu^4 = \mu\mu aa \text{ et } \mu = \pm \frac{\nu\nu}{a}.$$

Sit $\nu = \lambda a$, erit $\mu = \pm \lambda\lambda a$, et integrale satisfaciens

$$z = e^{\lambda a(y \pm \lambda x)}.$$

Corollarium 2.

413. Poni etiam potest

$$z = e^{\mu x} \cos. (\nu y + \alpha),$$

unde fit

$$\nu^4 = \mu\mu aa$$

ut ante, ita ut alia forma integralium particularium sit

$$z = e^{\pm \lambda\lambda ax} \cos. (\lambda ay + \alpha).$$

Hujusmodi formulae infinitae conjunctae integrale completum quasi exhaurire sunt putandae.

Corollarium 3.

414. Eaedem solutiones reperiuntur, ponendo generalius $z = XY$, unde fit

$$\frac{X \partial^4 Y}{\partial y^4} = \frac{aaY \partial\partial X}{\partial x^2}$$

qua aequatione ita repraesentata

$$\frac{\partial^4 Y}{Y \partial y^4} = \frac{aa \partial\partial X}{X \partial x^2},$$

utrumque membrum eidem constanti aequari debet.

Scholion.

415. Aequatio autem ad quam totum negotium reduximus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = b \left(\frac{\partial z}{\partial x}\right)$$

ex earum est numero, quae nullo modo in generi resolvi posse videntur, ita ut in solutionibus particularibus acquiescere debeamus. Aequatio autem proposita non in mera speculatione est posita, sed quando laminarum elasticarum vibrationes quam minimae in genere investigantur; ad hujusmodi aequationem quarti gradus resolvendam pervenietur, quae etiam causa est, quod haec quaestio non perinde atque cordarum vibrantium in genere adhuc resolvi potuerit. Simili autem modo facile intelligitur, hanc aequationem quarti gradus

$$\left(\frac{\partial^4 z}{\partial y^4}\right) = aa \left(\frac{\partial\partial z}{\partial x^2}\right) + 2ab \left(\frac{\partial z}{\partial x}\right) + bbz$$

reduci ad hanc geminatam secundi gradus

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \pm a \left(\frac{\partial z}{\partial x}\right) \pm bz,$$

neque difficile est alios casus a posteriori eruere, ubi hujusmodi reductio ad gradum inferiorem locum invenit.

CAPUT III.

DE

INTEGRATIONE AEQUATIONUM HOMOGENEARUM UBI SINGULI TERMINI FORMULAS DIFFERENTIALES EJUSDEM GRADUS CONTINENT.

Problema 69.

416.

Aequationis homogeneae secundi gradus

$$A\left(\frac{\partial\partial z}{\partial x^2}\right) + B\left(\frac{\partial\partial z}{\partial x \partial y}\right) + C\left(\frac{\partial\partial z}{\partial y^2}\right) = 0$$

integralem, seu indolem functionis z investigare, denotantibus litteris A, B, C quantitates quascunque constantes.

Solutio.

Hanc aequationem voco homogeneam, quia formulis differentialibus secundi gradus constat, neque praeterea alias quantitates variabiles involvit. Ad hanc resolvendam observo ei satisfacere hujusmodi aequationem homogeneam primi gradus

$$\left(\frac{\partial z}{\partial x}\right) + \alpha\left(\frac{\partial z}{\partial y}\right) = f = \text{Const.}$$

hac enim duplici modo per x et y differentiata oritur

$$\text{I.} \quad \left(\frac{\partial\partial z}{\partial x^2}\right) + \alpha\left(\frac{\partial\partial z}{\partial x \partial y}\right) = 0,$$

$$\text{II.} \quad \left(\frac{\partial\partial z}{\partial x \partial y}\right) + \alpha\left(\frac{\partial\partial z}{\partial y^2}\right) = 0,$$

Jam illa per A hac vero per $\frac{C}{\alpha}$ multiplicata junctim propositam producent; si fuerit

$$A\alpha + \frac{C}{\alpha} = B, \text{ seu}$$

$$A\alpha\alpha - B\alpha + C = 0;$$

unde duplex valor pro α resultat, quorum uterque per aequationem assumtam dabit partem functionis quaesitae z. Cum igitur sit

$$\frac{\partial z}{\partial x} = f - \alpha \left(\frac{\partial z}{\partial y}\right), \text{ erit}$$

$$\partial z = f\partial x + (\partial y + \alpha\partial x)\left(\frac{\partial z}{\partial y}\right),$$

patet $\left(\frac{\partial z}{\partial y}\right)$ functionem esse debere ipsius $y - \alpha x$, qua posita $= \Gamma' : (y - \alpha x)$, erit

$$z = fx + \Gamma : (y - \alpha x),$$

denotante f constantem quamcunque. Quocirca aequationis propositae solutio ita se habebit. Formetur primo aequatio algebraica

$$Auu + Bu + C = 0,$$

cujus factores simplices sint

$$u + \alpha \text{ et } u + \beta,$$

ita ut sit

$$Auu + Bu + C = A(u + \alpha)(u + \beta),$$

tum integrale quaesitum erit

$$z = fx + \Gamma : (y - \alpha x) + \Delta : (y - \beta x),$$

ubi cum prima pars fx jam in binis functionibus indefinitis contineri sit censenda, ob

$$fx = \frac{f(y - \alpha x) - f(y - \beta x)}{\beta - \alpha},$$

succinctius ita exprimetur

$$z = \Gamma : (y - \alpha x) + \Delta : (y - \beta x),$$

quod ob binas functiones arbitrarias utique pro completo est habendum: unico casu excepto, quo est $\beta = \alpha$. Pro quo casu statuamus $\beta = \alpha + \partial\alpha$, et cum sit

$$\Delta : [y - (\alpha + \partial\alpha)\, x] = \Delta : (y - \alpha x) - x\partial\alpha\Gamma' : (y - \alpha\, x),$$

quia pars prior jam in membro priori continetur, et loco posterioris scribere licet $x\Gamma : (y - \alpha x)$, erit pro casu $\beta = \alpha$, seu $BB = 4\,AC$, integrale

$$z = \Gamma : (y - \alpha x) + x\Gamma : (y - \alpha x).$$

Corollarium 1.

417. Pro casu $\beta = \alpha$ manifestum est, integrale etiam hoc modo exprimi posse

$$z = \Gamma : (y - \alpha x) + y\Gamma : (y - \alpha x),$$

quae autem forma ab illa non discrepat.

Corollarium 2.

418. Si $C = 0$, ut sit

$$A \left(\frac{\partial\partial z}{\partial x^2}\right) + B \left(\frac{\partial\partial z}{\partial x \partial y}\right) = 0,$$

hincque

$$Auu + Bu = Au\left(u + \frac{B}{A}\right), \text{ fit}$$

$$\alpha = 0 \text{ et } \beta = \frac{B}{A},$$

et integrale

$$z = \Gamma : y + \Delta : \left(y - \frac{B}{A}\, x\right) = \Gamma : y + \Delta : (Ay - Bx).$$

Simili modo aequationis

$$B \left(\frac{\partial\partial z}{\partial x \partial y}\right) + C \left(\frac{\partial\partial z}{\partial y^2}\right) = 0$$

integrale est

$$z = \Gamma : x + \Delta : (Cx - By).$$

Corollarium 3.

419. Porro hujus aequationis

$$aa\left(\frac{\partial\partial z}{\partial x^2}\right) + 2ab\left(\frac{\partial\partial z}{\partial x \partial y}\right) + bb\left(\frac{\partial\partial z}{\partial y^2}\right) = 0, \text{ ob}$$

$$aauu + 2abu + bb = aa\left(u + \frac{b}{a}\right)^2,$$

est integrale

$$z = \Gamma : (ay - bx) + x\Delta : (ay - bx).$$

Scholion.

420. Harum integralium forma nulla laborat difficultate, quamdiu aequatio

$$Auu + Bu + C = 0$$

duas habet radices reales, sive sint inaequales sive aequales; quando autem hae radices fiunt imaginariae, ut sit

$$\alpha = \mu + \nu\sqrt{-1} \text{ et } \beta = \mu - \nu\sqrt{-1},$$

tum functiones arbitrariae omni fere usu destituuntur. Etsi enim indoles functionum Γ et Δ lineis curvis utcunque ductis repraesentatur, ut $\Gamma : v$ et $\Delta : v$ denotent in iis applicatas abscissae v convenientes, nullo modo patet, quomodo valores

$$\Gamma : (p + q\sqrt{-1}) \text{ et } \Delta : (p - q\sqrt{-1}),$$

exhiberi debeant, etiamsi imaginaria se mutuo tollant. In quo ingens cernitur discrimen inter functiones continuas et discontinuas, cum in illis semper valores ita expressi

$$\Gamma : (p + q\sqrt{-1}) + \Gamma : (p - q\sqrt{-1}) \text{ et}$$

$$\frac{\Delta : (p + q\sqrt{-1}) - \Delta : (p - q\sqrt{-1})}{\sqrt{-1}},$$

realiter exhiberi queant, id quod si Γ et Δ significent functiones discontinuas nullo modo succedit. His igitur casibus solutio generalis hic inventa ad solas functiones continuas restringenda videtur, quandoquidem discontinuae applicationi et executioni adversantur.

Problema 70.

421. Proposita hac aequatione tertii gradus homogenea

$$A\left(\frac{\partial^3 z}{\partial x^3}\right) + B\left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) + C\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) + D\left(\frac{\partial^3 z}{\partial y^3}\right) = 0,$$

ejus integrale completum invenire.

Solutio.

Huic quoque aequationi, uti in praecedente problemate, satisfacere aequationem differentialem simplicem primi gradus, satis luculenter perspicitur, ex quo integrale particulare talem habebit formam

$$z = \Gamma : (y + nx).$$

Colligantur hinc singulae formulae differentiales tertii gradus, quae erunt

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = + n^3 \Gamma''' : (y + nx), \quad \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = + n^2 \Gamma''' : (y + nx),$$

$$\left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = + n \Gamma''' : (y + nx), \quad \left(\frac{\partial^3 z}{\partial y^3}\right) = + \Gamma''' : (y + nx),$$

quibus substitutis, quoniam divisio per

$$\Gamma''' : (y + nx),$$

succedit, nascitur ista aequatio

$$An^3 + Bn^2 + Cn + D = 0,$$

cujus tres radices si fuerint $n = \alpha$, $n = \beta$, $n = \gamma$, evidens est, aequationi propositae satisfacere hanc formam

$$z = \Gamma : (y + \alpha x) + \Delta : (y + \beta x) + \Sigma : (y + \gamma x),$$

quae cum tres functiones arbitrarias complectatur, dubium non est, quin ea sit integrale completum. Hoc tantum notetur, si duae radices sint aequales puta $\gamma = \beta$, integrale fore

$$z = \Gamma : (y + \alpha x) + \Delta : (y + \beta x) + x \Sigma : (y + \beta x),$$

sin autem adeo omnes tres fuerint inter se aequales

$$\gamma = \beta = \alpha,$$

tum erit integrale quaesitum

$$z = \Gamma : (y + \alpha x) + x \Delta : (y + \alpha x) + xx \Sigma : (y + \alpha x).$$

Quodsi duae radices fuerint imaginariae, eadem erunt tenenda, quae modo ante sunt observata.

Corollarium 1.

422. Ultimus casus, quo tres radices sunt aequales, etiam inde est manifestus, quodsi loco variabilium x et y binae novae

$$t = x \text{ et } u = y + \alpha x,$$

introducantur, aequatio proposita contrahatur in hanc formam $\left(\frac{\partial^3 z}{\partial t^3}\right) = 0$, cujus integrale manifesto est

$$z = \Gamma : u + x \Delta : u + xx \Sigma : u.$$

Corollarium 2.

423. Hinc ergo etiam intelligitur, quomodo in aequationibus homogeneis altioris gradus, si aequationes algebraicae inde formatae plures habeant radices aequales, integralia futura sint comparata. Ita ut etiam tum neque casus radicum aequalium neque integralium ulli difficultati sit obnoxius.

Scholion.

424. Casus autem binarum radicum imaginariarum, quibus functiones arbitrariae nullum usum habere videntur, ratione functionum continuarum quae satisfaciunt, uberiorem evolutionem merentur. Formulae autem his casibus in integrale ingredientes semper ad hanc formam reduci possunt

$$\Gamma : v(\cos.\Phi + \sqrt{-1}.\sin.\Phi) + \Delta : v(\cos.\Phi - \sqrt{-1}.\sin.\Phi),$$

unde primum si functiones sint potestates, hujusmodi valores colliguntur

$$A\, v^n \cos.\, n\Phi + B\, v^n \sin.\, n\Phi, \text{ seu}$$
$$A\, v^n \cos.\,(n\Phi + \alpha),$$

quotcunque enim hujusmodi valores, constantes A, n et α utcunque mutando adhiberi possunt. Deinde si functiones denotent logarithmos, prodeunt tales valores $A\, lv + B\Phi$. Tertio si functiones, sint exponentiales, oriuntur hi

$$e^{v \cos.\Phi}\,[A \cos.\,(v \sin.\,\Phi) + B \sin.\,(v \sin.\,\Phi)]$$
$$= A\, e^{v \cos.\Phi} \cos.\,(v \sin.\,\Phi + \alpha).$$

et generalius

$$A\, e^{v^n \cos.\, n\Phi} \cos.\,(v^n \sin.\, n\Phi + \alpha).$$

Plurimae autem aliae hujusmodi formulae ex doctrina imaginariorum elici possunt, quae utcunque cum his combinatae, pro parte integrali ex binis radicibus imaginariis nata usurpari poterunt, unde infinita functionum multitudo nascitur, quae solutionem completam mentiri videtur, neque tamen pro completa perinde haberi potest, atque usu venit iis casibus, quibus omnes radices sunt reales. Hic autem observetur, nullum adhuc problema mechanicum seu physicum occurrisse, quod ab hujusmodi casu penderet.

Problema 71.

425. Proposita hujusmodi aequatione homogenea gradus cujuscunque

$$A\left(\frac{\partial^\lambda z}{\partial x^\lambda}\right) + B\left(\frac{\partial^\lambda z}{\partial x^{\lambda-1}\,\partial y}\right) + C\left(\frac{\partial^\lambda z}{\partial x^{\lambda-2}\,\partial y^{}}\right) + \text{etc.} = 0\,,$$

ejus integrale invenire.

Solutio.

Formetur hinc aequatio algebraica ordinis λ

$$An^\lambda + B\,n^{\lambda-1} + Cn^{\lambda-2} + \text{etc.} = 0,$$

cujus radices numero λ sint

$$n = \alpha,\ n = \beta,\ n = \gamma,\ n = \delta\,,\ \text{etc.}$$

quae si omnes fuerint inaequales, integrale completum aequationis propositae erit

$$z = \Gamma : (y + \alpha x) + \Delta : (y + \beta x) + \Sigma : (y + \gamma x).$$
$$+ \Theta : (y + \delta x) + \text{etc.}$$

quarum functionum disparium numerus erit $= \lambda$. Sin autem eveniat, ut inter has radices duae pluresve reperiantur aequales, scilicet $\beta = \alpha$, $\gamma = \alpha$, tum functiones has radices aequales involventes respective multiplicari debent per terminos progressionis geometricae hujus 1, x, x^2, etc. vel hujus 1, y, y^2, etc. ita ut functionum arbitrariarum numerus non minuatur. De radicibus autem imaginariis perpetuo ea sunt notanda quae ante observavimus, nisi forte functiones arbitrarias formularum imaginariarum excludere nolimus.

Corollarium 1.

426. Casu radicum aequalium perinde est, utra serie geometrica utamur, siquidem functiones neque sint ipsius x neque ipsius y tantum. Sin autem hae functiones fuerint vel ipsius x vel ipsius y tantum, tum alterius variabilis diversae progressione geometrica uti oportet.

Corollarium 2.

427. Si in aequatione algebraica termini initiales A, B, C, etc. evanescant, ut radicum numerus exponente λ minor esse videatur, tum radices deficientes pro infinite magnis sunt habendae, quibus functiones ipsius x tantum respondebunt, in integrale introducendae.

Corollarium 3.

428. Ita si fuerit $A = 0$, $B = 0$ et $C = 0$, tres radices α, β, γ in infinitum excrescere sunt censendae, ex quibus nascetur pars integralis

$$\Gamma : x + y\Delta : x + y^2\Sigma : x.$$

Scholion.

429. Quoniam haec pars calculi integralis vix excoli coepit, ideoque hujus generis investigationes adhuc prorsus sunt reconditae, de hac sectione plura proferre non licet, ideoque his partem primam libri secundi, quae in investigatione functionum binarum variabilium ex data quadam differentialium relatione versatur, concludere cogor. Multo autem pauciora circa partem al-

teram hujus libri in medium afferre conceditur, ubi calculus integralis ad functiones trium variabilium accommodatur, hancque ob causam ne operae quidem erit pretium, istam partem in sectiones subdividere multo minus sequentes partes attingere.

CALCULI INTEGRALIS

LIBER POSTERIOR.

PARS ALTERA.

INVESTIGATIO FUNCTIONUM TRIUM VARIABILIUM EX DATA DIFFERENTIALIUM RELATIONE.

CAPUT I.

DE FORMULIS DIFFERENTIALIBUS FUNCTIONUM TRES VARIABILES INVOLVENTIUM.

Problema 72.

430.

Si v sit functio quaecunque trium quantitatum variabilium x, y et z, ejus formulas differentiales primi gradus exhibere.

Solutio.

Cum v sit functio trium variabilium x, y et z, si ea more solito differentietur, ejus differentiale in genere ita reperietur expressum

$$\partial v = p\partial x + q\partial y + r\partial z.$$

Tribus scilicet id constabit partibus, quarum prima $p\partial x$ seorsim invenitur, si in differentiatione sola quantitas x ut variabilis tractetur, binis reliquis y et z ut constantibus spectatis. Simili modo pars secunda $q\partial y$ impetratur differentiatione functionis v ita instituta, ut sola quantitas y pro variabili, binae reliquae vero x et z pro constantibus habeantur, quod idem de parte tertia $r\partial z$ est tenendum, quae est differentiale ipsius v variabilitatis solius quantitatis z ratione habita. Hinc patet, quomodo per differentiationem quantitates istae p, q et r seorsim sint inveniendae, quas hic formulas differentiales primi gradus functionis v appellabo, et ne novis litteris in calculum introducendis sit opus, eas naturae suae convenienter ita indicabo

$$p = \left(\frac{\partial v}{\partial x}\right), \quad q = \left(\frac{\partial v}{\partial y}\right), \quad r = \left(\frac{\partial v}{\partial z}\right).$$

Quaelibet ergo functio v trium variabilium x, y et z tres habet formulas differentiales primi gradus ita designandas

$$\left(\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial v}{\partial y}\right), \quad \left(\frac{\partial v}{\partial z}\right),$$

in quarum qualibet unicae variabilis ratio habetur, dum binae reliquae ut constantes spectantur, et quoniam differentialia per divisionem tolluntur, hae formulae differentiales ad classem quantitatum finitarum sunt referendae.

Corollarium 1.

431. Ex tribus formulis differentialibus functionis v inventis ejus differentiale solito more sumtum ita constatur, ut sit

$$\partial v = \partial x \left(\frac{\partial v}{\partial x}\right) + \partial y \left(\frac{\partial v}{\partial y}\right) + \partial z \left(\frac{\partial v}{\partial z}\right),$$

cujus ergo formae vicissim integrale est ipsa illa functio v, vel etiam eadem quantitate quacunque sive aucta sive minuta.

Corollarium 2.

432. Si trium variabilium x, y et z functio v fuerit data ejus formulae differentiales singulae

$$\left(\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial v}{\partial y}\right), \quad \left(\frac{\partial v}{\partial z}\right),$$

iterum erunt functiones certae earundem variabilium x, y et z per differentiationem facile inveniendae. Interim tamen evenire potest, ut una pluresve variabilium ex hujusmodi formulis differentialibus prorsus excedant.

Scholion 1.

433. Nihil etiam impedit, quominus quantitas v ut functio trium variabilium x, y et z spectari possit, etiamsi forte duas tantum involvat, dum scilicet ratio compositionis ita est comparata, ut tertia quasi casu excesserit; quod eo minus est mirandum, cum idem in functionibus tam unius quam duarum variabilium evenire possit. Quoniam enim functiones unius variabilis commodissime per applicatas cujuspiam lineae curvae repraesentari solent, siquidem pro curvae natura applicatae ejus ut certae functiones abscissae x spectari possunt, casu quo linea curva abit in lineam rectam axi parallelam, etsi tum applicata quantitati constanti aequatur, propterea tamen ex illa idea generali, qua ut functio abscissae x spectatur, neutiquam excluditur; neque enim si quaeratur, qualis sit functio y ipsius x? incongrue is respondere est censendus, qui dicat hanc functionem y aequari quantitati constanti. Quod deinde ad functiones binarum variabilium x et y attinet, quas semper per intervalla, quibus singula cujusdam superficiei puncta a quopiam plano distant, repraesentare licet, dum binae variabiles x et y in hoc plano accipiuntur, manifestum est utique superficiem ita comparatam esse posse, ut functio illa, vel per solam x, vel per solam y determinetur. Quin etiam si superficies fuerit plana ipsique illi plano parallela, functio illa adeo abit in quantitatem constantem; neque propterea minus tanquam functio binarum variabilium considerari debebit. Quamobrem etiam quando tractatio circa functiones trium variabilium versatur, in eo genere etiam ejusmodi functiones, quae tantum vel per binas vel unicam trium variabilium x, y et z determinantur, vel adeo ipsae sunt quantitates constantes.

Scholion 2.

434. In calculo differentiali jam est ostensum, functionum plures variabiles involventium differentialia inveniri, si unaquaeque variabilium seorsim, tanquam sola esset variabilis, spectetur, atque omnia differentialia inde nata in unam summam conjiciantur. Quodsi ergo differentiatio hoc modo instituatur, singulae istae operationes, deleto tantum differentiali, praebebunt formulas differentiales, quas his signis

$$\left(\frac{\partial v}{\partial x}\right),\ \left(\frac{\partial v}{\partial y}\right) \text{ et } \left(\frac{\partial v}{\partial z}\right)$$

indicamus: simulque intelligitur, quomodo etiam functionum quatuor pluresve variabiles involventium formulae differentiales sint inveniendae. Circa functiones autem trium variabilium x, y et z exempla aliquot subjungamus, quibus earum ternas formulas differentiales exhibebimus.

Exemplum 1.

435. *Si functio trium variabilium sit*

$$v = \alpha x + \beta y + \gamma z,$$

ejus formulae differentiales ita se habebunt.

Cum per differentiationem prodeat

$$\partial v = \alpha \partial x + \beta \partial y + \gamma \partial z,$$

manifestum est fore

$$\left(\frac{\partial v}{\partial x}\right) = \alpha,\ \left(\frac{\partial v}{\partial y}\right) = \beta,\ \left(\frac{\partial v}{\partial z}\right) = \gamma,$$

sicque omnes tres formulas differentiales esse constantes.

Exemplum 2.

436. *Si functio trium variabilium sit*

$$v = x^\lambda y^\mu z^\nu$$

ejus formulae differentiales ita se habebunt.

Differentiatione more solito peracta fit

$$\partial v = \lambda x^{\lambda-1} y^{\mu} z^{\nu} \partial x + \mu x^{\lambda} y^{\mu-1} z^{\nu} \partial y + \nu x^{\lambda} y^{\mu} z^{\nu-1} \partial z,$$

unde perspicuum est fore formulas differentiales

$$\left(\frac{\partial v}{\partial x}\right) = \lambda x^{\lambda-1} y^{\mu} z^{\nu},\ \left(\frac{\partial v}{\partial y}\right) = \mu x^{\lambda} y^{\mu-1} z^{\nu},\ \left(\frac{\partial v}{\partial z}\right) = \nu x^{\lambda} y^{\mu} z^{\nu-1},$$

quae ergo singulae sunt novae functiones omnium trium variabilium x, y, z, nisi exponentes λ, μ, ν sint vel nihilo vel unitati aequales.

Exemplum 3.

437. *Si functio v duas tantum involvat variabiles x et y, tertia z in ejus compositionem non ingrediente, formulae differentiales ita habebunt.*

Quia functio v duas tantum variabiles x et y implicat, ejus differentiale hujusmodi formam induet

$$\partial v = p\partial x + q\partial y + 0\,\partial z,$$

tertia scilicet parte ex variabilitate ipsius z orta evanescente, unde habebimus

$$\left(\frac{\partial v}{\partial x}\right) = p,\ \left(\frac{\partial v}{\partial y}\right) = q,\ \left(\frac{\partial v}{\partial z}\right) = 0.$$

Corollarium.

438. Hinc ergo vicissim patet, si fuerit $\left(\frac{\partial v}{\partial z}\right) = 0$, tum fore v functionem quamcunque binarum variabilium x et y, quam in posterum ita indicabimus $v = \Gamma:(x, y)$, denotante $\Gamma:(x, y)$ functionem quamcunque binarum variabilium x et y.

Scholion.

439. Mox ostendemus, quando functio trium variabilium ex data quadam relatione seu conditione formularum differentialium

investiganda, proponitur, qualibet integratione introduci functionem quamcunque arbitrariam binarium variabilium, atque adeo in hoc consistere criterium, quo haec pars culculi integralis a praecedentibus distinguitur. Quemadmodum enim, dum natura functionum unicae variabilis ex data differentialium conditione iuvestigatur, in quo universus liber primus est occupatus, per quamlibet integrationem quantitas constans arbitraria in calculum invehitur, ita in parte praecedente hujus secundi libri vidimus, si functiones binarum variabilium ex data formularum differentialium relatione investigari debeant, tum ad essentiam hujus tractationis id pertinere, quod qualibet integratione non quantitas constans sed adeo functio unius variabilis prorsus arbitraria in calculum introducatur; etsi enim plerumque hae functiones veluti $\Gamma : (\alpha x + \beta y)$ ambas variabiles x et y implicabant, tamen ibi tota quantitas $\alpha x + \beta y$ ut unica spectatur, cujus functionem quamcunque illa formula $\Gamma : (\alpha x + \beta y)$ denotat. Nunc igitur, ubi de functionibus trium variabilium agitur, probe notandum est, qualibet integratione functionem arbitrariam duarum adeo variabilium in calculum introduci: ex quo simul indolem integrationum, quae circa functiones plurium variabilium versantur, colligere licet.

Problema 62.

440. Si sit v functio quaecunque trium variabilium x, y et z, ejus formulas differentiales secundi altiorumque graduum exhibere.

Solutio.

Cum ejus formulae differentiales primi gradus sint tres

$$\left(\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial v}{\partial y}\right), \quad \left(\frac{\partial v}{\partial z}\right),$$

quaelibet instar novae functionis considerata iterum tres suppeditabit formulas differentiales, quae autem ob

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) = \left(\frac{\partial\partial v}{\partial y\partial x}\right)$$

reducentur ad sex sequentes

$$\left(\frac{\partial\partial v}{\partial x^2}\right), \left(\frac{\partial\partial v}{\partial y^2}\right), \left(\frac{\partial\partial v}{\partial z^2}\right), \left(\frac{\partial\partial v}{\partial x\partial y}\right), \left(\frac{\partial\partial v}{\partial y\partial z}\right), \left(\frac{\partial\partial v}{\partial x\partial z}\right),$$

ex quarum denominatoribus intelligitur, quaenam trium quantitatum x, y, z in utraque differentiatione pro sola variabili haberi debeat. Simili modo evidens est formulas differentiales tertii gradus dari decem sequentes

$$\left(\frac{\partial^3 v}{\partial x^3}\right), \left(\frac{\partial^3 v}{\partial x^2\partial y}\right), \left(\frac{\partial^3 v}{\partial x\partial y^2}\right),$$
$$\left(\frac{\partial^3 v}{\partial y^3}\right), \left(\frac{\partial^3 v}{\partial y^2\partial z}\right), \left(\frac{\partial^3 v}{\partial y\partial z^2}\right), \left(\frac{\partial^3 v}{\partial x\partial y\partial z}\right),$$
$$\left(\frac{\partial^3 v}{\partial z^3}\right), \left(\frac{\partial^3 v}{\partial z^2\partial x}\right), \left(\frac{\partial^3 v}{\partial z\partial x^2}\right).$$

Formularum porro differentialium quarti gradus numerus est 15, quinti 21 etc. secundum numeros triangulares; simulque ex cujusque forma perspicuum est, quomodo ejus valor ex data functione v per repetitam differentiationem, in qualibet unicam variabilem considerando, elici debeat.

Corollarium 1.

441. En ergo omnes formulas differentiales cujusque gradus, quas ex qualibet functione trium variabilium derivare licet per differentiationem, quae porro ut functiones trium variabilium spectari possunt.

442. Quemadmodum ergo ex hujusmodi functione data omnes ejus formulae differentiales ope calculi differentialis inveniuntur, ita vicissim ex data quapiam, formula differentiali, vel duarum pluriumve relatione quadam, ope calculi integralis ipsa illa functio, unde eae nascuntur, investigari debet.

Scholion 1.

443. In calculo quidem differentiali parum refert, utrum functio differentianda unam pluresve variabiles involvat, cum praecepta differentiandi pro quovis variabilium numero maneant eadem; quam ob causam etiam calculum differentialem secundum hanc functionum varietatem in diversas partes distingui non erat opus. Longe secus autem accidit in calculo integrali, quem secundum hanc functionum varietatem omnino in partes dividi necesse est, quippe quae partes tam ratione propriae indolis quam ratione praeceptorum maxime inter se discrepant. Quemadmodum igitur hanc partem circa functiones trium variabilium occupatam tractari conveniat, exponendum videtur. Ac primo quidem ii casus commodissime evolventur, quibus unius cujusdam formulae differentialis valor datur, ex quo indolem functionis quaesitae definiri oporteat, quoniam haec investigatio nulla laborat difficultate. Deinde hujusmodi quaestiones aggrediar, quibus relatio quaepiam inter duas pluresve formulas differentiales proponitur: ubi quidem plurimum refert, cujusnam gradus eae fuerint, siquidem ex primo gradu plures casus expedire licet, dum ex altioribus vix adhuc quicquam in medium afferri potest: hunc ergo ordinem in ista tractatione observabo.

Scholion 2.

444. Videri hic posset, ad functiones trium variabilium definiendas, duas adeo conditiones seu relationes inter formulas differentiales admitti posse, neque unica praescripta quaestionem esse determinatam. Quodsi enim ponatur

$$\partial v = p\partial x + q\partial y + r\partial z,$$

ubi litterae p, q, r vicem gerunt formularum differentialium primi gradus, atque verbi gratia hae duae proponantur conditiones, ut sit

$$q = p \quad \text{et} \quad r = p,$$

ac propterea

$$\partial v = p\,(\partial x + \partial y + \partial z),$$

manifestum est solutionem dari posse scilicet

$$v = \Gamma : (x + y + z).$$

Verum ad hanc objectionem respondeo, in hoc exemplo casu evenire, ut binae conditiones simul consistere possint, altera enim parumper immutata, ut manente $q = p$ esse debeat $r = px$, ideoque

$$\partial v = p\,(\partial x + \partial y + x\partial z),$$

perspicuum est, nullum pro p valorem exhiberi posse, per quem formula differentialis

$$\partial x + \partial y + x\partial z$$

multiplicata integrabilis reddatur, quod unicum exemplum sufficit ad demonstrandum, duabus conditionibus praescribendis hujusmodi quaestiones evadere plusquam determinatas, neque propterea solutionem admittere nisi certis casibus, quibus quasi altera conditio jam in altera involvitur. Quocirca semper unica relatio inter formulas differentiales proposito omnino sufficit problemati determinando, quod idcirco, quia per integrationem functio arbitraria indefinita ingreditur, aeque parum pro indeterminato est habendum ac problemata calculi integralis communis, quorum solutio constantem arbitrariam introducit.

CAPUT II.

DE
INVENTIONE FUNCTIONUM TRIUM VARIABILIUM EX DATO CUJUSPIAM FORMULAE DIFFERENTIALIS VALORE.

Problema 74.

445.

Dato valore cujuspiam formulae differentialis primi gradus, investigare ipsam functionem trium variabilium, ex qua illa formula differentialis nascitur.

Solutio.

Sit v functio quaesita trium variabilium x, y et z, et S earundem functio data quaecunque, cui formula differentialis $\left(\frac{\partial v}{\partial x}\right)$ debeat esse aequalis. Cum igitur sit $\left(\frac{\partial v}{\partial x}\right) = S$, erit posita sola quantitate x variabili, binis reliquis vero y et z ut constantibus spectatis, $\partial v = S\partial x$, ideoque

$$v = \int S\partial x + \text{Const.}$$

ubi notandum est in integratione formulae $S\partial x$ ambas quantitates y et z pro constantibus haberi, et loco constantis functionem quamcunque ipsarum y et z scribi debere, ex quo functio quaesita ita exhiberi poterit

$$v = \int S\partial x + \Gamma : (y \text{ et } z),$$

hic scilicet $\Gamma : (y$ et $z)$ quantitatem quamcunque ex binis quantitatibus y et z, una cum constantibus utcunque conflatam denotat.

Simili modo si proponatur $(\frac{\partial v}{\partial y}) = S$, erit

$$v = \int S\partial y + \Gamma : (x \text{ et } z),$$

et haec aequatio $(\frac{\partial v}{\partial z}) = S$ integrata praebet

$$v = \int S\partial z + \Gamma : (x \text{ et } y).$$

Corollarium 1.

446. Hic jam abunde intelligitur, integratione hujusmodi functionum loco constantis introduci functionem arbitrariam duarum quantitatum variabilium, atque adeo in hoc characterem harum integrationum esse constituendum.

Corollarium 2.

447. Hic ergo istud problema solutum dedimus, quo quaeritur functio v trium variabilium x, y, z, ut posito

$$\partial v = p\partial x + q\partial y + r\partial z,$$

fiat vel $p = S$, vel $q = S$, vel $r = S$ existente S functione quacunque data easdem variabiles, vel duas, vel unicam involvente.

Corollarium 3.

448. Quodsi igitur esse debeat $(\frac{\partial v}{\partial x}) = 0$, seu $p = 0$, functio quaesita erit $v = \Gamma : (y$ et $z)$, et ut fiat $(\frac{\partial v}{\partial y}) = 0$ erit $v = \Gamma : (x$ et $z)$, tum vero ut fiat $(\frac{\partial v}{\partial z}) = 0$, necesse est fit $v = \Gamma : (x$ et $y)$.

Scholion 1.

449. Quemadmodum in praecedente parte functiones arbitrariae unius variabilis per applicatas curvarum quarumcunque sive regularium sive etiam irregularium repraesentari poterant, ita in hac parte functiones binarum variabilium arbitrariae per superficiem pro lubitu descriptam repraesentari possunt. Ita si super plano, in quo binae coordinatae x et y more solito assumuntur, superficies quaecunque expansa concipiatur, tertia coordinata distantiam cujusvis superficiei puncti ab illo plano designans, functionem quamcunque binarum variabilium x et y repraesentabit. Hocque modo aptissime vera idea hujusmodi functionum constitui videtur, cum ex ea non solum ratio harum functionum regularium sed etiam irregularium perspiciatur.

Scholion 2.

450. Hic etiam notari convenit, hujusmodi functiones binarum variabilium infinitis diversis modis etiam designari posse. Variatis enim in plano memorato binis coordinatis x et y, in binas alias t et u, ut sit

$$t = \alpha x + \beta y \text{ et } u = \gamma x + \delta y,$$

manifestum est functionem binarum variabilium t et u seu $\Gamma:(t \text{ et } u)$ convenire cum functione ipsarum x et y seu $\Gamma:(x \text{ et } y)$; si enim loco t et u illi valores pro x et y substituantur utique prodit functio duas tantum variabiles x et y involvens. Atque multo generalius si t aequatur functioni cuipiam datae ipsarum x et y, pariterque u hujusmodi alii functioni, tum $\Gamma:(t \text{ et } u)$ facta substitutione abibit in functionem ipsarum x et y ita exprimendam $\Delta:(x \text{ et } y)$; non enim necesse est ut idem functionis character Γ rationem compositionis quasi denotans utrinque sit idem, cum hic in genere de functionibus quibuscunque agatur. Quare si in sequentibus forte ejusmodi functiones occurrant

$$\Gamma:(ax + by \text{ et } fxx + gyy), \text{ vel}$$
$$\Gamma:[\sqrt{(xx + yy)} \text{ et } l\tfrac{x}{y}], \text{ etc.}$$

earum loco semper haec forma simplex $\Gamma:(x$ et $y)$ scribi potest.

Scholion 3.

451. Solutionis, quam dedimus, consideratio nobis suppeditat sequentes reflexiones. Primo posito

$$\partial v = p\partial x + q\partial y + r\partial z,$$

si debeat esse $p = (\frac{\partial v}{\partial x}) = 0$, fiet

$$\partial v = q\partial y + r\partial z,$$

unde patet v ejusmodi esse quantitatem, cujus differentiale hanc habiturum sit formam $q\partial y + r\partial z$; quod fieri nequit, nisi quantitas v fuerit functio binarum variabilium y et z tantum, tertia x penitus exclusa; et quia circa quantitates q et r nulla conditio praescribitur, recte pronunciamus, loco quantitatis v accipi posse functionem quamcunque binarum variabilium y et z, seu esse $v = \Gamma:(y$ et $z)$, quam eandem solutionem consideratio formulae $(\frac{\partial v}{\partial x}) = 0$ suggessit. Deinde si esse debeat generalius $(\frac{\partial v}{\partial x}) = p = S$, denotante S quantitatem quamcunque ex variabilibus x, y, z conflatam, habebimus

$$\partial v = S\partial x + q\partial y + r\partial z,$$

quae aequatio ita resolvitur. Quaeratur primo integrale formulae $S\partial x$ sola quantitate x ut variabili spectata, quod sit $= V$; haecque quantitas per omnes tres variabiles differentiata praebeat

$$\partial V = S\partial x + Q\partial y + R\partial z,$$

ex quo cum sit

$$S\partial x = \partial V - Q\partial y - R\partial z, \text{ erit}$$
$$\partial v = \partial V + (q - Q)\partial y + (r - R)\partial z, \text{ seu}$$
$$\partial.(v - V) = (q - Q)\partial y + (r - R)\partial z,$$

unde ut ante patet, quantitatem $v - V$ functioni cuicunque binarum variabilium y et z aequari posse. Quare ob $V = \int S \partial x$, prodit ut ante

$$v = \int S \partial x + \Gamma : (y \text{ et } z);$$

hocque ratiocinium, quo isthuc pervenimus, diligenter notari meretur, cum etiam in parte prima eximium usum praestare possit. Proposita enim aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = aa \left(\frac{\partial\partial z}{\partial x^2}\right),$$

quia est

$$\partial \cdot \left(\frac{\partial z}{\partial x}\right) = \partial x \left(\frac{\partial\partial z}{\partial x^2}\right) + \partial y \left(\frac{\partial\partial z}{\partial x \partial y}\right), \text{ et}$$

$$\partial \cdot \left(\frac{\partial z}{\partial y}\right) = \partial x \left(\frac{\partial\partial z}{\partial x \partial y}\right) + \partial y \left(\frac{\partial\partial z}{\partial y^2}\right),$$

erit

$$a\partial \cdot \left(\frac{\partial z}{\partial x}\right) + \partial \cdot \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right)(a\partial x + aa\partial y)$$
$$+ \left(\frac{\partial\partial z}{\partial x \partial y}\right)(a\partial y + \partial x),$$

seu

$$a\partial \cdot \left(\frac{\partial z}{\partial x}\right) + \partial \cdot \left(\frac{\partial z}{\partial y}\right) = (\partial x + a\partial y)\left[a\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(\frac{\partial\partial z}{\partial x \partial y}\right)\right],$$

cujus posterioris membri integrale manifesto est $F : (x + ay)$, hincque

$$\left(\frac{\partial z}{\partial y}\right) = - a \left(\frac{\partial z}{\partial x}\right) + a\Gamma' : (x + ay),$$

quo una integratio absoluta est censenda. Quare cum sit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right),$$

habebitur

$$\partial z = \left(\frac{\partial z}{\partial x}\right)(\partial x - a\partial y) + a\partial y \Gamma' : (x + ay).$$

Sit $\left(\frac{\partial z}{\partial x}\right) = p$ et $x - ay = t$, ut fiat

$$\partial z = p\partial t + a\partial y \Gamma' : (t + 2ay),$$

pro duabus variabilibus t et y, hincque

$$z = \tfrac{1}{2}\Gamma : (t + 2ay) + \int \partial t \,[p - \tfrac{1}{2}\Gamma' : (t + 2ay)]$$
$$= \Gamma : (x + ay) + \Delta : (x - ay),$$

quia

$$\Delta : t = \Delta (x - ay) \quad \text{et} \quad \Gamma : (t + 2ay) = \Gamma : (x + ay).$$

Problema 75.

452. Investigare indolem functionis trium variabilium x, y, z, cujus formula quaedam differentialis secundi gradus aequetur datae cuipiam functioni S.

Solutio.

Denotet v functionem quaesitam, et cum ejus sex dentur formulae differentiales secundi gradus, ponamus primo esse debere $\left(\frac{\partial\partial v}{\partial x^2}\right) = S$, et integratione semel instituta prodit

$$\left(\frac{\partial v}{\partial x}\right) = \int S\partial x + \Gamma : (y \text{ et } z),$$

iterumque integrando

$$v = \int \partial x \int S\partial x + x\Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z),$$

ubi in formulae $\int \partial x \int S \partial x$ duplici integratione sola quantitas x ut variabilis spectatur, quemadmodum jam supra est inculcatum. Similis autem omnino est intgratio aequationum

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = S \quad \text{et} \quad \left(\frac{\partial\partial v}{\partial z^2}\right) = S.$$

Pro reliquis formulis differentialibus secundi gradus sufficit hanc unam $\left(\frac{\partial\partial v}{\partial x \partial y}\right) = S$ resolvisse; quae primo per solam variabilem x integrata dabit

$$\left(\frac{\partial v}{\partial y}\right) = \int S\partial x + f : (y \text{ et } z).$$

Deinde altera integratione per solam variabilem y instituta colligitur

$$v = \int \partial y \int S\partial x + \int \partial y\, f : (y \text{ et } z) + \Delta : (x \text{ et } z),$$

ubi primum observo, partem primam nullo discrimine ordinis inter binas variabiles x et y habito ita $\int\int S\partial x\partial y$ exprimi posse. Deinde quaecunque fuerit $f:(y \text{ et } z)$ functio ipsarum y et z, si ea per ∂y multiplicetur, et spectata z ut constante integretur, evidens est denuo functionem ipsarum y et z prodire, et quia illa nullomodo determinatur, etiam hanc fore indeterminatam ideoque arbitrariam, unde statuere poterimus

$$v = \int\int S\partial x\partial y + \Gamma:(y \text{ et } z) + \Delta:(x \text{ et } z).$$

Corollarium 1.

453. Hic observo per integrationem formulae

$$\int\partial y\, f:(y \text{ et } z)$$

jam sponte formulam $\Delta:(x \text{ et } z)$ invehi; cum enim ibi sola quantitas y ut variabilis spectetur, loco quantitatis constantis per integrationem adjiciendae functio quaecunque ipsarum x et z scribi poterit.

Corollarium 2.

454. Quodsi functio illa data S evanescat, sequentes integrationes provenient

si $\left(\frac{\partial\partial v}{\partial x^2}\right) = 0$, erit $v = x\Gamma:(y \text{ et } z) + \Delta:(y \text{ et } z)$,

si $\left(\frac{\partial\partial v}{\partial y^2}\right) = 0$, erit $v = y\Gamma:(x \text{ et } z) + \Delta:(x \text{ et } z)$,

si $\left(\frac{\partial\partial v}{\partial z^2}\right) = 0$, erit $v = z\Gamma:(x \text{ et } y) + \Delta:(x \text{ et } y)$,

si $\left(\frac{\partial\partial v}{\partial x\partial y}\right) = 0$, erit $v = \Gamma:(x \text{ et } z) + \Delta:(y \text{ et } z)$,

si $\left(\frac{\partial\partial v}{\partial x\partial z}\right) = 0$, erit $v = \Gamma:(x \text{ et } y) + \Delta:(y \text{ et } z)$,

si $\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0$, erit $v = \Gamma:(x \text{ et } y) + \Delta:(y \text{ et } z)$.

Corollarium 3.

455. Quia hic duplici opus est integratione, atque etiam duae functiones arbitrariae, utraque binarum variabilium, in calculum sunt invectae; hoc certissimum est criterium, haec integralia inventa esse completa.

Scholion.

456. Alio etiam modo haec eadem integralia erui possunt, qui nititur principio supra (. 451.) indicato, quod si fuerit

$$\partial v = S\,\partial x + q\partial y + r\partial z, \text{ fore}$$
$$v = \int S\partial x + f : (y \text{ et } z).$$

Secundum hoc principium ergo si fuerit $(\frac{\partial\partial v}{\partial x^2}) = S$, erit

$$\partial \cdot (\frac{\partial v}{\partial x}) = S\partial x + \partial y\,(\frac{\partial\partial v}{\partial x\partial y}) + \partial z\,(\frac{\partial\partial v}{\partial x\partial z}),$$

qua forma cum illa collata, loco v habemus $(\frac{\partial v}{\partial x})$ et loco q et r has formulas

$$(\frac{\partial\partial v}{\partial x\partial y}) \text{ et } (\frac{\partial\partial v}{\partial x\partial z}),$$

ex quo integrale erit

$$(\frac{\partial v}{\partial x}) = \int S\partial x + f : (y \text{ et } z).$$

Cum jam porro sit

$$\partial v = (\frac{\partial v}{\partial x})\,\partial x + (\frac{\partial v}{\partial y})\,\partial y + (\frac{\partial v}{\partial z})\,\partial z, \text{ erit}$$

$$\partial v = \partial x \int S\partial x + \partial x\, f : (y \text{ et } z) + \partial y\,(\frac{\partial v}{\partial y}) + \partial z\,(\frac{\partial v}{\partial z}),$$

unde pariter manifesto sequitur

$$v = \int \partial x \int S\partial x + x\Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z).$$

Pari modo operatio est instituenda pro aequatione $(\frac{\partial\partial v}{\partial x\partial y}) = S$, inde enim fit

$$\partial \cdot (\frac{\partial v}{\partial y}) = S\partial x + \partial y\,(\frac{\partial\partial v}{\partial y^2}) + \partial z\,(\frac{\partial\partial v}{\partial y\partial z}),$$

cujus integrale est

$$\left(\frac{\partial v}{\partial y}\right) = \int S \partial x + f : (y \text{ et } z) :$$

altera integratio instituatur in hac forma

$$\partial v = \partial y \int S \partial x + \partial y f : (y \text{ et } z) + \partial x \left(\frac{\partial v}{\partial x}\right) + \partial z \left(\frac{\partial v}{\partial z}\right),$$

unde ob

$$\int \partial y f : (y \text{ et } z) = \Gamma : (y \text{ et } z),$$

obtinetur ut ante

$$v = \int\int S \partial x \partial y + \Gamma : (y \text{ et } z) + \Delta : (x \text{ et } z).$$

Problema 76.

457. Investigare indolem functionis trium variabilium x, y et z, cujus quaedam formula differentialis tertii gradus aequetur datae cuipiam quantitati S, ex illis variabilibus et constantibus utcunque compositae.

Solutio.

Posita functione quaesita $= v$, percurramus non tam singulas ejus formulas differentiales tertii gradus, qnam eas quarum ratio est diversa.

Sit igitur primo $\left(\frac{\partial^3 v}{\partial x^3}\right) = S$, et prima integratio statim dat

$$\left(\frac{\partial \partial v}{\partial x^2}\right) = \int S \partial x + 2\Gamma : (y \text{ et } z),$$

tum vero altera

$$\left(\frac{\partial v}{x \varrho}\right) = \int \partial x \int S \partial x + 2x\Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z),$$

unde tandem colligitur

$$v = \int \partial x \int \partial x \int S \partial x + xx\Gamma : (y \text{ et } z) + x\Delta : (y \text{ et } z)$$
$$+ \Sigma : (y \text{ et } z).$$

Sit secundo $(\frac{\partial^3 v}{\partial x^2 \partial y}) = S$, et binae priores integrationes ut ante dant

$$(\frac{\partial v}{\partial y}) = \int \partial x \int S \partial x + x \Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z),$$

quia nunc ut vidimus, pro $\int \partial y \Gamma : (y \text{ et } z)$ scribere licet $\Gamma : (y \text{ et } z)$, per tertiam integrationem invenimus

$$v = \int^3 S \partial x^2 \partial y + x \Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z) + \Sigma : (x \text{ et } z).$$

In his autem duobus casibus omnes formulae differentiales tertii gradus, variabilibus permutandis, continentur, sola excepta ultima hac $(\frac{\partial^3 v}{\partial x \partial y \partial z})$, quam idcirco seorsim tractari oportet.

Sit igitur $(\frac{\partial^3 v}{\partial x \partial y \partial z}) = S$, et prima integratione per solam variabilem x instituta obtinetur

$$(\frac{\partial \partial v}{\partial y \partial z}) = \int S \partial x + f : (y \text{ et } z);$$

nunc secundo integretur per solam variabilem y, ac reperietur

$$(\frac{\partial v}{\partial z}) = \int\int S \partial x \partial y + \Gamma : (y \text{ et } z) + \Delta : (x \text{ et } z);$$

unde tandem tertia integratio per z dabit

$$v = \int^3 S \partial x \partial y \partial z + \Gamma : (y \text{ et } z) + \Delta : (x \text{ et } z) + \Sigma : (x \text{ et } y),$$

sicque problema perfecte est resolutum.

Corollarium 1.

458. Quoniam hic triplici opus erat integratione, integralia inventa etiam tres functiones arbitrarias complectuntur, easque singulas binarum variabilium, quemadmodum natura integralium completorum postulat.

Corollarium 2.

459. Si quantitas data S evanescat, integralia haec sequenti modo se habebunt

si fuerit $(\frac{\partial^3 v}{\partial x^3}) = 0$, erit

$$v = xx\Gamma : (y \text{ et } z) + x\Delta : (y \text{ et } z) + \Sigma : (y \text{ et } z),$$

si fuerit $(\frac{\partial^3 v}{\partial x^2 \partial y}) = 0$, erit

$$v = x\Gamma : (y \text{ et } z) + \Delta : (y \text{ et } z) + \Sigma : (x \text{ et } z),$$

si fuerit $(\frac{\partial^3 v}{\partial x \partial y \partial z}) = 0$, erit

$$v = \Gamma : (y \text{ et } z) + \Delta : (x \text{ et } z) + \Sigma : (x \text{ et } y).$$

Scholion.

460. Eadem integralia etiam altera methodo supra exposita inveniri possunt, superfluumque foret singulas operationes hic apponere. Aeque parum autem opus erit has investigationes ad formulas differentiales altiorum graduum prosequi, cum lex progressionis functionum arbitrarium singulas integralium partes constituentium, cum per se tum per ea quae supra sunt exposita, satis sit manifesta. Quare huic capiti, quo una quaedam formula differentialis quantitati datae aequari debet, plane est satisfactum. Antequam autem ulterius progredior, duos adhuc casus satis late patentes proponam, quorum resolutio facile ad praecedentes jam tractatas calculi integralis partes reducitur, quam propterea hic tanquam concessam assumere licet, siquidem difficultates, quae in iis occurrunt, non ad praesens institutum sunt referendae.

Problema 77.

461. Si in relationem propositam ex qua naturam functionis trium variabilium x, y et z definiri oportet, aliae formulae differentiales non ingrediantur, nisi quae ex unica variabili x oriuntur, quae sunt

$$(\frac{\partial v}{\partial x}), \quad (\frac{\partial\partial v}{\partial x^2}), \quad (\frac{\partial^3 v}{\partial x^3}), \quad \text{etc.}$$

functionem quaesitam investigare.

Solutio.

Cum aequatio propositam continens relationem alias formulas differentiales praeter memoratas non comprehendat, in ea binae quantitates y et z pro constantibus habentur, ideoque etiam in singulis integrationibus tanquam tales tractari possunt. Hinc aequatio proposita duas tantum variabiles x et v involvere est censenda, et rejectis formularum differentialium vinculis, habebitur aequatio differentialis ad librum primum referenda, in qua, si ad altiores gradus exsurgat, elementum ∂x constans sumtum est putandum. Quodsi ergo praeceptorum ibidem traditorum ope haec aequatio integrari queat, tum loco constantium per singulas integrationes ingressarum substituantur functiones arbitrariae binarum variabilium y et z, veluti

$$\Gamma:(y \text{ et } z),\ \Delta:(y \text{ et } z),\ \text{etc.}$$

sicque habebitur solutio completa problematis propositi.

Corollarium 1.

462. Praeter plurimos igitur integrabilitatis casus in libro I. expositos, etiam sequentes aequationes differentiales quamtumvis alti gradus resolutionem admittent

$$S = Av + B\left(\frac{\partial v}{\partial x}\right) + C\left(\frac{\partial\partial v}{\partial x^2}\right) + D\left(\frac{\partial^3 v}{\partial x^3}\right) + \text{etc., et}$$

$$S = Av + Bx\left(\frac{\partial v}{\partial x}\right) + Cx^2\left(\frac{\partial\partial v}{\partial x^2}\right) + Dx^3\left(\frac{\partial^3 v}{\partial x^3}\right) + \text{etc.}$$

Corollarium 2.

463. Vinculis enim abjectis ejusmodi habentur aequationes differentiales, quales in extremis capitibus libri I. integrare docuimus. Tantum opus est, ut loco constantium per integrationes ingressarum scribantur tales functiones

$\Gamma:(y \text{ et } z)$, $\Delta:(y \text{ et } z)$, $\Sigma:(y \text{ et } z)$, etc.
ut hoc pacto integralia completa obtineantur.

Scholion.

464. Huc etiam referri possunt ejusmodi relationes propositae, in quibus formulae differentiales bina elementa ∂x et ∂y involventes ita continentur, ut hoc ∂y ubique eundem habeat dimensionum numerum, cujusmodi sunt

$$\left(\frac{\partial v}{\partial y}\right), \left(\frac{\partial\partial v}{\partial x\partial y}\right), \left(\frac{\partial^3 v}{\partial x^2\partial y}\right), \left(\frac{\partial^4 v}{\partial x^3\partial y}\right), \text{ etc. vel}$$
$$\left(\frac{\partial\partial v}{\partial y^2}\right), \left(\frac{\partial^3 v}{\partial x\partial y^2}\right), \left(\frac{\partial^4 v}{\partial x^2\partial y^2}\right), \left(\frac{\partial^5 v}{\partial x^3\partial y^2}\right), \text{ etc.}$$

ipsa autem tum quantitas v nusquam occurrat. Si enim tum pro priori casu ponatur $\left(\frac{\partial v}{\partial y}\right) = u$, pro posteriori vero $\left(\frac{\partial\partial v}{\partial y^2}\right) = u$, relatio ad casum problematis revocabitur, alias formulas differentiales non continens praeter

$$\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial\partial u}{\partial x^2}\right), \left(\frac{\partial^3 u}{\partial x^3}\right),$$

et ipsam forte functionem u. Quare si aequationem per praecepta supra tradita integrare, indeque functionem u definire licuerit, tum restituendo loco u vel $\left(\frac{\partial v}{\partial y}\right)$, vel $\left(\frac{\partial\partial v}{\partial y^2}\right)$, ut fiat $\left(\frac{\partial v}{\partial y}\right) = S$, vel $\left(\frac{\partial\partial v}{\partial y^2}\right) = S$, etiam hinc per praecepta hujus capitis ipsa functio v determinabitur. Quin etiam hoc modo resolvi poterunt aequationes hujusmodi tantum formulas differentiales complectentes

$$\left(\frac{\partial^{\mu+\nu} v}{\partial y^\mu \partial z^\nu}\right), \left(\frac{\partial^{\mu+\nu+1} v}{\partial x\partial y^\mu \partial z^\nu}\right), \left(\frac{\partial^{\mu+\nu+2} v}{\partial x^2 \partial y^\mu \partial z^\nu}\right); \text{ etc.}$$

ubi omnia tria elementa ∂x, ∂y, ∂z occurrunt; posito enim $\left(\frac{\partial^{\mu+\nu} v}{\partial y^\partial \partial z^\nu}\right) = u$, tota aequatio alias formulas non continebit praeter

$$\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial\partial u}{\partial x^2}\right), \left(\frac{\partial^3 u}{\partial x^3}\right), \text{ etc.}$$

unam cum ipsa functione u, sicque ad casum hujus problematis erit referenda, ex cujus resolutione si prodierit

$$u = S = \left(\frac{\partial^{\mu+\nu} v}{\partial y^{\mu} \partial z^{\nu}}\right),$$

existente jam S functione cognita, investigatio ipsius functionis v jam nulla amplius laborat difficultate. Datur autem praeterea alius casus ad libri II. partem priorem reducibilis, quem sequenti problemate sum expediturus.

Problema 78.

465. Si in relationem propositam, ex qua trium variabilium x, y, z functionem v definiri oportet, aliae formulae differentiales non ingrediuntur, nisi quae ex variabilitate binarum x et y tantum nascuntur, tertio elemento ∂z penitus excluso, functionem v investigare.

Solutio.

Quoniam in aequationem resolvendam, qua relatio proposita continetur, quantitas z non ut variabilis ingreditur, quotcunque integrationes fuerint instituendae, in iis ita quantitas y tanquam esset constans tractari debet. Hujus ergo aequationis resolutio ad partem praecedentem est referenda, cum functio binarum tantum variabilium x et y ex formularum differentialium relatione data sit investiganda; quodsi itaque negotium successerit et integrale fuerit inventum, in eo totidem occurrent functiones arbitrariae unius variabilis certo modo ex x et y conflatae, quot integrationibus fuerit opus. Sit $\Gamma : t$ hujusmodi functio, ubi t per x et y dari assumitur: ac nunc ut ista solutio ad praesens institutum accommodetur, ubi quantitas z variabilibus annumeratur, loco cujusque functionis arbitrariae $\Gamma : t$ scribatur hic $\Gamma : (t$ et $z)$, functio scilicet duarum variabilium, sicque habebitur integrale completum.

Corollarium 1.

466. Si ergo haec proposita fuerit aequatio

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = a\,a\left(\frac{\partial\partial v}{\partial x^2}\right),$$

quia in parte praecedente invenimus

$$v = \Gamma:(x + a\,y) + \Delta:(x - a\,y),$$

pro casu praesente, quo v debet esse functio trium variabilium x, y et z, integrale ita se habebit

$$v = \Gamma:(\overline{x + a\,y} \text{ et } z) + \Delta:(\overline{x - a\,y} \text{ et } z).$$

Corollarium 2.

467. Hic scilicet meminisse oportet, formam

$$\Gamma:(\overline{x + a\,y} \text{ et } z),$$

designare functionem quamcunque binarum variabilium, quarum altera sit $= x + a\,y$, altera vero $= z$; unde ipsam functionem per applicatam ad certam superficiem relatam repraesentare licebit.

Scholion.

468. Non solum autem aequationes in problemate descriptae ad partem praecedentem calculi integralis reducentur, sed etiam innumerabiles aliae, quae facta quadam substitutione ad eam formam revocantur. Veluti si in aequatione proposita aliae formulae differentiales non occurrant, nisi in quibus omnibus unica dimensio elementi ∂z reperitur, quae sunt

$$\left(\frac{\partial v}{\partial z}\right),\ \left(\frac{\partial\partial v}{\partial x\,\partial z}\right),\ \left(\frac{\partial\partial v}{\partial y\,\partial z}\right),\ \left(\frac{\partial^3 v}{\partial x^2\,\partial z}\right),\ \left(\frac{\partial^3 v}{\partial x\,\partial y\,\partial z}\right),\ \left(\frac{\partial^3 v}{\partial y^2\,\partial z}\right), \text{ etc.}$$

manifestum est posito $\left(\frac{\partial v}{\partial z}\right) = u$, aequationem illam in aliam transformari, ex qua jam functionem u investigari oporteat, eamque ad casum in problemate expositum referri. Quare si inde indoles functionis u definiri potuerit, ut sit $u = S$, restat ut haec aequatio $\left(\frac{\partial v}{\partial z}\right) = S$ resolvatur, unde ut ante vidimus, fit

$$v = \int S \partial z + \Gamma : (x \text{ et } y).$$

Hoc idem tenendum est, si aequatio proposita ope substitutionis

$$\left(\frac{\partial \partial v}{\partial z^2}\right) = u, \text{ vel } \left(\frac{\partial^3 v}{\partial z^3}\right) = u, \text{ etc.}$$

ad casum problematis reduci queat. Quin etiam per se est perspicuum, si ope transformationis cujuscunque aequatio proposita ad casum problematis reduci queat; tales autem transformationes supra plures exposui, dum vel loco functionis quaesitae v alia u introducitur ponendo $v = S u$, vel ipsae variabiles x, y et z in alias p, q, r mutantur, quae ad illas certam teneant rationem, quod negotium pro casu duarum variabilium supra fusius explicavi; hocque ita perspicuum est, ut similis reductio ad hunc casum trium variabilium facile accommodari queat. In sequentibus tamen forte ejusmodi transformationes occurrent; ad alios ergo casus, ubi omnis generis formulae differentiales occurrunt, progredior, vix ultra prima elementa rem producturus.

CAPUT III.

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM PRIMI GRADUS.

Problema 79.

469.

Si pro functione v trium variabilium x, y, z, posito

$$\partial v = p\,\partial x + q\,\partial y + r\,\partial z,$$ fuerit

$$\alpha p + \beta q + \gamma r = 0,$$

indolem functionis v definire.

Solutio.

Cum sit

$$\gamma\,\partial v = \gamma p\,\partial x + \gamma q\,\partial y - (\alpha p + \beta q)\,\partial z,$$ erit

$$\gamma\,\partial v = p(\gamma\,\partial x - \alpha\,\partial z) + q(\gamma\,\partial y - \beta\,\partial z),$$

ideoque ponendo

$$\gamma x - \alpha z = t \text{ et } \gamma y - \beta z = u,$$

habebitur

$$\gamma\,\partial v = p\,\partial t + q\,\partial u;$$

unde patet quantitatem v aequari functioni cuicunque binarum variabilium t et u, ita ut sit

$$v = \Gamma:(t \text{ et } u),$$

et restitutis valoribus assumtis

$$v = \Gamma:(\overline{\gamma x - \alpha z} \text{ et } \overline{\gamma y - \beta z}),$$

quae ergo est solutio problematis, si inter formulas differentiales proponatur haec conditio, ut sit

$$\alpha\left(\frac{\partial v}{\partial x}\right)+\beta\left(\frac{\partial v}{\partial y}\right)+\gamma\left(\frac{\partial v}{\partial z}\right)=0,$$

cujus itaque aequationis integrale clarius ita exhibetur

$$v=\Gamma:\left(\frac{x}{\alpha}-\frac{z}{\gamma}\text{ et }\frac{y}{\beta}-\frac{z}{\gamma}\right).$$

Corollarium 1.

470. Evidens est, hoc integrale etiam ita exprimi posse

$$v=\Gamma:\left(\frac{x}{\alpha}-\frac{y}{\beta}\text{ et }\frac{y}{\beta}-\frac{z}{\gamma}\right);$$

quandoquidem in genere ut supra observavimus, est

$$\Gamma:(x\text{ et }y)=\Delta:(t\text{ et }u),$$

siquidem t et u utcunque per x et y determinentur.

Corollarium 2.

471. Quin etiam affirmare licet, constitutis his tribus formulis

$$\frac{x}{\alpha}-\frac{y}{\beta},\quad \frac{y}{\beta}-\frac{z}{\gamma},\quad \frac{z}{\gamma}-\frac{x}{\alpha},$$

quantitatem v esse functionem quamcunque trium harum formularum; siquidem unaquaeque jam per binas reliquas datur, ac propterea v nihilominus functioni duarum tantum quantitatum variabilium aequatur.

Problema 80.

472. Si posito

$$\partial v=p\,\partial x+q\,\partial y+r\,\partial z,$$

haec conditio requiratur, ut sit

$$px + qy + rz = nv, \text{ seu}$$

$$nv = x\left(\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}\right),$$

indolem hujus functionis v investigare.

Solutio.

Ex conditione praescripta capiatur valor $r = \frac{nv - px - qy}{z}$, quo substituto fit

$$\partial v - \frac{nv\partial z}{z} = p\left(\partial x - \frac{x\partial z}{z}\right) + q\left(\partial y - \frac{y\partial z}{z}\right), \text{ seu}$$

$$\partial v - \frac{nv\partial z}{z} = pz\,\partial.\frac{x}{z} + qz\,\partial.\frac{y}{z}.$$

Quo primum membrum integrabile reddatur, multiplicetur per $\frac{1}{z^n}$ ita ut jam habeamus

$$\partial.\frac{v}{z^n} = \frac{pz}{z^n}\partial.\frac{x}{z} + \frac{qz}{z^n}\partial.\frac{y}{z}.$$

Cum nunc quantitates p et q non sint d[illegible]inatae, quoniam in genere ex tali aequatione

$$\partial V = P\partial X + Q\partial Y$$

sequitur

$$V = \Gamma:(X \text{ et } Y),$$

pro nostro casu colligimus

$$\frac{v}{z^n} = \Gamma:\left(\frac{x}{z} \text{ et } \frac{y}{z}\right), \text{ seu}$$

$$v = z^n\,\Gamma:\left(\frac{x}{z} \text{ et } \frac{y}{z}\right).$$

Si scilicet functio quaecunque binarum quantitatum $\frac{x}{z}$ et $\frac{y}{z}$ per z^n, seu etiam quod eodem redit per x^n vel y^n multiplicetur, oritur valor idoneus pro functione v conditioni praescriptae satisfaciens.

Corollarium 1.

473. Perspicuum autem est, formam $\Gamma : (\frac{x}{z} \text{ et } \frac{y}{z})$ exprimere ejusmodi functionem, in qua tres variabiles x, y, z ubique constituant nullum dimensionum numerum, ac vicissim omnes hujusmodi functiones in forma illa contineri.

Corollarium 2.

474. Multiplicatione autem porro facta per z^n, oritur functio homogenea trium variabilium x, y, z, cujus dimensionum numerus est $= n$; unde solutio nostri problematis ita enunciari potest, ut quantitas quaesita v sit functio homogenea trium variabilium x, y et z, dimensionum numero existente $= n$.

Corollarium 3.

475. Quodsi ergo conditio praescripta sit

$$p x + q y + r z = 0, \text{ seu}$$

$$x \left(\frac{\partial v}{\partial x}\right) + y \left(\frac{\partial v}{\partial y}\right) + z \left(\frac{\partial v}{\partial z}\right) = 0,$$

quantitas v erit functio homogenea nullius dimensionis trium variabilium x, y et z.

Scholion.

476. Simili modo solutio succedit, si conditio praescripta postulet, ut sit

$$\alpha p x + \beta q y + \gamma r z = n v, \text{ seu}$$

$$\alpha x \left(\frac{\partial v}{\partial x}\right) + \beta y \left(\frac{\partial v}{\partial y}\right) + \gamma z \left(\frac{\partial v}{\partial z}\right) = n v,$$

tum enim ob

$$r = \frac{n v - \alpha p x - \beta q y}{\gamma z}, \text{ fit}$$

$$\partial v - \frac{n v \partial z}{\gamma z} = p \left(\partial x - \frac{\alpha x \partial z}{\gamma z}\right) + q \left(\partial y - \frac{\beta y \partial z}{\gamma z}\right),$$

quae aequatio sequenti forma exhibeatur

$$\frac{\gamma\partial v}{v} - \frac{n\partial z}{z} = \frac{px}{v}\left(\frac{\gamma\partial x}{x} - \frac{\alpha\partial z}{z}\right) + \frac{qy}{v}\left(\frac{\gamma\partial y}{y} - \frac{\beta\partial z}{z}\right),$$

ex qua concludimus, integrale primi membri $\gamma\, l v - n\, l z$ aequari functioni cuicunque binarum quantitatum

$$\gamma\, l x - \alpha\, l z \text{ et } \gamma\, l y - \beta\, l z,$$

et logarithmorum numeris sumtis fore

$$\frac{v^\gamma}{z^n} = \Gamma : \left(\frac{x^\gamma}{z^\alpha} \text{ et } \frac{y^\gamma}{z^\beta}\right).$$

Ponamus $\alpha = \frac{1}{\lambda}$, $\beta = \frac{1}{\mu}$ et $\gamma = \frac{1}{\nu}$, ut conditio praescripta sit

$$\frac{px}{\lambda} + \frac{qy}{\mu} + \frac{rz}{\nu} = n\, v,$$

et solutio reducetur ad hanc formam

$$v = z^{\nu n}\, \Delta : \left(\frac{x^\lambda}{z^\nu} \text{ et } \frac{y^\mu}{z^\nu}\right).$$

Quodsi porro scribamus

$$x^\lambda = X,\ y^\mu = Y \text{ et } z^\nu = Z, \text{ fiet}$$

$$v = Z^n\, \Delta : \left(\frac{X}{Z} \text{ et } \frac{Y}{Z}\right),$$

ideoque quantitas quaesita v est functio homogenea, in qua tres variabiles X, Y et Z ubique eundem dimensionum numerum $= n$ adimplent.

Problema 81.

477. Si posito

$$\partial v = p\, \partial x + q\, \partial y + r\, \partial z,$$

haec conditio praescribatur ut sit

$$p\, x + q\, y + r\, z = n\, v + S,$$

existente S functione quacunque data variabilium x, y, z, investigare naturam functionis quaesitae v.

Solutio.

Cum conditio praescripta praebet

$$r = \frac{nv + S - px - qy}{z}, \text{ erit}$$

$$\partial v - \frac{nv\partial z}{z} = \frac{S\partial z}{z} + p\left(\partial x - \frac{x\partial z}{z}\right) + q\left(\partial y - \frac{y\partial z}{z}\right),$$

seu

$$\partial \cdot \frac{v}{z^n} = \frac{S\partial z}{z^{n+1}} + \frac{p}{z^{n-1}}\,\partial \cdot \frac{x}{z} + \frac{q}{z^{n-1}}\,\partial \cdot \frac{y}{z}.$$

Sit $x = tz$ et $y = uz$, ut jam S fiat functio trium variabilium t, u et z, et formula differentialis $\frac{S\partial z}{z^{n+1}}$ ita integretur, ut quantitates t et u constantes habeantur, quo integrali posito $= V$, erit

$$v = Vz^n + z^n \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right),$$

ubi pars posterior significat functionem homogeneam trium variabilium x, y, z, numero dimensionum existente $= n$.

Corollarium 1.

478. Si S sit quantitas constans $= C$, erit

$$V = \int \frac{C\partial z}{z^{n+1}} = -\frac{C}{nz^n},$$

hincque primum integralis membrum

$$Vz^n = -\frac{C}{n},$$

ex quo perspicuum est eundem valorem proditurum fuisse, quantitatibus x, y, z inter se permutatis.

Corollarium 2.

479. Si S sit functio homogenea ipsarum x, y, z, dimensionum numero existente $= m$, quia tum posito $x = tz$ et $y = uz$, sit $S = Mz^m$, ita ut M tantum quantitates t et u involvat, ideo

que pro constante sit habenda, prodit

$$V = \int Mz^{m-n-1}\partial z = \frac{Mz^{m-n}}{m-n} = \frac{S}{(m-n)z^n},$$

sicque primum integralis membrum erit $= \frac{S}{m-n}$.

Corollarium 3.

480. At si hoc casu sit $m = n$, fit

$$V = Mlz + C = Mlaz,$$

et primum integralis membrum

$$Mz^n laz = Slaz.$$

Pari jure id autem erit

$$Slby \text{ vel } Slax;$$

id quod satis est manifestum, cum horum valorum differentia fiat functio homogenea n dimensionum, ideoque in altero integralis membro contineatur.

Scholion.

481. Principium hujus solutionis in hoc lemmate latissime patente continetur, quod si fuerit

$$\partial V = S\partial Z = P\partial X + Q\partial Y,$$

ubi S denotat functionem datam, P et Q vero functiones indefinitas, futurum sit

$$V = \int S\partial Z + \Gamma : (X \text{ et } Y),$$

at hic non sufficit indicasse in integratione formulae $S\partial Z$, solam quantitatem Z pro variabili haberi, sed insuper notari convenit, binas X et Y tanquam constantes tractari debere. Quare si forte S sit proposita functio aliarum trium variabilium x, y, z, ex quibus hae X, Y, Z, quarum ratio hic est habenda, certo modo nascantur, primum loco x, y, z istae X, Y et Z introduci debent, ut fiat S funtio harum X, Y et Z; tum vero demum binis X et Y pro constantibus solaque Z pro variabili sumta, integrale $\int S\partial z$ est capiendum. Ita in casu problematis pro integrali $\int \frac{S\partial z}{z^{n+1}}$, quantitates $\frac{x}{z}$ et $\frac{y}{z}$ ut constantes sunt spectandae, sola z pro variabili sumta; ex quo in functione S statui oportet $x = tz$ et $y = uz$, ut S fiat functio ipsarum z, $t = \frac{x}{z}$ et $u = \frac{y}{z}$, quarum binae posteriores pro constantibus sunt habendae. Hoc ergo casu insignis error committeretur, si quis, sumta z variabili, reliquas x et y ut constantes tractare voluerit, quoniam ambae x et y etiam variabilem z involvere sunt censendae. Quod autem variabilibus permutatis primum integralis membrum idem resultare debeat, ut sit

$$z^n \int \frac{S\partial z}{z^{n+1}} = x^n \int \frac{S\partial z}{x^{n+1}},$$

inde patet, quod posito $x = tz$ et $\partial x = t\partial z$, ob t constantem sumendam fiat

$$x^n \int \frac{S\partial x}{x^{n+1}} = t^n z^n \int \frac{St\partial z}{t^{n+1} z^{n+1}} = z^n \int \frac{S\partial z}{z^{n+1}};$$

in utraque enim integratione rationes variabilium $\frac{x}{z}$, $\frac{y}{z}$, $\frac{x}{y}$, pro constantibus sunt habendae, hincque in reductione facta quantitas $t = \frac{x}{z}$ recte ut constans spectatur.

Problema 82.

482. Si posito

$$\partial v = p\partial x + q\partial y + r\partial z,$$

haec conditio praescribatur, ut esse debeat

$$pL + qM + rN = 0,$$

existentibus L, M, N functionibus datis respective variabilium x,

y et z, nempe L ipsius x, M ipsius y et N ipsius z tantum, naturam functionis quaesitae v definire.

Solutio.

Ob $r = \frac{-pL - qM}{N}$ aequatio principalis fit

$$\partial v = p(\partial x - \tfrac{L\partial z}{N}) + q(\partial y - \tfrac{M\partial z}{N}), \text{ vel}$$

$$\partial v = pL(\tfrac{\partial x}{L} - \tfrac{\partial z}{N}) + qM(\tfrac{\partial y}{M} - \tfrac{\partial z}{N}).$$

Statuatur

$$t = \int \tfrac{\partial x}{L} - \int \tfrac{\partial z}{N} \text{ et } u = \int \tfrac{\partial y}{M} - \int \tfrac{\partial z}{N},$$

ut fiat

$$\partial v = pL\partial t + qM\partial u;$$

et manifestum est, quantitatem v aequari debere functioni cuicunque binarum variabilium t et u, quas ita quoque describere licet, ut positis formulis tribus integralibus $\int \frac{\partial x}{L}$, $\int \frac{\partial y}{M}$, et $\int \frac{\partial z}{N}$, pro t et u sumi oporteat differentias inter binas earum.

Scholion 1.

483. Solutio etiam successisset, dummodo $\frac{L}{N}$ fuisset functio ipsarum x et z, et $\frac{M}{N}$ ipsarum y et z tantum; tum enim multiplicatores P et Q ad integrationem apti quaeri debuissent, ut fieret

$$P(\partial x - \tfrac{L\partial z}{N}) = \partial t \text{ et } Q(\partial y - \tfrac{M\partial z}{N}) = \partial u,$$

et ob

$$\partial v = \tfrac{p\partial t}{P} + \tfrac{q\partial u}{Q}, \text{ foret}$$

$$v = \Gamma:(t \text{ et } u).$$

Permutandis vero variabilibus x, y et z, etiam alii casus resolubiles prodeunt. Quando autem quantitates L, M, N aliter sunt comparatae, via non patet certa ad solutionem perveniendi, quae certe haud parum abstrusa videtur, cum pro hoc casu satis simplici

$$(y + z)p + (x + z)q + (x + y)z = 0$$

per plures ambages tandem ad hanc pervenerim solutionem, ut posito

$$t = (x + y + z)(x - z)^2 \text{ et } u = (x + y + z)(y - z)^2,$$

fiat $v = \Gamma : (t \text{ et } u)$; quoniam igitur binae quantitates t et u, quarum functio quaecunque loco v posita conditioni satisfacit, hoc casu tantopere sunt complicatae, generaliter multo minus solutionem expectare licebit.

Scholion 2.

484. Ad plures autem alios casus solutio extendi potest. Si functiones datae L, M, N ita fuerint comparatae, ut alias E, F, G, H reperire liceat, quibus fiat

$$E\left(\partial x - \frac{L\partial z}{N}\right) + F\left(\partial y - \frac{M\partial z}{N}\right) = \partial t \text{ et}$$

$$G\left(\partial x - \frac{L\partial z}{N}\right) + H\left(\partial y - \frac{M\partial z}{N}\right) = \partial u,$$

tum enim posito

$$p = PE + QG \text{ et } q = PF + QH, \text{ fiet}$$

$$\partial v = P\partial t + Q\partial u,$$

ubi P et Q sunt functiones indefinitae loco p et q introductae, quantitas v aequabitur functioni cuicunque binarum variabilium t et u, seu erit

$$v = \Gamma : (t \text{ et } u).$$

Totum ergo negotium huc redit, ut pro datis functionibus L, M, N, functiones E, F et G, H inveniantur, quod quidem semper praestari posse videtur, sed haec ipsa quaestio plerumque difficilior evadit quam ipsa proposita. Sufficit autem binas ejusmodi functiones

E et F indeque quantitatem t investigasse; quia deinceps permutandis variabilibus x, y, z, una cum respondentibus L, M, N, sponte idoneus valor pro u elicitur. Ita in exemplo ante allato

$$L = y + z, \quad M = x + z, \quad N = x + y,$$

postquam invenerimus

$$t = (x + y + z)(x - z)^2,$$

sola permutatio statim praebet

$$u = (x + y + z)(y - z)^2,$$

vel etiam

$$u = (x + y + z)(x - y)^2,$$

Problema 83.

485. Si posito

$$\partial v = p\partial x + q\partial y + r\partial z$$

haec conditio praescribatur, ut sit $pqr = 1$, naturam functionis v investigare.

Solutio.

Ob $r = \frac{1}{pq}$, erit

$$\partial v = p\partial x + q\partial y + \frac{\partial z}{pq},$$

unde colligimus

$$v = px + qy + \frac{z}{pq} - \int (x\partial p + y\partial q - \frac{z\partial p}{ppq} - \frac{z\partial q}{pqq}),$$

qua transformatione id sumus assecuti, ut formula integralis bina tantum differentialia ∂p et ∂q involvat. His igitur in locum principalium introductis concludimus, illam formulam integralem aequari debere functioni cuicunque binarum variabilium p et q. Sit S talis functio, ut fiat

$$v = px + qy + \frac{z}{pq} - S,$$

et jam superest, ut cum litterae p et q in calculo retineantur aliae duae elidentur, id quod inde est petendum, quod sit

$$\partial S = \left(x - \frac{z}{ppq}\right)\partial p + \left(y - \frac{z}{pqq}\right)\partial q,$$

ideoque

$$x - \frac{z}{ppq} = \left(\frac{\partial S}{\partial p}\right) \text{ et } y - \frac{z}{pqq} = \left(\frac{\partial S}{\partial q}\right):$$

Nunc igitur solutio ita se habebit. Introductis his ternis variabilibus p, q et z, sumtaque binarum p et q functione quacunque S, capiatur

$$x = \frac{z}{ppq} + \left(\frac{\partial S}{\partial p}\right) \text{ et } y = \frac{z}{pqq} + \left(\frac{\partial S}{\partial q}\right),$$

ac tum functio quaesita v ita definietur, ut sit

$$v = \frac{3z}{pq} + p\left(\frac{\partial S}{\partial p}\right) + q\left(\frac{\partial S}{\partial q}\right) - S.$$

Vel si malimus v per ipsas tres variabiles x, y, z exprimere, ex binis aequationibus

$$x = \frac{z}{ppq} + \left(\frac{\partial S}{\partial p}\right) \text{ et } y = \frac{z}{pqq} + \left(\frac{\partial S}{\partial q}\right)$$

quaerantur valores ipsarum p et q, quibus in functione S substitutis erit

$$v = px + qy + \frac{z}{pq} - S,$$

sicque quaesito erit satisfactum.

Corollarium 1.

486. Si functio S sumatur quantitas constans C, ob

$$ppq = \frac{z}{x} \text{ et } pqq = \frac{z}{y}, \text{ erit}$$

$$pq = \sqrt[3]{\frac{zz}{xy}}, \text{ hincque}$$

$$p = \sqrt[3]{\frac{yz}{xx}} \text{ et } q = \sqrt[3]{\frac{xz}{yy}}; \text{ unde fit}$$

$$v = 3\sqrt[3]{xyz} - C,$$

qui est valor particularis problemati satisfaciens.

Corollarium 2.

487. Quoniam in conditione praescripta

$$pqr = 1 \,; \quad \text{seu} \quad \left(\frac{\partial v}{\partial x}\right) \left(\frac{\partial v}{\partial y}\right) \left(\frac{\partial v}{\partial z}\right) = 1$$

tantum differentialia trium variabilium x, y et z occurrunt, eas quantitatibus constantibus quibusvis augere licet, unde nascitur solutio aliquanto latius patens

$$v = 3 \sqrt[3]{(x + a)(y + b)(z + c)} - C.$$

Scholion 1.

488. Alius datur praeterea casus facilem evolutionem admittens, ponendo $S = 2c\sqrt{pq}$, unde colligitur

$$p = \frac{\sqrt{y}}{\sqrt{x}} \sqrt[3]{\frac{z}{\sqrt{xy} - c}} \quad \text{et} \quad q = \frac{\sqrt{x}}{\sqrt{y}} \sqrt[3]{\frac{z}{\sqrt{xy} - c}},$$

ideoque $S = 2c \sqrt[3]{\frac{z}{\sqrt{xy} - c}}$.

Assequimur ergo

$$v = 3 \sqrt[3]{z (\sqrt{xy} - c)^2},$$

et permutandis variabilibus simili modo habebimus

$$v = 3 \sqrt[3]{y(\sqrt{xz} - b)^2} \quad \text{et} \quad v = 3 \sqrt[3]{x(\sqrt{yz} - a)^2},$$

ubi porro pro x, y, z scribere licet $x + f$, $y + g$, $z + h$. Caeterum patet solutionem generalem perinde succedere, si quantitas r functioni cuicunque ipsarum p et q aequari debeat, seu si inter p, q, r aequatio quaecunque proponatur.

Scholion 2.

489. Quodsi enim posito

$$\partial v = p\partial x + q\partial y + r\partial z,$$

inter ternas formulas

$$p = \left(\frac{\partial v}{\partial x}\right), \quad q = \left(\frac{\partial v}{\partial y}\right), \quad r = \left(\frac{\partial v}{\partial z}\right)$$

aequatio proponatur quaecunque, quae differentiata praebeat

$$P \partial p + Q \partial q + R \partial r = 0,$$

tum facto

$$S = \int (x \partial p + y \partial q + z \partial r),$$

ut sit

$$v = px + qy + rz - S,$$

sumatur functio quaecunque trium quantitatum p, q, r, quae sit V, haecque differentiata praebeat

$$\partial V = L \partial p + M \partial q + N \partial r,$$

tum vero est

$$0 = P u \partial p + Q u \partial q + R u \partial r,$$

ideoque

$$\partial V = (L + P u) \partial p + (M + Q u) \partial q + (N + R u) \partial r,$$

quae forma ob novam introductam variabilem u latissime patet. Statuatur jam $S = V$, fietque

$$x = L + P u, \quad y = M + Q u, \quad z = N + R u,$$

ita ut nunc praeter variabiles p, q, r, quarum una per binas reliquas datur, nova habeatur u, ex quibus jam tres x, y et z ita definivimus, ut per eas vicissim hae p, q, r et u determinentur, tum vero erit

$$v = px + qy + rz - V.$$

Quare pro V sumta quacunque functione trium quantitatum p, q, r, inter quas ejusmodi conditio praescribitur, ut sit

$$P \partial p + Q \partial q + R \partial r = 0,$$

sumatur

$$x = Pu + \left(\frac{\partial V}{\partial p}\right), \quad y = Qu + \left(\frac{\partial V}{\partial q}\right), \quad z = Ru + \left(\frac{\partial V}{\partial r}\right),$$

eritque

$$v = (Pp + Qq + Rr)u + p\left(\frac{\partial V}{\partial p}\right) + q\left(\frac{\partial V}{\partial q}\right) + r\left(\frac{\partial V}{\partial r}\right) - V,$$

quae solutio praecedenti ideo est anteferenda, quod in hanc tres quantitates p, q, r aequaliter ingrediuntur.

Problema 84.

490. Si posito

$$\partial v = p\,\partial x + q\,\partial y + r\,\partial z,$$

haec conditio praescribatur, ut esse debeat $pqr = \frac{v^3}{xyz}$, naturam functionis v definire.

Solutio.

Ponamus $p = \frac{Pv}{x}$, $q = \frac{Qv}{y}$, $r = \frac{Rv}{z}$, et ob conditionem praescriptam debet esse $PQR = 1$; tum vero erit

$$\frac{\partial v}{v} = \frac{P\partial x}{x} + \frac{Q\partial y}{y} + \frac{R\partial z}{z}.$$

Statuamus nunc

$$lv = V, \quad lx = X, \quad ly = Y, \quad lz = Z,$$

et habebimus hanc aequationem

$$\partial V = P\,\partial X + Q\,\partial Y + R\,\partial Z,$$

pro qua esse debet $PQR = 1$, quae quaestio cum non discrepet a problemate praecedente, eadem solutio huc quoque facillime transferetur.

Scholion.

491. Plures casus, quos forte in hoc capite expedire liceat, hic non evolvo, cum quia usus nondum perspicitur, tum vero im-

primis, quoniam hujus partis calculi integralis prorsus adhuc incognitae prima tantum principia adumbrare constitui. Pro formulis autem differentialibus altiorum graduum, quae in conditionem praescriptam ingrediantur, vix quicquam proferre licet, praeter quasdam observationes ad aequationes homogeneas pertinentes, quibus ergo hanc partem calculi integralis sum finiturus, simulque toti operi finem impositurus.

CAPUT IV.

DE AEQUATIONUM DIFFERENTIALIUM HOMOGENEARUM RESOLUTIONE.

Problema 85.

492.

Si v aequetur functioni cuicunque binarum quantitatum t et u, ita per tres variabiles x, y et z determinatarum, ut sit

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

ejus formulas differentiales omnium graduum inde definire.

Solutio.

Cum v sit functio quantitatum

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

ejus formulae differentiales ex his duabus variabilibus natae innotescent, scilicet

$$\left(\frac{\partial v}{\partial t}\right),\ \left(\frac{\partial v}{\partial u}\right),\ \left(\frac{\partial\partial v}{\partial t^2}\right),\ \left(\frac{\partial\partial v}{\partial t\partial u}\right),\ \left(\frac{\partial\partial v}{\partial u^2}\right),$$

hinc autem statim colligimus

$$\left(\frac{\partial v}{\partial x}\right) = \alpha\left(\frac{\partial v}{\partial t}\right),\ \left(\frac{\partial v}{\partial y}\right) = \gamma\left(\frac{\partial v}{\partial u}\right),\ \left(\frac{\partial v}{\partial z}\right) = \beta\left(\frac{\partial v}{\partial t}\right) + \delta\left(\frac{\partial v}{\partial u}\right),$$

formulas scilicet differentiales primi gradus. Pro formulis autem differentialibus secundi gradus adipiscimur

$$\left(\frac{\partial\partial v}{\partial x^2}\right) = \alpha\alpha\left(\frac{\partial\partial v}{\partial t^2}\right),\ \left(\frac{\partial\partial v}{\partial y^2}\right) = \gamma\gamma\left(\frac{\partial\partial v}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial v}{\partial z^2}\right) = \beta\beta\left(\frac{\partial\partial v}{\partial t^2}\right) + 2\beta\delta\left(\frac{\partial\partial v}{\partial t\partial u}\right) + \delta\delta\left(\frac{\partial\partial v}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) = \alpha\gamma\left(\frac{\partial\partial v}{\partial t\partial u}\right),\ \left(\frac{\partial\partial v}{\partial x\partial z}\right) = \alpha\beta\left(\frac{\partial\partial v}{\partial t^2}\right) + \alpha\delta\left(\frac{\partial\partial v}{\partial t\partial u}\right),$$

et $\left(\frac{\partial\partial v}{\partial y\partial z}\right) = \beta\gamma\left(\frac{\partial\partial v}{\partial t\partial u}\right) + \gamma\delta\left(\frac{\partial\partial v}{\partial u^2}\right).$

Simili modo ad tertium gradum ascendimus

$$\left(\frac{\partial^3 v}{\partial x^3}\right) = \alpha^3 \left(\frac{\partial^3 v}{\partial t^3}\right), \quad \left(\frac{\partial^3 v}{\partial y^3}\right) = \gamma^3 \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial z^3}\right) = \beta^3 \left(\frac{\partial^3 v}{\partial t^3}\right) + 3\,\beta^2 \delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + 3\,\beta\,\delta^2 \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \delta^3 \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right) = \alpha\,\alpha\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right), \quad \left(\frac{\partial^3 v}{\partial x \partial y^2}\right) = \alpha\,\gamma\,\gamma \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

$$\left(\frac{\partial^3 v}{\partial x^2 \partial z}\right) = \alpha\,\alpha\,\beta \left(\frac{\partial^3 v}{\partial t^3}\right) + \alpha\,\alpha\,\delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right),$$

$$\left(\frac{\partial^3 v}{\partial y^2 \partial z}\right) = \beta\,\gamma\,\gamma \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \gamma\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x \partial z^2}\right) = \alpha\,\beta\,\beta \left(\frac{\partial^3 v}{\partial t^3}\right) + 2\,\alpha\,\beta\,\delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + \alpha\,\delta\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

$$\left(\frac{\partial^3 v}{\partial y \partial z^2}\right) = \beta\,\beta\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + 2\,\beta\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \gamma\,\delta\,\delta \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x \partial y \partial z}\right) = \alpha\,\beta\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + \alpha\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

unde facile patet, quomodo has formulas differentiales ad altiores gradus continuari oporteat.

Scholion 1.

493. Hoc problema fortasse generalius concipi debuisse videbitur, quantitates t et u ita per tres variabiles x, y, z definiendo, ut esset

$$t = \alpha x + \beta y + \gamma z \quad \text{et} \quad u = \delta x + \varepsilon y + \zeta z,$$

verum cum haec hypothesis in eum tantum finem sit facta, ut v fieret functio ipsarum t et u, evidens est tum quoque v spectari posse ut functionem harum duarum quantitatum $\varepsilon t - \beta u$ et $\delta t - \alpha u$, quarum illa ab y haec vero ab x erit libera. Quocirca hypothesis assumta latissime patere est censenda, exceptio tamen forte hinc admittenda videbitur, si fuerit

$$t = x + z \quad \text{et} \quad u = x - z,$$

quia hic ipsius u valor non continetur, verum etiam hoc casu quantitas v ut functio ipsarum $t + u$ et $t - u$ spectata fiet functio ipsarum x et z, qui casus utique in hypothesi continetur, sumtis $\beta = 0$ et $\gamma = 0$.

Scholion 2.

494. Hoc problema ideo praemisi, quia alias aequationes differentiales tractare hic non sustineo, nisi quibus ejusmodi valor satisfaciet, ut v aequetur functioni cuicunque binarum novarum variabilium t et u, quae ab principalibus x, y, z ita pendeant, ut sit quemadmodum assumsi

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Hujusmodi autem aequationes, quibus hoc modo satisfieri potest, esse homogeneas, facile patet, ita ut aequatio resolvenda constet nonnisi formulis differentialibus ejusdem gradus, singulis per constantes quantitates multiplicatis, et inter se additis, qua appellatione aequationum homogenearum jam in parte praecedente sum usus. Proposita ergo hujusmodi aequatione homogenea, loco singularum formulam differentialium per elementa ∂x, ∂y, ∂z formatarum substituantur valores hic inventi per elementa ∂t et ∂u formati, et tum singula membra, quatenus certam formulam differentialem ex elementis ∂t et ∂u natam complectuntur, seorsim ad nihilum redigantur; indeque rationes $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ determinentur; quandoquidem quaestio non tam circa has ipsas quantitates, quam earum rationes versatur. Quoniam igitur duae tantum res investigationi relinquuntur, si pluribus aequationibus fuerit satisfaciendum, ejusmodi aequationes homogeneae hac ratione resolvi nequeunt, nisi casu quo plures illae aequationes ad duas tantum revocentur, id quod in sequentibus clarius explicabitur.

Problema 86.

495. Proposita aequatione homogenea primi gradus

$$A\left(\frac{\partial v}{\partial x}\right) + B\left(\frac{\partial v}{\partial y}\right) + C\left(\frac{\partial v}{\partial z}\right) = 0,$$

investigare naturam functionis v trium variabilium x, y et z.

Solutio.

Fingatur $v = \Gamma:(t \text{ et } u)$, existente

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

et facta substitutione ex problemate praecedente aequatio nostra in duas partes dividetur

$$\left(\frac{\partial v}{\partial t}\right)(A\alpha + C\beta) + \left(\frac{\partial v}{\partial u}\right)(B\gamma + C\delta) = 0,$$

quarum utraque seorsim ad nihilum reducta praebet

$$\frac{\beta}{\alpha} = \frac{-A}{C} \text{ et } \frac{\delta}{\gamma} = \frac{-B}{C},$$

unde fit

$$t = Cx - Az \text{ et } u = Cy - Bz.$$

Quare aequationis propositae integrale completum erit

$$v = \Gamma:(\overline{Cx - Az} \text{ et } \overline{Cy - Bz}),$$

quod etiam concinnius ita exhiberi potest

$$v = \Gamma:\left(\frac{x}{A} - \frac{z}{C} \text{ et } \frac{y}{B} - \frac{z}{C}\right).$$

Corollarium 1.

496. Permutandis variabilibus hoc integrale etiam ita exprimi posse evidens est

$$v = \Gamma:\left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{y}{B} - \frac{z}{C}\right), \text{ vel}$$

$$v = \Gamma:\left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{x}{A} - \frac{z}{C}\right),$$

quoniam est

$$\frac{x}{A} - \frac{y}{B} = \left(\frac{x}{A} - \frac{z}{C}\right) - \left(\frac{y}{B} - \frac{z}{C}\right).$$

Corollarium 2.

497. Quin etiam constitutis ex aequatione proposita his tribus formulis

$$\frac{x}{A} - \frac{y}{B}, \ \frac{x}{A} - \frac{z}{C}, \ \frac{y}{B} - \frac{z}{C},$$

functio quaecunque ex iis utcunque conflata valorem idoneum pro v suppeditabit. Quoniam enim harum binarum formularum unaquaeque est differentia binarum reliquarum, talis functio duas tantum variabiles complecti est censenda.

Corollarium 3.

498. Perinde est quanam harum trium formarum integralium utamur, quando autem binae novae variabiles t et u inter se fuerint aequales, tum alia est utendum. Veluti si esset $C = 0$, prima forma $v = \Gamma : (z \text{ et } z)$, utpote functio solius z foret inutilis, et integrale completum esset futurum

$$v = \Gamma : (\overline{\tfrac{x}{A} - \tfrac{y}{B}} \text{ et } z), \text{ seu}$$
$$v = \Gamma : (\overline{Bx - Ay} \text{ et } z).$$

Problema 87.

499. Proposita aequatione homogenea secundi gradus

$$A\left(\frac{\partial\partial v}{\partial x^2}\right) + B\left(\frac{\partial\partial v}{\partial y^2}\right) + C\left(\frac{\partial\partial v}{\partial z^2}\right) + 2D\left(\frac{\partial\partial v}{\partial x\,\partial y}\right) + 2E\left(\frac{\partial\partial v}{\partial x\,\partial z}\right) + 2F\left(\frac{\partial\partial v}{\partial y\,\partial z}\right) = 0,$$

casus investigare, quibus ejus integrale hac forma $\Gamma : (t \text{ et } u)$ exprimi potest, existente

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Solutio.

Facta substitutione secundum formulas in problemate 85. traditas, aequatio proposita in tria membra sequentia resolvetur

$$\left.\begin{array}{l} \left(\frac{\partial\partial v}{\partial t^2}\right)(A\alpha\alpha + C\beta\beta + 2E\alpha\beta) \\ \left(\frac{\partial\partial v}{\partial t\,\partial u}\right)(2C\beta\delta + 2D\alpha\gamma + 2E\alpha\delta + 2F\beta\gamma) \\ \left(\frac{\partial\partial v}{\partial u^2}\right)(B\gamma\gamma + C\delta\delta + 2F\gamma\delta) \end{array}\right\} = 0,$$

quorum singula seorsim nihilo debent aequari. At primum praebet

$$\frac{\beta}{\alpha} = \frac{-E + \sqrt{(EE - AC)}}{C},$$

ultimum vero

$$\frac{\delta}{\gamma} = \frac{-F + \sqrt{(FF - BC)}}{C},$$

qui valores in media, quae ita referatur

$$\frac{C\beta\delta}{\alpha\gamma} + D + \frac{E\delta}{\gamma} + \frac{F\beta}{\alpha} = 0,$$

substituti suppeditant hanc aequationem

$$EF - CD = \sqrt{(EE - AC)(FF - BC)},$$

qua aequatione conditio inter coëfficientes A, B, C, D, E, F continetur, ut solutio hic applicata locum invenire possit. Haec autem aequatio evoluta dat

$$CCDD - 2CDEF + BCEE + ACFF - ABCC = 0,$$

unde fit

$$C = \frac{2DEF - BEE - AFF}{DD - AB},$$

quia factor C per multiplicationem est ingressus. Quoties autem haec conditio habet locum, ut sit

$$AFF + BEE + CDD = ABC + 2DEF,$$

toties haec expressio algebraica ex aequatione proposita formanda

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$$

in duos factores potest resolvi, neque ergo aliis casibus solutio hic adhibita locum habere potest. Quo ergo hos casus solutionem admittentes rite evolvamus, ponamus hujus formae factores esse

$$(ax + by + cz)(fx + gy + hz),$$

quod ergo eveniet, si fuerit

$$A = af, \quad B = bg, \quad C = ch,$$
$$2D = ag + bf, \quad 2E = ah + cf, \quad 2F = bh + cg,$$

unde utique fit

$$AFF + BEE + CDD = ABC + 2DEF.$$

Hinc autem pro solutione colligitur

$$\text{vel } \frac{\beta}{\alpha} = \frac{-a}{c}, \quad \text{vel } \frac{\beta}{\alpha} = \frac{-f}{h}, \quad \text{et}$$

$$\text{vel } \frac{\delta}{\gamma} = \frac{-b}{c}, \quad \text{vel } \frac{\delta}{\gamma} = \frac{-g}{h},$$

ubi observari oportet, pro fractionibus $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ valores sibi subscriptos conjungi oportere, ita ut sit

$$\text{vel } t = cx - az, \quad \text{et } u = cy - bz,$$

$$\text{vel } t = hx - fz, \quad \text{et } u = hy - gz.$$

Quocirca pro his casibus solutionem admittentibus integrale completum erit

$$v = \Gamma : (\overline{cx - az} \text{ et } \overline{cy - bz}) + \Delta : (\overline{hx - fz} \text{ et } \overline{hy - gz}),$$

seu

$$v = \Gamma : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) + \Delta : (\tfrac{x}{f} - \tfrac{z}{h} \text{ et } \tfrac{y}{g} - \tfrac{z}{h}).$$

Corollarium 1.

500. Hoc ergo modo aliae aequationes homogeneae secundi gradus resolvi nequeunt, nisi quae in hac forma continentur

$$af\left(\frac{\partial\partial v}{\partial x^2}\right) + bg\left(\frac{\partial\partial v}{\partial y^2}\right) + ch\left(\frac{\partial\partial v}{\partial z^2}\right) + (ag + bf)\left(\frac{\partial\partial v}{\partial x \partial y}\right)$$

$$+ (ah + cf)\left(\frac{\partial\partial v}{\partial x \partial z}\right) + (bh + cg)\left(\frac{\partial\partial v}{\partial y \partial z}\right) = 0,$$

tum vero integrale completum erit

$$v = \Gamma : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} + \tfrac{z}{c}) + \Delta : (\tfrac{x}{f} - \tfrac{z}{h} \text{ et } \tfrac{y}{g} - \tfrac{z}{h}).$$

Corollarium 2.

501. Quo autem facilius dignoscatur, utrum aequatio quaepiam proposita

$$A\left(\frac{\partial\partial v}{\partial x^2}\right) + B\left(\frac{\partial\partial v}{\partial y^2}\right) + C\left(\frac{\partial\partial v}{\partial z^2}\right) + 2\,D\left(\frac{\partial\partial v}{\partial x\partial y}\right) + 2\,E\left(\frac{\partial\partial v}{\partial x\partial z}\right) + 2\,F\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0$$

eo reduci possit nec ne? formetur inde haec forma algebraica

$$A\,xx + B\,yy + C\,zz + 2\,D\,xy + 2\,E\,xz + 2\,F\,yz,$$

quae si resolvi patiatur in duos factores rationales

$$(ax + by + cz)\,(fx + gy + hz),$$

ejus integrale completum hinc statim exhiberi potest.

Corollarium 3.

502. Unicus tantum casus quo duo isti factores inter se fiunt aequales, exceptionem postulat, quoniam tum binae functiones inventae in unam coalescerent. Verum ex superioribus colligitur, si hoc eveniat ut sit $f = a$, $g = b$ et $h = c$, integrale completum ita exprimi

$$z = x\,\Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right).$$

Scholion 1.

503. Quibus ergo casibus aequatio homogenea secundi gradus resolutionem admittit, iis quoque in se complectitur duas aequationes homogeneas primi gradus

$$a\left(\frac{\partial v}{\partial x}\right) + b\left(\frac{\partial v}{\partial y}\right) + c\left(\frac{\partial v}{\partial z}\right) = 0, \text{ et}$$

$$f\left(\frac{\partial v}{\partial x}\right) + g\left(\frac{\partial v}{\partial y}\right) + h\left(\frac{\partial v}{\partial z}\right) = 0,$$

quippe quarum utraque illi satisfacit, et harum integralia completa junctim sumta illius integrale completum suppeditant. Hinc alia via aperitur aequationum homogenearum secundi gradus integralia inveniendi, fingendo aequationem primi gradus ipsis satisfacientem

$$a\left(\frac{\partial v}{\partial x}\right) + b\left(\frac{\partial v}{\partial y}\right) + c\left(\frac{\partial v}{\partial z}\right) = 0\,;$$

tum ex hac per triplicem differentiationem tres novae formentur

$$a\left(\frac{\partial\partial v}{\partial x^2}\right) + b\left(\frac{\partial\partial v}{\partial x\partial y}\right) + c\left(\frac{\partial\partial v}{\partial x\partial z}\right) = 0,$$
$$a\left(\frac{\partial\partial v}{\partial x\partial y}\right) + b\left(\frac{\partial\partial v}{\partial y^2}\right) + c\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0,$$
$$a\left(\frac{\partial\partial v}{\partial x\partial z}\right) + b\left(\frac{\partial\partial v}{\partial y\partial z}\right) + c\left(\frac{\partial\partial v}{\partial z^2}\right) = 0,$$

quarum prima per f, secunda per g et tertia per h multiplicatae et in unam summam collectae, ipsam illam aequationem generalem producunt, cujus integrale supra exhibuimus. Ea ergo quasi productum ex binis aequationibus homogeneis primi gradus spectari poterit, ex quibus conjunctis integrale completum formatur.

Scholion 2.

504. Infinitae ergo aequationes homogeneae secundi gradus hic excluduntur, quae hoc modo integrationem respuunt, seu ad aequationes primi gradus reduci nequeunt; qui casus exclusi omnes ex hoc criterio agnoscuntur, si non fuerit

$$AFF + BEE + CDD = ABC + 2DEF.$$

Hujus generis est ista aequatio $\left(\frac{\partial\partial v}{\partial x\partial y}\right) = \left(\frac{\partial\partial v}{\partial z^2}\right)$, quae ergo tale integrale, cujusmodi hic assumsimus non admittit, neque etiam alia patet via ejus integrale completum investigandi. Integralia autem particularia facile innumera exhiberi possunt, et quae adeo functiones arbitrarias complectuntur, sed tantum unius quantitatis variabilis, quae in praesenti instituto nonnisi integralia particularia constituere sunt censendae. Si enim ponatur

$$v = \Gamma : (\alpha x + \beta y + \gamma z),$$

facta substitutione fieri debet $\alpha\beta = \gamma\gamma$, seu sumto $\gamma = 1$, debet esse $\alpha\beta = 1$; quare innumerabiles adeo hujusmodi formulae conjunctae satisfaciunt, ut sit

$$v = \Gamma : \left(\tfrac{\alpha}{\beta} x + \tfrac{\beta}{\alpha} y + z\right) + \Delta : \left(\tfrac{\gamma}{\delta} x + \tfrac{\delta}{\gamma} y + z\right)$$
$$+ \Sigma : \left(\tfrac{\varepsilon}{\zeta} x + \tfrac{\zeta}{\varepsilon} y + z\right) + \text{etc.}$$

ubi pro α, β, γ, δ, etc. numeros quoscunque accipere licet: quamvis autem infinitae hujusmodi formulae diversae conjunguntur, tamen integrale nonnisi pro particulari haberi potest. Ex quo intelligitur integrationem completam istius aequationis $\left(\frac{\partial\partial v}{\partial x \partial y}\right) = \left(\frac{\partial\partial v}{\partial z^2}\right)$ maximi esse momenti, methodumque eo perveniendi fines analyseos non mediocriter esse prolaturam. Aequationes autem homogeneae tertii gradus multo majorem restrictionem exigunt, ut integratio completa hoc modo succedat; uti sequenti problemate ostendetur.

Problema 88.

505. Aequationum homogenearum tertii gradus eos casus definire, quibus integrale completum per formam assumtam exhiberi, seu ad formam aequationum homogenearum primi gradus reduci potest.

Solutio.

In aequatione homogenea tertii gradus fingatur contineri haec primi gradus

$$a \left(\frac{\partial v}{\partial x}\right) + b \left(\frac{\partial v}{\partial y}\right) + c \left(\frac{\partial v}{\partial z}\right) = 0,$$

quae ut satisfaciat aequationi tertii gradus

$$\left.\begin{aligned} A \left(\frac{\partial^3 v}{\partial x^3}\right) + B \left(\frac{\partial^3 v}{\partial y^3}\right) + C \left(\frac{\partial^3 v}{\partial x^3}\right) + D \left(\frac{\partial^3 v}{\partial x^2 \partial y}\right) + E \left(\frac{\partial^3 v}{\partial x \partial y^2}\right) \\ + F \left(\frac{\partial^3 v}{\partial x^2 \partial z}\right) + G \left(\frac{\partial^3 v}{\partial x \partial z^2}\right) \\ + H \left(\frac{\partial^3 v}{\partial y^2 \partial z}\right) + I \left(\frac{\partial^3 v}{\partial y \partial z^2}\right) \\ + K \left(\frac{\partial^3 v}{\partial x \partial y \partial z}\right) \end{aligned}\right\} = 0,$$

necesse est ut expressio haec algebraica

$$Ax^3 + By^3 + Cz^3 + Dxxy + Fxxz + Hyyz + Kxyz$$
$$+ Exyy + Gxzz + Iyzz$$

factorem habeat $ax + by + cz$, nisi autem alter factor denuo in duos simplices sit resolubilis, ad aequationem homogeneam secundi gradus referetur, quae solutionem respuit. Quare ut integratio completa succedat necesse est, istam expressionem tribus constare factoribus simplicibus, qui sint

$$(ax + by + cz)(fx + gy + hz)(kx + my + nz),$$

hincque aequatioins generalis coëfficientes ita se habebunt

$$A = afk,\quad D = afm + agk + bfk,\quad H = bgn + bhm + cgm,$$
$$B = bgm,\quad E = agm + bfm + bgk,\quad I = bhn + cgn + chm,$$
$$C = chn,\quad F = afn + ahk + cfk,\quad K = agn + ahm + bfn$$
$$G = ahn + cfn + chk,\quad + bhk + cfm + cgk,$$

ac tum integrale completum erit

$$v = \Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{f} - \frac{z}{h} \text{ et } \frac{y}{g} - \frac{z}{h}\right)$$
$$+ \Sigma : \left(\frac{x}{k} - \frac{z}{n} \text{ et } \frac{y}{m} - \frac{z}{n}\right),$$

quilibet scilicet factor simplex praebet functionem arbitrariam duarum variabilium.

Corollarium 1.

506. In qualibet harum functionum variabiles x, y, z inter se permutare licet; quin etiam quaelibet quasi ex tribus variabilibus conflata spectari potest, prima nempe ex his

$$\frac{x}{a} - \frac{y}{b},\quad \frac{y}{b} - \frac{z}{c} \text{ et } \frac{z}{c} - \frac{x}{a},$$

similique modo de caeteris.

Corollarium 2.

507. Si duo factores fuerint aequales $f = a$, $g = b$, $h = c$, quo casu duae priores functiones in unam coalescerent, earum loco scribi debet

$$x\Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right);$$

at si omnes tres fuerint aequales, ut insuper sit

$$k = a,\quad m = b,\quad n = c,$$

integrale completum erit

$$\begin{aligned} v = xx\,\Gamma &: \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) \\ + x\,\Delta &: \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) \\ + \Sigma &: \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right). \end{aligned}$$

Corollarium 3.

508. Quemadmodum hic duas priores partes per xx et x multiplicavimus, ita eas quoque per yy et y item zz et z multiplicare possemus, perinde enim est quanam variabili hic utamur, dum ne sit ea, quae forte sola post signum functionis occurit, scilicet si esset $a = 0$, et functiones quantitatum x et $\frac{y}{b} - \frac{z}{c}$ capi debeant, tum multiplicatores xx et x excludi deberent.

Scholion 1.

509. Simili modo patet aequationes homogeneas quarti gradus hac methodo resolvi non posse, nisi in quatuor ejusmodi aequationes simplices resolvi, et quasi earum producta spectari queant. Etsi enim hic revera nulla resolutio in factores locum habeat, tamen ex allatis exemplis clare perspicitur, quemadmodum ex aequatione differentiali homogenea cujuscunque gradus expressio algebraica ejusdem gradus ternas variabiles x, y, z involvens debeat formari;

quae si in factores simplices formae $ax + by + cz$ resolvi queat, simul inde aequationis differentialis integrale completum facile exhibebitur, cum quilibet factor functionem duarum variabilium suppeditet, integralis partem constituentem; ita ut etiam haec pars seorsim sumta aequationi differentiali satisfaciat et pro integrali particulari haberi possit. At si illa expressio algebraica ita fuerit comparata, ut factores quidem habeat simplices sed non tot, quot dimensiones, singuli quidem integralia particularia praebebunt, quae autem junctim sumta non integrale completum suppeditabunt. Veluti si proponatur haec aequatio differentialis tertii gradus

$$a\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right) + b\left(\frac{\partial^3 v}{\partial x \partial y^2}\right) - a\left(\frac{\partial^3 v}{\partial x \partial z^2}\right) - b\left(\frac{\partial^3 v}{\partial y \partial z^2}\right) = 0,$$

quia forma algebraica

$$axxy + bxyy - axzz - byzz$$

factorem habet simplicem $ax + by$, illi utique satisfaciet valor $v = \Gamma : (\frac{x}{a} - \frac{y}{b}$ et $z)$, pro integrali autem completo adhuc desunt duae functiones arbitrariae, integrale completum hujus aequationis $\left(\frac{\partial\partial v}{\partial x \partial y \partial z}\right) - \left(\frac{\partial\partial v}{\partial z^2}\right) = 0$ continentes, ex qua quippe alter factor $xy - zz$ illius expressionis nascitur. Quoties ergo hae expressiones algebraicae ex aequationibus differentialibus homogeneis altiorum graduum formatae resolutionem in factores, etsi non simplices, admittant; hinc saltem discimus, quomodo earum integratio ad aequationes inferiorum graduum revocari possit, quod in hujusmodi arduis investigationibus sine dubio maximi est momenti.

Scholion 2.

510. Haec sunt quae de functionibus trium variabilium ex data quadam differentialium relatione iuvestigandis proferre potui,

in quibus utique nonnisi prima elementa hujus scientiae continentur, quorum ulterior evolutio sagacitati Geometrarum summo studio est commendanda. Tantum enim abest, ut hae speculationes pro sterilibus sint habendae, ut potius pleraque, quae adhuc in Theoria motus fluidorum desiderantur, ad has Analyseos partes sublimiores sint referenda; quarum propterea utilitas neutiquam parti priori calculi integralis postponenda videtur. Eo magis autem hae partes posteriores excoli merentur, quod Theoria fluidorum adeo circa functiones quatuor variabilium versetur, quarum naturam ex aequationibus differentialibus secundi gradus investigari oportet, quam partem ob penuriam materiae ne attingere quidem volui. In hac autem Theoria resolutio hujus aequationis

$$\left(\frac{\partial\partial v}{\partial t^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \left(\frac{\partial\partial v}{\partial y^2}\right) + \left(\frac{\partial\partial v}{\partial z^2}\right)$$

maxime est momenti, ubi litterae x, y, z ternas coordinatas, t vero tempus elapsum exprimunt, harumque quatuor variabilium functio quaeritur, quae loco v substituta illi aequationi satisfaciat. Ex hactenus autem allatis facile colligitur, integrale completum hujus aequationis duas complecti debere functiones arbitrarias, quarum utraque sit functio trium variabilium, aliasque solutiones omnes minus late patentes pro incompletis esse habendas. Facili autem negotio innumeras solutiones particulares exhibere licet, veluti si ponamus

$$v = \Gamma : (\alpha x + \beta y + \gamma z + \delta t),$$

reperitur

$$\delta\delta = \alpha\alpha + \beta\beta + \gamma\gamma,$$

quod cum infinitis modis fieri possit, infinitae hujusmodi functiones additae valorem idoneum pro v exhibebunt. Deinde etiam satisfaciunt isti valores

$$v = \frac{\Gamma : [t \pm \sqrt{(xx + yy + zz)}]}{\sqrt{(xx + yy + zz)}},$$

$$v = \frac{\Gamma : [x \pm \sqrt{(tt - yy - zz)}]}{\sqrt{(tt - yy - zz)}},$$

$$v = \frac{\Gamma : [y \pm \surd (tt - xx - zz)]}{\surd (tt - xx - zz)},$$

$$v = \frac{\Gamma : [z \pm \surd (tt - xx - yy)]}{\surd (tt - xx - yy)},$$

quorum quidem investigatio jam est difficilior. Cum autem hae functiones tantum sint unius variabilis, integralia maxime particularia exhibent, quae adeo etiamnum forent particularia, si pro v functiones binarum variabilium haberentur, quales autem ne suspicari quidem licet. Quare cum integrale completum duas adeo functiones arbitrarias trium variabilium complecti debeat, facile intelligitur quantopere adhuc ab hoc scopo simus remoti.

APPENDIX
DE
CALCULO
VARIATIONUM.

48 *

CAPUT I.

DE
CALCULO VARIATIONUM IN GENERE.

Definitio 1.

1.

Relatio inter binas variabiles variari dicitur, si valor, quo altera inde per alteram determinatur, incremento infinite parvo augeri concipiatur, quod incrementum variationem ejus quantitatis, cui adjicitur, vocabimus.

Explicatio.

2. Primum ergo hic consideratur relatio inter binas variabiles x et y quaecunque, aequatione quacunque inter easdem expressa, qua pro singulis valoribus ipsi x tributis valores ipsius y convenientes determinentur, tum vero singuli valores ipsius y particulis infinite parvis utcunque augeri concipiantur, ita ut hi valores variati a veris, quos ex relatione proposita sortiuntur, infinite parum discrepent, atque hoc modo relatio illa inter x et y variari dicitur, simulque particulae illae infinite parvae valoribus veris ipsius y adjunctae variationes appellentur. Imprimis autem hic notandum est has variationes, quibus singuli valores ipsius y augeri concipiuntur, neque inter se statui aequales, neque ullo modo a se in-

vicem pendentes, sed ita arbitrio nostro permitti, ut omnes praeter unam vel aliquas certis valoribus ipsius y respondentes plane ut nullas spectare liceat. Nulli scilicet legi hae variationes adstrictae sunt concipiendae neque relatio inter x et y data ullam determinationem in istas variationes inferre est censenda, quas ut prorsus arbitrarias spectare oportet.

Corollarium 1.

3. Hinc patet variationes toto coelo differre a differentialibus, etiamsi utraque sint infinite parva ideoque plane evanescant, variatio enim afficit eundem valorem ipsius y, eidem valori ipsius x convenientem, dum ejus differentiale ∂y simul sequentem valorem $x + \partial x$ respicit.

Corollarium 2.

4. Si enim ex relatione inter x et y proposita ipsi x conveniat y, ipsi $x + \partial x$ vero valor ipsius y conveniens ponatur y', tum est $\partial y = y' - y$; at variatio ipsius y neutiquam pendet a valore sequente y', quin potius utrique y et y' pro lubitu suam variationem seorsim tribuere licet.

Scholion.

5. Haec variationum idea quae per se tam nimis vaga quam sterilis videri queat, maxime illustrabitur, si ejus originem et quo pacto ad eam est perventum, accuratius exposuerimus. Perduxit autem eo potissimum quaestio de curvis inveniendis, quae certa quadam maximi minimive proprietate sint praeditae, unde ne rem in genere considerando obscuritas offundatur, problema contemplemur, quo linea curva quaeritur, super qua grave delabens e dato puncto citissime ad aliud punctum datum descendat. Atque hic quidem ex natura maximorum et minimorum statim constat,

curvam ita debere esse comparatam, ut si ejus loco alia curva quaecunque infinite parum ab illa discrepans substituatur, tempus descensus super ea idem prorsus sit futurum. Solutionem ergo ita institui oportet, ut dum curva quaesita tanquam data spectatur, calculus quoque ad aliam curvam infinite parum ab ea discrepantem accommodetur, indeque discrimen quod in temporis expressionem redundat, supputetur; tum enim hoc ipsum discrimen nihilo aequale positum naturam curvae quaesitae declarabit. Curvae autem istae infinite parum a quaesita discrepante commodissime ita considerantur, ut applicatae singulis abscissis respondentes particulis infinite parvis vel augeantur vel minuantur, hoc est, ut variationes recipere concipiantur. Vulgo quidem sufficit hujusmodi variationem in unica applicata constituisse, nihil autem impedit, quominus pluribus atque adéo omnibus applicatis tales variationes assignentur cum semper ad eandem solutionem perduci sit necesse. Hoc autem modo non solum vis methodi multo luculentius illustratur, sed etiam inde solutiones quaestionum hujus generis pleniores obtinentur, unde etiam quaestiones ad alias conditiones spectantes enucleare licet. Quam ob causam omnino necessarium videtur, ut calculus variationum in amplissima extensione, cujus quidem est capax, pertractetur.

Definitio 2.

6. *Pro data relatione inter binas variabiles quantitates utraque earum variari dicitur, si utraque seorsim incremento infinite parvo augeri concipiatur; unde patet quomodo intelligendum sit, si utrique variabili sua tribuatur variatio.*

Explicatio.

7. Si proposita sit aequatio quaecunque inter binas variabiles x et y, qua earum relatio mutua exprimitur, haec relatio per

definitionem duplici modo variari potest, altero quo manentibus valoribus x, singulis y variatio tribuitur, altero vero quo manentibus valoribus y, singuli x variari concipiuntur. Nihil igitur prohibet, quo minus utraque variabilis simul suas variationes recipere intelligatur, quas adeo ita capere licet, ut nullo plane nexu inter se cohaereant, duplex ergo hic variatio consideratur, cum in definitione prima unica tantum sit admissa. Rem autem hic ita generaliter contemplamur, ut neutra variatio ulli legi sit adstricta, neque etiam variationes ipsius y ullo modo a variationibus ipsius x pendeant.

Corollarium 1.

8. Ex casu ergo quo duplex variatio statuitur, casus prior tanquam species nascitur, si variationes alterius variabilis plane rejiciantur, unde manifestum est casum definitionis secundae in se complecti casum primae.

Corollarium 2.

9. Hinc magis elucet, quemadmodum data relatio inter binas variabiles infinitis modis variari possit, simulque intelligitur, quoniam has variationes nulli legi adstrictas assumimus, omnes omnino illius relationis variationes possibiles hac ratione indicari.

Scholion 1.

10. Variationes quidem alterutri tantum variabili inductae jam omnes variationes possibiles, quae in propositam relationem inter binas variabiles cadere possunt, comprehendunt, ut superfluum videri possit calculum ad duplicem variationem accommodari, verum si indolem rei, usumque cui destinatur, attentius contemplemur, duplicis variationis consideratio neutiquam supervacanea deprehendetur, id quod per Geometriam evidentissime sequentem in modum illustrabitur. Cum relatio quaecunque inter binas variabiles distinctissime per lineam curvam in plano descriptam repraesente-

tur, sit A Y M linea curva, aequatione inter coordinatas $AX = x$ et $XY = y$ definita, quae ergo datam illam relationem exhibeat; jam igitur quaelibet linea curva alia Aym ab illa infinite parum discrepans relationem illam variatam repraesentabit, quae quomodocunque se habeat, semper ita considerari potest, ut eidem abscissae $AX = x$ conveniat applicata variata Xv, existente particula Yv ejus variatione, quae consideratio quoque pro plerisque circa maxima et minima prolatis quaestionibus sufficit, ubi adeo curva AM in nonnullis tantum elementis variari solet concipi. At si quaestio ita sit comparata ut inter omnes curvas, quas a dato puncto A ad datam quampiam curvam CD usque ducere licet, ea definiatur AYM cui maximi minimive proprietas quaedam conveniat, tum eadem proprietas in aliam quamcunque curvam proximam Aym etiam in alio lineae CD puncto m terminatam aeque competere debet, sicque pro ultimo curvae quaesitae puncto M tam abscissa AP quam applicata PM variationem recipere est censenda, et hujusmodi quidem, quae naturae lineae CD sit consentanea. Quo igitur calculus ad talem variationem ultimo elemento inductam accommodari queat, omnino necesse est, ut pro singulis curvae AM punctis intermediis Y generalissime tam abscissae $AX = x$ quam applicatae $XY = y$ variationes tribuantur quaecunque, illiusque variatio statuatur particula Xx hujus vero $= xy - XY$, ex quo indoles simulque usus hujusmodi duplicis variationis clarissime perspicitur.

Scholion 2.

11. Quemadmodum consideratio ultimi puncti curvae investigandae nobis hanc insignem dilucidationem suppeditavit, ita etiam subinde primo puncto variationem tribui oportet. Veluti si inter omnes lineas, quas a data quadam curva AB ad aliam quandam itidem datam CD ductas concipere licet ea sit quaerenda, quae

maximi minimive cujuspiam proprietate sit praedita, tum multo magis erit necessarium tam singulis abscissis AX quam applicatis XY variationes quascunque nulla lege adstrictas in calculo assignari, ut deinceps tam ad initii G curvae quaesitae, quam ejus finis M variationem transferri possint. Quanquam autem haec illustratio ex Geometria est desumta, tamen facile intelligitur ideam variationum inde petitam multo latius patere, atque in Analysi absoluta summo usu non esse carituram. Celeberrimus autem de la Grange, acutissimus Geometra Taurinensis, cui primas speculationes de calculo variationum acceptas referre debemus, hanc methodum adeo ingeniosissime transtulit ad lineas non continuas veluti ad polygonorum genus referendas, in quo negotio hae duplices variationes ipsi summam praestiterunt utilitatem.

Definitio 3.

12. *Relatio inter tres variabiles, duabus aequationibus determinata, variari dicitur, si earum vel una, vel duae, vel omnes tres particulis infinite parvis augeantur, quae earum variationes appellantur.*

Explicatio.

13. Cum tres proponantur variabiles quantitates veluti x, y et z, inter quas duae aequationes dari concipiuntur, ex unaquaque earum binas reliquas determinare licet, ita ut tam y quam z tanquam functio ipsius x spectari possit. Hoc autem modo definiri solet linea curva non in eodem plano descripta, dum singula ejus puncta per has ternas coordinatas x, y et z more solito assignantur. Quodsi jam talis curva alia quacunque sibi proxima comitetur, ut differentia sit infinite parva, haec nova curva propositae erit variata, ac relatio illa inter ternas variabiles x, y, z variata ejus na-

turam exprimere est concipienda. Ex quo prout bina puncta proxima alterum in ipsa curva proposita, alterum in variata cominus assumtum inter se comparantur, fieri potest ut pro variata vel omnes tres coordinatae prodeant diversae, vel duae tantum, vel saltem unica, harumque differentiae a coordinatis principalis curvae earum variationes repraesentabunt; quas autem hic ita generalissime contemplari convenit, ut ad omnes omnino curvas proximas extendantur, sive eae per totum tractum a curva proposita fuerint diversae, sive tantum in quibusdam portionibus ab ea aberrent; ita ut etiam lineae non continuae dummodo principali sint proximae, hinc non excludantur. Neque enim hae curvae variatae ulli continuitatis legi sunt adstringendae, ut omnes plane curvas possibiles infinite parum a principali aberrantes in se complectantur.

Corollarium 1.

14. Cum puncto ergo quocunque curvae propositae seu principalis comparatur qunctum puodpiam curvae variatae infinite parum ab illo dissitum, et hincque coordinatarum variationes definiri intelliguntur.

Corollarium 2.

15. Quia porro ex assumta variabili una x, binae reliquae y et z ideoque punctum curvae propositae determinatur, etiam variationes singularum coordinatarum tanquam functiones ipsius x spectare licet, dummodo earum quantitas ut infinite parva spectetur.

Corollarium 3.

16. Tres ergo quascunque functiones ipsius x utcunque inter se diversas concipere licet, quae per factores infinite parvos multiplicatae idoneae erunt ad ternas variationes coordinatarum re-

praesentandas. Quod idem de ternis quibuscunque variabilibus est tenendum, etiamsi non ad geometriam referantur.

Corollarium 4.

17. Simili quoque modo si relatio tantum inter duas variabiles proponatur, earum variationes tanquam functiones alterius variabilis spectari possunt, modo sint infinite parvae, sed quod eodem redit, per quantitatem infinite parvam multiplicatae.

Scholion 1.

18. Consideratio autem geometrica maxime est ideonea ad has speculationes illustrandas, quae in genere consideratae nimis abstractae atque etiam vagae videri queant. Casus igitur trium variabilium quarum relatio duabus aequationibus definiri assumitur, luculentissime per curvam non in eodem plano descriptam explicatur, dum illis variabilibus ternae coordinatae designantur. Quodsi enim de hujusmodi curvis quaestio instituatur, ut inter eas definiatur ea quae maximi minimive proprietate quapiam sit praedita, necesse est ut eadem proprietas in omnes alias curvas ab ea infinite parum aberrantes aeque competat, id quod ex variationibus debite in calculum introductis est dijudicandum. Cuinam autem usui summa generalitas in variationibus hic stabilita sit futura, inde intelligere licet, si loco duarum curvarum A B et C D datae sint duae quaecunque superficies a quarum illa ad hanc ejusmodi lineam curvam duci oporteat, quae maximi minimive quapiam gaudeat proprietate. Tum enim ternarum coordinatarum variationes ita generales considerari oportet, ut curvae quaesitae puncto ad initium in superficiem AB translato, variationes ibi ad eandem superficiem accommodari possint, idque simili modo in fine ad superficiem C D fieri queat. Ex quo perspicuum est, in genere tres variationes in calculum introduci debere, ut eas tam pro initio quam pro fine

curvae investigandae ad superficies terminatrices transferre liceat, quippe quarum indoles in utroque termino relationem mutuam inter variationes determinabit.

Scholion 2.

19. Quemadmodum hic tres variabiles sumus contemplati, quarum relatio duabus aequationibus determinatur, ita etiam calculus variabilium ad quatuor pluresve extendi potest, siquidem relatio per tot aequationes exprimatur ut per unicam variabilem reliquae omnes determinationem suam nanciscantur, etiamsi hujus casus illustratio non amplius ex Geometria tribus tantum dimensionibus inclusa peti queat, nisi forte tempus in subsidium vocare velimus, fluvium continuum a superficie AB ad superficiem CD profluentem sed temporis lapsu jugiter immutatum considerantes, ita ut tum etiam temporis momentum sit assignandum, quo quaepiam fluvii vena a superficie AB ad superficiem CD porrecta maximi vel minimi proprietate quadam sit praedita. Ad quas variabiles si insuper celeritatis mutabilitatem adjiciamus, haec majori variationum numero illustrando inservire poterunt. Imprimis autem hinc intelligitur, etiamsi omnes variabiles per unicam determinari assumantur, rationem investigationis tamen ab ea ubi duae tantum variabiles admittuntur, maxime discrepare, propterea quod singulis suae variationes a reliquis non pendentes tribui debent; neque enim inde, quod inter variabiles ipsas certa quaedam relatio agnoscitur, ideo quoque earum variationes ulli relationi adstrictae sunt censendae. Veluti ex casu ante allato manifestum est, ubi curva inter binas superficies AB et CD porrecta et certa maximi minimive proprietate praedita utique ita est in se determinata, ut sumta coordinatarum una, binae reliquae determinentur; nihilo vero minus curvae variatae omnes quae in omnes plagas ab illa deflectere possunt, pro singulis coordinatis recipiunt variationes neutiquam a se invicem

pendentes, solo initio ac fine excepto, ubi eas ad datas superficies accommodari oportet.

Definitio 4.

20. *Relatio inter ternas variabiles unica aequatione definita, ut una earum aequetur functioni binarum reliquarum, variari dicitur, si vel una vel omnes tres illae variabiles particulis infinite parvis augeantur, quae earum variationes vocantur.*

Explicatio.

21. Quoniam hic relatio inter ternas variabiles unica aequatione definiri ponitur, duabus pro arbitrio sumtis tertia demum determinatur, ita ut pro functione duarum variabilium sit habenda. Ea ergo relatione non quaedam linea curva, si rem ad figuras transferre velimus, indicatur, sed tota quaedam superficies, cujus natura aequatione inter ternas coordinatas exprimitur, ex quo intelligitur, eadem relatione variata aliam superficiem ab illa infinite parum dissidentem repraesentari, quae variatio ita latissime patere debet, ut variatio vel tantum ad quampiam superficiei portionem restringi vel per totam extendi possit. Prout igitur cum quovis superficiei datae puncto aliud punctum superficiei variatae illi quidem proximum comparatur, fieri potest, ut non solum trium coordinatarum una sed etiam duae vel adeo omnes tres varientur; unde quo tractatio in omni amplitudine instituatur, conveniet statim singulis coordinatis suas tribui variationes, quas propterea ita comparatas esse oportet, ut tanquam functiones binarum variabilium spectari possint, cum binis demum determinatis superficiei punctum determinetur.

Corollarium 2.

22. Si igitur tres variabiles seu coordinatae sint x, y et z, quemadmodum ex relatione binis x et y pro lubitu valores tribuere

licet, unde z valorem determinatum obtineat, itidem variatio ipsius z ab utraque illarum x et y pendere censenda est, quandoquidem sive alterutra sive ambae mutentur, alia variatio ipsius z resultare debet.

Corollarium 2.

23. Quod hic de variatione unius z observatum est, perinde de binis reliquis est intelligendum,. ita ut singularum variationes sint tanquam functiones binarum variabilium considerandae; quoniam vero inter x et y et z aequatio datur, perinde est, quarumnam binarum functiones concipiantur, quia functio ipsarum y et z per aequationem ad functionem ipsarum x et y revocari potest, si scilicet loco z suus valor per x et y expressus substituatur.

Scholion 1.

24. Hac variationum institutione erit utendum, si superficies fuerit investiganda, quae maximi minimive quapiam proprietate sit praedita, quandoquidem calculum tum ita instrui oportet ut eadem proprietas in superficies illi proximas ac variatas aeque competat. Deinde cum in curvis maximi minimive proprietate praeditis amborum terminorum ratio praescribi soleat, ut vel in datis punctis, vel ad datas lineas curvas, vel adeo superficies terminentur, similis conditio hic est admittenda, ut superficies quaerenda circumquaque definiatur, vel data quadam superficie circumscribatur; cujus posterioris conditionis ut ratio haberi possit, omnino necesse est, ut omnibus tribus coordinatis variationes generalissimae a se invicem neutiquam pendentes tribuantur, quo eae deinceps in extrema ora ad naturam superficiei terminantis, accommodari queant. Hic quidem fatendum est, methodum maximorum et minimorum vix adhuc ad hujusmodi investigationes esse promotam, tantasque difficultates hic occurrere, ad quas superandas multo majora

Analyseos incrementa requiri videntur. Verum ob hanc ipsam causam eo magis erit enitendum ut principia hujus methodi, quae calculo variationum continentur, solide stabiliantur, simulque clare ac distincte proponantur.

Scholion 2.

25. Vix opus esse arbitror hic animadvertere, istum calculum simili modo ad plures tribus variabiles amplificari posse, etiamsi quaestiones geometricae non amplius dilucidationem suppeditent; ipsa enim Analysis non uti Geometria certo dimensionum numero limitari est censenda. Quando autem plures variabiles considerantur, ante omnia perpendi convenit, utrum earum relatio mutua unica tantum aequatione exprimatur, an pluribus? quae tot esse possunt, ut multitudo unitate tantum a numero variabilium deficiat, quo casu omnes tanquam functiones unius spectare licet. Sin autem paucioribus aequationibus constet relatio, singulae variabiles erunt functiones duarum pluriumve variabilium, et quolibet quoque casu variationes singulis tributae tanquam functiones totidem variabilium tractari debent, siquidem hunc calculum generalissime expedire velimus.

Definitio 5.

26. *Calculus variationum est methodus inveniendi variationem, quam recipit expressio ex quotcunque variabilibus utcunque conflata, dum variabilibus vel omnibus vel aliquibus variationes tribuantur.*

Explicatio.

27. In hac definitione nulla fit mentio relationis, quam hactenus inter variabiles dari assumsimus, cum enim hic calculus potissimum in hac ipsa relatione investiganda sit occupatus, quae scilicet maximi minimive proprietate sit praedita, quamdiu ea adhuc est incognita, ejus rationem in calculo neutiquam habere licet, sed potius eum ita tractari convenit, quasi variabiles nulla plane

relatione inter se essent connexae. Calculum igitur ita instrui convenit, ut si singulis variabilibus, quae in calculum ingrediuntur, variationes tribuantur quaecunque, omnis generis expressionum, quae utcunque ex iis fuerint conflatae, variationes inde oriundae investigari doceantur, quibus in genere inventis tum demum ejusmodi quaestiones evolvendae occurrunt, qualem relationem inter variabiles statui oporteat, ut variatio illa inventa sit vel nulla, uti in investigatione maximorum seu minimorum usu venit, vel alio certo quodam modo sit comparata, prout natura quaestionum exegerit. Hoc modo si istius calculi praecepta tradantur, nihil impedit, quo minus etiam ejusmodi quaestiones tractentur, in quibus statim relatio quaedam inter variabiles tanquam data assumitur ac certae cujusdam expressionis ex iis formatae variatio ex variabilium variationibus nata desideratur. Ex quo intelligitur, huc calculum ad quaestiones plurimas diversissimi generis accomodari posse.

Corollarium 1.

28. Quaestiones ergo in hoc calculo tractandae huc redeunt, ut proposita expressione qnacunque ex quotcunque variabilibus utcunque conflata, ejus incrementum definiatur, si singulae variabiles suis variationibus augeantur.

Corollarium 2.

29. Similis igitur omnino est calculus variationum calculo differentiali, dum in utroque variabilibus incrementa infinite parva tribuuntur. Quatenus autem uti jam observavimus, variationes a differentialibus discrepant, adeoque simul cum iis consistere possunt, catenus summum discrimen inter utrumque calculum est agnoscendum.

Scholion.

30. Ex observationibus supra allatis discrimen hoc maxime

fit manifestum, ubi enim calculus refertur ad lineam curvam, quam cum alia sibi proxima comparari oportet, per differentialia a puncto quovis curvae ad alia puncta ejusdem curvae progredimur, quando autem ab hac curva ad alteram sibi proximam transilimus, transitus quatenus est infinite parvus, fit per variationes. Idem evenit in superficiebus ad alias sibi proximas relatis, ubi differentialia in eadem superficie concipiuntur, variationibus vero ab una in alteram transilitur. Eadem omnino est ratio, si res analytice consideretur sine ullo respectu ad figuras geometricas, ubi semper variationes quantitatem variabilium a suis differentialibus sollicite distingui oportet, quem in finem variationes signo diverso indicari conveniet.

Hypothesis.

31. *Variationem cujusque quantitatis variabilis littera* δ *eidem quantitati praefixa in posterum designabimus. ita ut* δx, δy, δz *designent variationes quantitatum* x, y, z; *ac si* V *fuerit expressio quaecunque ex iis conflata, ejus variatio hoc modo* δV *nobis indicabitur.*

Corollarium 1.

32. Significat ergo δx incrementum illud infinite parvum, quo quantitas x augeri concipitur, ut ejusdem valor variatus prodeat; ex quo vicissim intelligitur valorem variatum ipsius x fore $x + \delta x$.

Corollarium 2.

33. Quatenus ergo expressio V ex variabilibus x, y et z conflatur, si earum loco scribantur valores variati

$$x + \delta x, \quad y + \delta y \quad \text{et} \quad z + \delta z,$$

atque a valore hoc modo pro V resultante subtrahatur ipsa V residuum erit variatio δV.

Corollarium 3.

34. Hactenus ergo omnia perinde se habent atque in calculo differentiali, ac si V fuerit functio quaecunque ipsarum x, y et z, sumto ejus differentiali more solito tantum ubique loco ∂ scribatur δ, et habebitur ejus variatio δV.

Scholion 1.

35. Quoties ergo V est functio quaecunque quantitatum variabilium x, y, z, ejus variatio iisdem regulis inde elicitur ac differentiale ejus, ex quo calculus variationum prorsus cum calculo differentiali congruere videri posset, cum sola signi diversitas levis sit momenti. Verum probe perpendendum est, hic non omnes quantitates, quarum variationes requiruntur, in genere functionum comprehendi posse; quamobrem etiam in definitione vocabulo expressionis sum usus, cui longe ampliorem significatum attribuo. Quatenus enim ad relationem mutuam variabilium respicere non licet, quia est incognita, eatenus ejusmodi expressiones seu formulae, in quas variabilium differentialia atque etiam integralia ingrediuntur, non amplius tanquam merae functiones variabilium spectari possunt, ac formularum tam differentialium quam integralium variatio peculiaria praecepta postulat; sicque totum negotium huc redit, ut quemadmodum formularum utriusque generis variationes investigari conveniat, doceamus, ex quo tractatio nostra evadit bipartita.

Scholion 2.

36. In ipsa autem tractatione maximum exoritur discrimen ex numero variabilium, qui si binarium superet, vix adhuc perspicitur, quomodo calculus sit expediendus. Cum enim pluribus introductis variabilibus, etiam differentialium consideratio longe aliter expendatur, dum plerumque binarum tantum differentialia ita inter se comparari solet, quasi reliquae variabiles manerent constantes, simi-

lis quoque ratio in variationibus erit habenda in quo etiamnunc tantae difficultates occurrunt, ut vix pateat quomodo eas superare liceat; ante omnia certe prima hujus calculi principia accuratissime evolvi erit necesse, ut ex intima rei natura calculi praecepta repetantur, in quo plerumque summae difficultates offendi solent. Primum igitur hunc calculum ad duas tantum variabiles accommodatum, quemadmodum is quidem adhuc tractari est solitus, explicare conabor, variationes tam formularum differentialium quam integralium investigaturus, tum vero si quid lucis ex ipsa hac tractatione affulserit, quoque ad tres pluresve variabiles contemplandas progrediar.

CAPUT II.

DE VARIATIONE FORMULARUM DIFFERENTIALIUM DUAS VARIABILES INVOLVENTIUM.

Theorema 1.

37.

Variatio differentialis semper aequalis est differentiali variationis, seu est $\delta \partial V = \partial \delta V$, *quaecunque fuerit quantitas* V, *quae dum per differentialia crescit, etiam variationem recipit.*

Demonstratio.

Quantitas variabilis V spectari potest tanquam applicata curvae cujuspiam, quae suis differentialibus per eandem curvam progrediatur, suis variationibus vero in aliam curvam illi proximam transiliat. Dum autem in ejusdem curvae punctum proximum promovetur, fit ejus valor $= V + \partial V$, qui sit $= V'$, ideoque $\partial V = V' - V$; ex quo variatio ipsius ∂V hoc est $\delta \partial V$ erit $= \delta V' - \delta V$. Verum $\delta V'$ est valor proximus, in quem δV suo differentiali auctum abit, ita ut sit $\delta V' = \delta V + \partial \delta V$, seu $\delta V' - \delta V = \partial \delta V$; unde evidens est fore $\delta \partial V = \partial \delta V$, seu variationem differentialis esse aequalem differentiali variationis, prorsus uti Theorema affirmat.

Corollarium 1.

38. Hinc variatio differentialis secundi $\partial \partial V$ ita definitur,

ut sit $\delta\partial\partial V = \partial\delta . \partial V$, at cum sit $\delta\partial V = \partial\delta V$, aequalitas erit inter has formulas

$$\delta\partial\partial V = \partial\delta\partial V = \partial\partial\delta V,$$

Corollarium 2.

39. Eodem modo pro differentialibus tertii ordinis erit

$$\delta\partial^3 V = \partial\delta\partial\partial V = \partial\partial\delta\partial V = \partial^3 \delta V,$$

et pro differentialibus quarti ordinis variatio ita se habebit ut sit

$$\delta\partial^4 V = \partial\delta\partial^3 V = \partial\partial\delta\partial\partial V = \partial^3 \delta\partial V = \partial^4\delta V,$$

similique modo pro altioribus gradibus.

Corollarium 3.

40. Si igitur variatio desideretur differentialis cujuscunque gradus, signum variationis δ, ubicunque libuerit, inter signa differentiationis ∂ inseri potest; in ultimo autem loco positum declarat, variationem differentialis cujusvis gradus aequalem esse differentiali ejusdem gradus ipsius variationis.

Corollarium 4.

41. Cum igitur sit $\delta\partial^n V = \partial^n\delta V$, res semper eo reducitur, ut variationis quantitatis V seu ipsius δV differentialia cujusque gradus capi possint; atque in hac reductione praecipua vis hujus novi calculi est constituenda.

Scholion 1.

42. Vis demonstrationis in hoc potissimum est sita, quod
Fig. 3. δV abeat in $\delta V'$, si quantitas V suo differentiali increscit, quod quidem ex natura differentialium per se est manifestum; interim tamen juvabit id per Geometriam illustrasse. Pro curva quacunque E F sint coordinatae $AX = x$ et $XY = y$, in qua si per in-

tervallum infinite parvum YY′ progrediamur, erit in differentialibus

$$AX' = x + \partial x \quad \text{et} \quad X'Y' = y + \partial y,$$

ideoque

$$\partial x = AX' - AX \quad \text{et} \quad \partial y = X'Y' - XY.$$

Nunc concipiamus aliam curvam ef illi proximam, cujus puncta y et y' cum illius punctis Y et Y′ comparentur, ad quae propterea per variationes transitus fiat; ac sumtis simili modo coordinatis erit

$$Ax = x + \delta x \quad \text{et} \quad xy = y + \delta y,$$

ideoque

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY,$$

tum vero erit

$$Ax' = x + \partial x + \delta(x + \partial x) \quad \text{et}$$
$$x'y' = y + \partial y + \delta(y + \partial y),$$

quatenus a puncto Y′ per variationem in punctum y' transilimus. Verum ad idem punctum y' quoque ex puncto y per differentiationem pervenimus, unde colligitur

$$Ax' = x + \delta x + \partial(x + \delta x) \quad \text{et}$$
$$x'y' = y + \delta y + \partial(y + \delta y).$$

His jam valoribus cum illis collatis, prodit

$$x + \partial x + \delta x + \delta\partial x = x + \delta x + \partial x + \partial\delta x \quad \text{et}$$
$$y + \partial y + \delta y + \delta\partial y = y + \delta y + \partial y + \partial\delta y,$$

unde manifesto sequitur fore

$$\delta\partial x = \partial\delta x \quad \text{et} \quad \delta\partial y = \partial\delta y.$$

Quae si attentius consideremus, principium, cui demonstratio innititur, huc redire comperimus, ut si quantitas variabilis primo per differentiationem deinde vero per variationem proferatur, idem proveniat, ac si ordine inverso primo per variationem tum vero per differentiationem promoveretur. Veluti in figura ex puncto Y primo per differentiationem pervenitur in Y′, hinc vero per variationem in y': inverso autem ordine primum ex puncto Y per variationem pervenitur in y, hinc vero per differentiationem in punctum y', idem quod ante.

Scholion 2.

43. Theorema hoc latissime patet, neque enim ad casum duarum variabilium tantum restringitur, sed veritati est etiam consentaneum, quotcunque variabiles in calculum ingrediantur, quandoquidem in demonstratione solius illius variabilis cujus tam differentiale quam variatio consideratur, ratio habetur sine ullo respectu ad
Fig. 4. reliquas variabiles. Ne autem hic ulli dubio locus relinquatur, consideremus superficiem quamcunque, cujus punctum quodvis Z per coordinatas ternas

$$AX = x,\ XY = y,\ \text{et}\ YZ = z$$

definiatur, a quo si ad aliud punctum proximum Z′ in eadem superficie progrediamur, hae coordinatae suis differentialibus increscent. Tum vero aliam quamcunque superficiem concipiamus proximam, cujus puncta z et z' cum illis Z et Z′ conferantur, quod fit per variationem. His positis perspicuum est, duplici modo ad punctum z' perveniri posse, altero per variationem ex puncto Z′ altero per differentiale ex puncto z, sicque fore

$$\begin{aligned} Ax' &= AX' + \delta . AX' = Ax + \partial . Ax, \\ x'y' &= X'Y' + \delta . X'Y' = xy + \partial . xy, \\ y'z' &= Y'Z' + \delta . Y'Z' = yz + \partial . yz, \end{aligned}$$

quod etiam de omnibus aliis quantitatibus variabilibus ad haec puncta referendis valet. Hinc autem luculenter sequitur fore

$$\delta\partial x = \partial\delta x,\ \delta\partial y = \partial\delta y,\ \delta\partial z = \partial\delta z.$$

Scholion 3.

44. Memorabile prorsus est, quod casu differentialium altioris ordinis signum variationis δ pro lubitu inter signa differentiationis ∂ inscribi possit, atque hinc intelligere licet, hanc permutabilitatem locum quoque esse habituram, etiamsi signum variationis δ perinde ac differentiationis ∂ aliquoties repetatur; quod fortasse in aliis speculationibus usu venire posset. Verum in praesenti instituto repetitio variationis δ nullo modo locum habere potest, quoniam lineam vel superficiem tantum cum unica alia sibi proxima comparamus; etsi enim haec generalissime consideratur, ut omnes possibiles itidem proximas in se complectatur, tamen tanquam unica spectatur, neque postquam e principali in proximam transiliverimus, novus transitus in aliam conceditur. Hinc ergo ejusmodi speculationes, quibus variationum variationes essent quaerendae, omnino excluduntur. Vicissim autem hic variationum differentialia cujusque ordinis admitti debent, et cum in formulis differentialibus, quae quidem significatum habent finitum, ratio differentialium tantum spectetur, quae si binae variabiles sint x et y, hujusmodi positionibus

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \text{etc.}$$

ad formas finitas revocari solent, harum quantitatum p, q, r, etc. variationes potissimum assignari necesse est.

Problema 1.

45. Datis binarum variabilium x et y variationibus δx et δy, formulae differentialis $p = \frac{\partial y}{\partial x}$ variationem definire.

Solutio.

Cum sit

$$\delta\partial y = \partial\delta y \quad \text{et} \quad \delta\partial x = \partial\delta x,$$

variatio quaesita δp per notas differentiationis regulas reperitur, dummodo loco signi differentiationis ∂ scribatur signum variationis δ, unde cum oriatur

$$\delta p = \frac{\partial x \delta \partial y - \partial y \delta \partial x}{\partial x^2},$$

erit per conversionem ante demonstratam

$$\delta p = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

ubi cum δx et δy sint variationes ipsarum x et y, hincque $\delta x + \partial \delta x$ et $\delta y + \partial \delta y$ variationes ipsarum $x + \partial x$ et $y + \partial y$, notandum est fore uti jam observavimus

$$\partial \delta x = \delta(x + \partial x) - \delta x \text{ et } \partial \delta y = \delta(y + \partial y) - \delta y.$$

Idem invenitur ex primis principiis, cum enim valor ipsius variatus sit $p + \delta p$, isque prodeat, si loco x et y earum valores variati, qui sunt $x + \delta x$ et $y + \delta y$, substituantur, erit

$$p + \delta p = \frac{\partial (y + \delta y)}{\partial (x + \delta x)} = \frac{\partial y + \partial \delta y}{\partial x + \partial \delta x},$$

unde ob $p = \frac{\partial y}{\partial x}$ fit

$$\delta p = \delta \cdot \frac{\partial y}{\partial x} = \frac{\partial y + \partial \delta y}{\partial x + \partial \delta x} - \frac{\partial y}{\partial x} = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

quoniam in denominatore particula $\partial x \partial \delta x$ prae ∂x^2 evanescit.

Corollarium 1.

46. Si dum per differentialia progredimur, variabiles x et y continuo auctas designemus per x', x'', x''', etc. y', y'', y''', etc. ut sit

$$x' = x + \partial x \text{ et } y' = y + \partial y, \text{ erit}$$
$$\partial \delta x = \delta x' - \delta x \text{ et } \partial \delta y = \delta y' - \delta y,$$

hincque

$$\delta p = \delta \cdot \frac{\partial y}{\partial x} = \frac{\partial x (\delta y' - \delta y) - \partial y (\delta x' - \delta x)}{\partial x^2}.$$

Corollarium 2.

47. Quoniam variationes ambarum variabilium x et y neutiquam a se invicem pendent, sed prorsus arbitrio nostro relinquuntur, si ipsi x nullas tribuamus variationes ut sit

$$\delta x = 0 \text{ et } \delta x' = 0, \text{ erit}$$
$$\delta p = \frac{\partial \delta y}{\partial x} = \frac{\delta y' - \delta y}{\partial x}.$$

Corollarium 3.

48. Si praeterea unicae variabili y variationem δy tribuamus, ut sit $\delta y' = 0$, erit $\delta p = -\frac{\delta y}{\partial x}$, quae hypothesis minime naturae refragatur, quia curvam proximam ita cum principali congruentem assumi licet, ut in unico tantum puncto ab ea discrepet.

Scholion.

49. Vulgo in solutione problematum isoperimetricorum aliorumque ad id genus pertinentium, curva variata ita congruens statui solet, ut tantum in uno quasi elemento discrepet. Ita si quaerenda sit curva EF certa quadam maximi minimive proprietate gaudens, unicum punctum Y in locum proximum y transferri solet, ut curva variata E M y Y′F tantum in intervallo minimo MY′ a quaesita deflectat ita, ut positis Fig. 5.

$$AX = x \text{ et } XY = y,$$

sit pro variata curva

$$Ax = x + \delta x \text{ et } xy = y + \delta y, \text{ seu}$$
$$\delta x = Ax - AX \text{ et } \delta y = xy - XY,$$

pro sequentibus vero punctis, ad quae differentialia ducunt, sit ubique

$$\delta x' = 0, \; \delta y' = 0, \; \delta x'' = 0, \; \delta y'' = 0, \text{ etc.}$$

itemque pro antecedentibus. Quin etiam ad calculi commodum variatio $Xx = \delta x$ nulla sumi solet, ut omnis variatio ad solum elementum δy perducatur, quo casu utique habebitur $\delta p = -\frac{\delta y}{\partial x}$, haecque unica variatio utique sufficit ad problemata hujus generis, quae quidem fuerint tractata, resolvenda. Verum si, uti hic instituimus, haec problemata latius extendimus, ut curva quaesita circa initium et finem certas determinationes recipere queat, utique necessarium est calculum variationum quam generalissime absolvere, atque in omnibus curvae punctis variationes indefinitas coordinatis tribuere. Quod etiam maxime est necessarium, si hujusmodi investigationes ad lineas curvas non continuas accommodare velimus.

Problema 2.

50. Datis binarum variabilium x et y variationibus δx et δy, si ponatur $\partial y = p\partial x$ et $\partial p = q\partial x$, invenire variationem quantitatis q, seu valorem ipsius δq.

Solutio.

Cum sit $q = \frac{\partial p}{\partial x}$, erit pro valore variato

$$q + \delta q = \frac{\partial (p + \delta p)}{\partial (x + \delta x)} = \frac{\partial p + \partial \delta p}{\partial x + \partial \delta x},$$

unde auferendo quantitatem $q = \frac{\partial p}{\partial x}$ relinquitur

$$\delta q = \frac{\partial x \partial \delta p - \partial p \partial \delta x}{\partial x^2},$$

quae variatio ergo etiam ex differentiatione formulae $q = \frac{\partial p}{\partial x}$ resultat, si more consueto differentiatio instituatur, loco vero signi differentialis ∂ scribatur signum variationis δ; ubi quidem meminisse juvabit esse,

$$\delta \partial x = \delta \partial x \quad \text{et} \quad \delta \partial p = \partial \delta p.$$

Supra autem invenimus, ob $p = \frac{\partial y}{\partial x}$ esse

$$\delta p = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

unde porro per consuetam differentiationem valor ipsius $\partial \delta p$ scilicet differentiale ipsius δp colligitur.

Corollarium 1.

51. Cum sit $\frac{\partial y}{\partial x} = p$ et $\frac{\partial p}{\partial x} = q$, erit primo

$$\delta p = \frac{\partial \delta y}{\partial x} - \frac{p \partial \delta x}{\partial x},$$

tum vero

$$\delta q = \frac{\partial \delta p}{\partial x} - \frac{q \partial \delta x}{\partial x}.$$

Pro usu autem futuro praestat hic particulam $\partial \delta p$ relinqui, quam ejus valorem ex praecedente formula erui.

Corollarium 2.

52. Interim tamen cum prior per differentiationem det

$$\partial \delta p = \frac{\partial \partial \delta y}{\partial x} - \frac{\partial \partial x \partial \delta y}{\partial x^2} - \frac{p \partial \partial \delta x}{\partial x} - q \partial \delta x + \frac{p \partial \partial x \partial \delta x}{\partial x^2},$$

hoc valore substituto prodit

$$\delta q = \frac{\partial \partial \delta y}{\partial x^2} - \frac{\partial \partial x \partial \delta y}{\partial x^3} - \frac{p \partial \partial \delta x}{\partial x^2} - \frac{2q \partial \delta x}{\partial x} + \frac{p \partial \partial x \partial \delta x}{\partial x^3}.$$

Corollarium 3.

53. Quod si soli variabili y variationes tribuantur, ut particulae δx et quae inde derivantur evanescant, habebimus

$$\delta p = \frac{\partial \delta y}{\partial x} \text{ et } \delta q = \frac{\partial \delta p}{\partial x^2} = \frac{\partial \partial \delta y}{\partial x^2} - \frac{\partial \partial x \partial \delta y}{\partial x^3},$$

ac si differentiale ∂x constans accipiatur, erit $\delta q = \frac{\partial \partial \delta y}{\partial x^2}$.

Scholion 1.

54. Quo haec facilius intelligantur, consideremus in curva EF, per relationem inter variabiles $AX = x$ et $XY = y$, plura Fig. 5.

puncta Y, Y′, Y″, etc. secundum differentialia continuo promota, ut sit

$$AX = x, \quad AX' = x + \partial x, \quad AX'' = x + 2\partial x + \partial\partial x,$$
$$AX''' = x + 3\partial x + 3\partial\partial x + \partial^3 x,$$
$$XY = y, \quad X'Y' = y + \partial y, \quad X''Y'' = y + 2\partial y + \partial\partial y,$$
$$X'''Y''' = y + 3\partial y + 3\partial\partial y + \partial^3 y,$$

quae differentialia cujusque ordinis indicantes ita brevitatis gratia repraesententur

$$AX = x, \quad AX' = x', \quad AX'' = x'', \quad AX''' = x''', \text{ etc.}$$
$$XY = y, \quad X'Y' = y', \quad X''Y'' = y'', \quad X'''Y''' = y''', \text{ etc.}$$

quibus singulis suae variationes nullo modo a se invicem pendentes tribui concipiantur, ita ut omnes istae variationes

$$\delta x, \quad \delta x', \quad \delta x'', \quad \delta x''', \quad \text{etc.}$$
$$\delta y, \quad \delta y', \quad \delta y'', \quad \delta y''', \quad \text{etc.}$$

a lubitu nostro pendentes tanquam cognitae spectari queant. His jam ita constitutis differentialia cujusque ordinis variationum in hunc modum repraesentabuntur, ut sit

$$\partial\delta x = \delta x' - \delta x, \quad \partial\partial\delta x = \delta x'' - 2\delta x' + \delta x,$$
$$\partial^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x,$$
$$\partial\delta y = \delta y' - \delta y, \quad \partial\partial\delta y = \delta y'' - 2\delta y' + \delta y,$$
$$\partial^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Quodsi jam unicum punctum curvae Y variari sumamus, erit

$$\partial\delta x = -\delta x, \quad \partial\partial\delta x = +\delta x, \quad \partial^3\delta x = -\delta x, \quad \text{etc.}$$
$$\partial\delta y = -\delta y, \quad \partial\partial\delta y = +\delta y, \quad \partial^3\delta y = -\delta y, \quad \text{etc.}$$

hincque

$$\delta p = -\frac{\delta y}{\partial x} + \frac{p\delta x}{\partial x} \quad \text{et}$$
$$\delta q = \frac{\partial y}{\partial x^2} + \frac{\partial\partial x\delta y}{\partial x^3} - \frac{p\delta x}{\partial x^2} + \frac{2q\delta x}{\partial x} - \frac{p\partial\partial x\delta x}{\partial x^3},$$

ubi omissis partibus reliquarum respectu evanescentibus, erit

$$\delta q = \delta y \cdot \frac{1}{\partial x^2} - \delta x \cdot \frac{p}{\partial x^2}.$$

Denique si soli applicatae $XY = y$ variatio tribuatur, habebitur

$$\delta p = -\frac{1}{\partial x}\delta y \quad \text{et} \quad \delta q = \frac{1}{\partial x^2} \cdot \delta y.$$

Scholion 2.

55. Hinc patet si in unico curvae puncto variatio statuatur, insigniter contra recepta differentialium principia impingi, cum variationum differentialia superiora neutiquam prae inferioribus evanescant sed jugiter eundem valorem retineant, atque adeo variationes quantitatum p et q in infinitum excrescant, siquidem infinite parva δx et δy ex eodem ordine quo differentialia ∂x et ∂y assumantur. Quin etiam hinc in calculo maxime cavendum est ne in enormes errores praecipitemur, cum calculi praecepta legi continuitatis innitantur, qua lineae curvae continuo puncti fluxu describi concipiuntur; ita ut in earum curvatura nusquam saltus agnoscatur. Quodsi autem unicum curvae punctom Y in y diducatur, reliquo Fig. 5.
curvae tractu praeter bina quasi elementa My et yY' invariato relicto, evidens est curvaturae ingentem irregularitatem induci, cum vulgares ca culi regulae non amplius applicari queant. Cui incommodo ut occurramus tutissimum erit remedium, ut singulis curvae punctis mente saltem variationes tribuantur, quae continuitatis quapiam lege contineantur, neque ante irregularitas in calculo admittatur, quam omnes differentiationes et integrationes fuerint peractae; hocque modo saltem species continuitatis in calculo retineatur. Quamvis ergo variationum differentialia

$$\partial\delta y,\ \partial\partial\delta y,\ \partial^3\delta y, \text{ etc. item}$$
$$\partial\delta x,\ \partial\partial\delta x,\ \partial^3\delta x, \text{ etc.}$$

forte in facta hypothesi ad simplices variationes revocare liceat, tamen expedit illas formas in calculo retineri ad easque sequentes integrationes accommodari, atque huc etiam redeunt operationes, quas olim, cum idem argumentum de inveniendis curvis maximi minimive proprietate praeditis tractassem, expedire docueram.

Problema 3.

56. Datis binarum variabilium x et y variationibus δx et δy, rationum inter differentialia cujuscunque gradus variationes investigare.

Solutio.

Quaestio huc redit ut positis continuo

$$\partial y = p\partial x,\ \partial p = q\partial x,\ \partial q = r\partial x,\ \partial r = s\partial x, \text{ etc.}$$

quantitatum p, q, r, s, etc. variationes assignentur, cum ad has quantitates omnes differentialium cujuscunque ordinis rationes, quae quidem finitis valoribus continentur, reducantur. Ac de harum quidem duabus primis p et q jam vidimus esse

$$\delta p = \frac{\partial \delta y}{\partial x} - \frac{p\partial \delta x}{\partial x} \text{ et } \delta q = \frac{\partial \delta p}{\partial x} - \frac{q\partial \delta x}{\partial x}.$$

Quoniam igitur porro est

$$r = \frac{\partial q}{\partial x} \text{ et } s = \frac{\partial r}{\partial x}, \text{ etc.}$$

harum variationes simili modo per differentiationis regulas inveniuntur

$$\delta r = \frac{\partial \delta q}{\partial x} - \frac{r\partial \delta x}{\partial x},\ \delta s = \frac{\partial \delta r}{\partial x} - \frac{s\partial \delta x}{\partial x}, \text{ etc.}$$

ubi si lubuerit loco $\partial \delta p$, $\partial \delta q$, $\partial \delta r$, etc. differentialia variationum δp, δq, δr, etc. ante inventarum substitui possunt. Hoc autem non solum in formulas nimis prolixas induceret, sed etiam uti ex sequentibus patebit, ne quidem est necessarium, cum hinc multo facilius omnes deductiones, quibus opus erit, institui queant.

Corollarium 1.

57. Si soli variabili y variationes tribuantur, seu manentibus abscissis x tantum applicatae y suis variationibus augeantur, habebimus

$$\delta p = \frac{\partial \delta y}{\partial x}, \quad \delta q = \frac{\partial \delta p}{\partial x}, \quad \delta r = \frac{\partial \delta q}{\partial x}, \quad \delta s = \frac{\partial \delta r}{\partial x}.$$

Corollarium 2.

58. Quodsi praeterea omnia ipsius x incrementa ∂x aequalia capiantur, seu elementum ∂x constans statuatur, substitutis differentialibus praecedentium formularum in sequentibus, obtinebitur

$$\delta p = \frac{\partial \delta y}{\partial x}, \quad \delta q = \frac{\partial\partial \delta y}{\partial x^2}, \quad \delta r = \frac{\partial^3 \delta y}{\partial x^3}, \quad \delta s = \frac{\partial^4 \delta y}{\partial x^4}, \text{ etc.}$$

Corollarium 2.

59. Si solis abscissis x variationes tribuantur, ut variatio δy cum omnibus derivatis evanescat, simulque elementum ∂x constans capiatur, singulae hae variationes ita se habebunt

$$\delta p = \frac{-p\,\partial \delta x}{\partial x}, \quad \delta q = \frac{-p\,\partial\partial \delta x}{\partial x^2} - \frac{2q\,\partial \delta x}{\partial x},$$

$$\delta r = \frac{-p\,\partial^3 \partial x}{\partial x^3} - \frac{3q\,\partial\partial \delta x}{\partial x^2} - \frac{3r\,\partial \delta x}{\partial x},$$

$$\delta s = \frac{-p\,\partial^4 \delta x}{\partial x^4} - \frac{4q\,\partial^3 \delta x}{\partial x^3} - \frac{6r\,\partial\partial \delta x}{\partial x^2} - \frac{4s\,\partial \delta x}{\partial x},$$

etc.

Corollarium 4.

60. Etiamsi ergo hoc casu elementum ∂x constans accipiatur, tamen hic occurrunt differentialia cujusque ordinis variationis δx, cujus rei ratio est, quod variationes valorum ipsius x continuo ulterius promotorum x', x'', etc. neutiquam a differentialibus pendere statuuntur.

Scholion.

61. Quando autem placuerit soli variabili x variationes tribuere, tum omnino praestat variabiles x et y inter se permutari, atque hujusmodi potius positionibus uti

$$\partial x = p\partial y, \quad \partial p = q\partial y, \quad \partial q = r\partial y, \text{ etc.}$$

quibus species differentialium tollatur, tum vero sumto elemento ∂y constante, similes formulae simpliciores pro variationibus quantitatum p, q, r, etc. reperiuntur, atque Corollario 2. Caeterum quo calculus ad omnes casus accommodari queat, semper expedit utrique variabili suas variationes tribui, etsi enim tum formae multo perplexiores prodeant, praecipue si evolvantur, tamen calculum prosequendo tam egregia se offerunt compendia, ut in fine calculus vix fiat operosior, neque hujus prolixitatis taedeat. Ad problemata ergo magis generalia ad hoc caput pertinentia progrediamur.

Problema 4.

62. Datis duarum variabilium x et y variationibus δx et δy, formulae cujuscunque finitae V tam ex illis variabilibus ipsis quam earum differentialibus cujuscunque ordinis conflatae variationem invenire.

Solutio.

Cum V sit quantitas valorem habens finitum, ponendo

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \partial r = s\partial x, \text{ etc.}$$

differentialia inde tollentur, prodibitque pro V functio ex quantitatibus finitis formata x, y, p, q, r, s, etc. Quaecunque ergo sit ratio compositionis, ejus differentiale semper hujusmodi habebit formam

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

horum membrorum numero existente eo majore, quo altiora differentialia ingrediuntur in V. Quodsi vero hujus formulae V variatio δV fuerit indaganda, ea obtinetur si loco quantitatum variabilium x, y, p, q, r, etc. eaedem suis variationibus auctae substituantur, et a forma resultante ipsa quantitas V auferatur, ex quo intelligitur, variationem ope consuetae differentiationis inveniri signo tantum differentialis ∂ in signum variationis δ mutato. Quare cum differentiale supra jam sit exhibitum, impetrabimus variationem quaesitam

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

quamadmodum autem variationes δp, δq, δr, δs, etc. per variationes sumtas δx et δy determinentur, jam supra est ostensum.

Corollarium 1.

63. Si hic substituamus valores ante inventos, obtinebimus variationem quaesitam ita expressam

$$\delta V = M\delta x + N\partial y + \frac{1}{\partial x}(P\partial\delta y + Q\partial\delta p + R\partial\delta q + S\partial\delta r + \text{etc.})$$
$$- \frac{\partial\delta x}{\partial x}(Pp + Qq + Rr + Ss + \text{etc.}).$$

Corollarium 2.

64. Si variabili x nulla plane tribuatur variatio, atque insuper elementum ∂x constans accipiatur, tum quantitatis propositae V variatio ita prodibit expressa

$$\delta V = N\delta y + \frac{P\partial\delta y}{\partial x} + \frac{Q\partial\partial\delta y}{\partial x^2} + \frac{R\partial^3\delta y}{\partial x^3} + \frac{S\partial^4\delta y}{\partial x^4} + \text{etc.}$$

Scholion.

65. In his formis saltem species homogeneitatis in differentialibus spectatur, siquidem δx et δy ad ordinem differentialium

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referantur, quod longe secus eveniret, si eo casu quo unicum curvae punctum variatur, statim vellemus loco differentialium variationum valores supra (§. 54.) exhibitos substituere, quo quippe pacto idea integrationis, qua hae formulae deinceps indigent, excluderetur. Caeterum patet quomodo inventio variationum ad consuetam differentiationem revocetur, dum totum discrimen in hoc tantum est situm, ut loco variationum δp, δq, δr, etc. valores jam ante assignati, quos quidem ipsos quoque per consuetam differentiationem elicuimus, substituantur. Conveniet autem hanc operationem aliquot exemplis illustrari, quo clarius indoles totius hujus tractationis percipiatur.

Exemplum 1.

66. *Formulae subtangentem exprimentis $\frac{y\partial x}{\partial y}$ variationem invenire.*

Ob $\partial y = p\partial x$ haec formula fit $\frac{y}{p}$, unde ejus variatio $\frac{\delta y}{p} - \frac{y\delta p}{pp}$, ubi loco δp valore substituto, fit ea

$$\frac{\delta y}{p} - \frac{y\partial\delta y}{pp\partial x} + \frac{y\partial\delta x}{p\partial x} = \frac{\partial x}{\partial y} \cdot \delta y - \frac{y\partial x}{\partial y^2} \cdot \partial\delta y + \frac{y}{\partial y}\partial\delta x,$$

quae postrema forma immediate ex differentiatione formulae propositae nascitur.

Exemplum 2.

67. *Formulae ipsam tangentem exprimentis $\frac{y\sqrt{(\partial x^2 + \partial y^2)}}{\partial y}$ variationem invenire.*

Posito $\partial y = p\partial x$ praebet hanc formam finitam

$$\frac{y}{p}\sqrt{(1 + pp)},$$

unde variatio quaesita est

$$\frac{\delta y}{p}\sqrt{(1 + pp)} - \frac{y\delta p}{pp\sqrt{(1 + pp)}}:$$

quae transformatur in hanc

$$\frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial y}\,\delta y - \frac{y\,\partial x}{\partial y^2 \sqrt{(\partial x^2 + \partial y^2)}}\,(\partial x \partial \delta y - \partial y \partial \delta x).$$

Exemplum 3.

68. *Formulae radium curvedinis exprimentis* $\frac{(\partial x^2 + \partial y)^{\frac{3}{2}}}{\partial x \partial \partial y}$ *variationem definire.*

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$ haec formula transit in hanc $\frac{(1+pp)^{\frac{3}{2}}}{q}$, cujus propterea variatio est

$$\frac{2p\delta p}{q}\sqrt{(1+pp)} - \frac{\delta q}{qq}(1+pp)^{\frac{3}{2}},$$

ubi quidem substitutioni valorum ante inventorum non immoror.

Problema 5.

69. Datis duarum quantitatum variabilium x et y variationibus δx et δy, formulae tam ex illis variabilibus quam earum differentialibus cujuscunque ordinis conflatae, sive fuerit infinita sive infinite parva, variationem investigare.

Solutio.

Positis ut hactenus $\partial y = p\partial x$, $\partial p = q\partial x$, $\partial q = r\partial x$, etc. formula semper reducetur ad hujusmodi formam $V\partial x^n$, ubi V sit functio finita quantitatum x, y, p, q, r, etc. exponens vero n sive positivus sive negativus, ita ut priori casu formula sit infinite parva, posteriori vero infinite magna. Ponamus igitur differentiationem ordinariam dare

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

unde simul ejus variatio habetur. Cum igitur formae propositae variatio sit

$$nV\partial x^{n-1}\,\partial\delta x + \partial x^n \delta V,$$

erit utique haec variatio quam quaerimus

$$nV\partial x^{n-1}\,\partial\delta x + \partial x^n\,(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}),$$

ubi ex superioribus hos valores substitui oportet

$$\delta p = \frac{\partial\delta y - p\partial\delta x}{\partial x};\quad \delta q = \frac{\partial\delta p - q\partial\delta x}{\partial x},$$

$$\delta r = \frac{\partial\delta q - r\partial\delta x}{\partial x},\quad \delta s = \frac{\partial\delta r - s\partial\delta x}{\partial x},$$

quae cum per se sint perspicua, nulla ampliori explicatione indigent; simulque hoc caput penitus absolutum videtur.

CAPUT III.

DE VARIATIONE FORMULARUM INTEGRALIUM SIMPLICIUM DUAS VARIABILES INVOLVENTIUM.

Definitio 6.

70.

Formulam integralem simplicem hic appello, quae nulla alia integralia in se involvit, sed simpliciter integrale refert formulae differentialis, praeter binas variabiles quaecunquae earum differentialia complectentis.

Corollarium 1.

71. Si ergo x et y sint binae variabiles, formula integralis $\int W$ erit simplex, si expressio W praeter has variabiles tantum earum differentialia, cujuscunque fuerint ordinis, contineat, neque praeterea alias formulas integrales in se implicet.

Corollarium 2.

72. Quod si ergo statuamus

$$\partial x = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \text{ etc.}$$

ut species differentialium tollatur, quoniam integratio requirit formulam differentialem, expressio illa W semper reducetur ad hujusmodi formam $V\partial x$, existente V functione quantitatum x, y, p, q, etc.

Corollarium 3.

73. Dum igitur formula integralis simplex sit hujusmodi $\int V\partial x$, ubi V est functio quantitatum x, y, p, q, r, etc. ejus indolem commodissime differentiale ejus repraesentabit, si dicamus esse

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

Scholion.

74. Distinguo hic formulas integrales simplices a complicatis, in quibus integratio proponitur ejusmodi formularum differentialium, quae jam ipsae unam pluresve formulas integrales involvunt. Veluti si littera s denotet integrale

$$\int \sqrt{(\partial y^2 + \partial y^2)} = \int \partial x \sqrt{(1 + pp)},$$

atque quantitas V praeter illas quantitates etiam hanc s contineat, formula integralis $\int V\partial x$ merito censetur complicata, cujus variatio singularia praecepta postulat deinceps exponenda. Hoc autem capite primo methodum formularum integralium simplicium variationes inveniendi tradere constitui.

Theorema 2.

75. *Variatio formulae integralis $\int W$ semper aequalis est integrali variationis ejusdem formulae differentialis, cujus integrale proponitur; seu est $\delta \int W = \int \delta W$.*

Demonstratio.

Cum variatio sit excessus, quo valor variatus cujusque quantitatis superat ejus valorem naturalem, perpendamus formulae propositae $\int W$ valorem variatum, quem obtinet, si loco variabilium x et y earundem valores suis variationibus δx et δy aucti substituantur. Cum autem tum quantitas W abeat in $W + \delta W$, formae propositae valor variatus erit

$$\int (W + \delta W) = \int W + \int \delta W,$$

unde cum sit

$$\delta \int W = \int (W + \delta W) - \int W,$$

habebimus

$$\delta \int W = \int \delta W,$$

unde patet variationem integralis aequari integrali variationis.

Idem etiam hoc modo ostendi potest. Ponatur $\int W = w$, ita ut quaerenda sit variatio δw. Quia ergo sumtis differentialibus est $\partial w = W$, capiantur nunc variationes, eritque

$$\delta \partial w = \delta W = \partial \delta w,$$

ob $\delta \partial w = \partial \delta w$. Nunc vero aequatio $\partial \delta w = \delta W$ denno integrata praebet

$$\delta w = \int \delta W = \delta \int W.$$

Corollarium 1.

76. Proposita ergo hac formula integrali $\int V \partial x$, ejus variatio $\delta \int V \partial x$ erit

$$\int \delta (V \partial x) = \int (V \delta \partial x + \partial x \delta V),$$

hincque ob $\delta \partial x = \partial \delta x$ habebitur

$$\delta \int V \partial x = \int V \partial \delta x + \int \partial x \delta V.$$

Corollarium 2.

77. Posito $\delta x = \omega$ ut sit $\partial \delta x = \partial \omega$, quia est

$$\int V \partial \omega = V \omega - \int \omega \partial V,$$

in priori membro differentiale variationis ∂x exuitur, fietque

$$\delta \int V \partial x = V \delta x - \int \partial V \delta x + \int \partial x \delta V,$$

ubi prima pars ab integratione est immunis.

Scholion.

78. Quemadmodum supra ostendimus, signa differentiationis ∂ cum signo variationis δ expressioni cuicunque praefixa inter se pro lubitu permutari posse, ita nunc videmus signum integrationis $\int$ cum signo variationis δ permutari posse, cum sit

$$\delta \int W = \int \delta W.$$

Atque hoc etiam ad integrationes repetitas patet, ut si proposita fuerit talis formula $\int\int W$, ejus variatio his modis exhiberi possit

$$\delta \int\int W = \int \delta \int W = \int\int \delta W,$$

ideoque variatio formularum integralium ad variationes expressionum nullam amplius integrationem involventium reducatur, pro quibus inveniendis jam supra praecepta sunt tradita.

Problema 6.

79. Propositis binarum variabilium x et y variationibus δx et δy, si positis

$$\partial y = p \partial x, \quad \partial p = q \partial x, \quad \partial q = r \partial x, \quad \text{etc.}$$

fuerit V functio quaecunque quantitatum x, y, p, q, r, etc. formulae integralis $\int V \partial x$ variationem investigare.

Solutio.

Modo vidimus (§. 77.) hujus formulae integralis variationem ita exprimi, ut sit

$$\delta \int V \partial x = V \delta x - \int \partial V \delta x + \int \partial x \delta V.$$

Jam ad variationem δV elidendam, cum sit V functio quantitatum x, y, p, q, r, etc. statuamus ejus differentiale esse

$$\partial V = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + \text{etc.}$$

ac simili modo ejus variatio ita erit expressa

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

quibus valoribus substitutis consequimur variationem quaesitam

$$\delta\int V\partial x = V\delta x + \int\partial x\,(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.})$$
$$-\int\delta x\,(M\partial x + N\partial y + P\partial P + Q\partial q + R\partial r + \text{etc.})$$

ubi cum partes ab M pendentes se destruant, erit partibus secundum litteras N, P, Q, R, etc. separatis variatio

$$\delta\int V\partial x = V\delta x + \int N\,(\partial x\delta y - \partial y\delta x) + \int P\,(\partial x\delta p - \partial p\delta x)$$
$$+ \int Q\,(\partial x\delta q - \partial q\delta x) + \int R\,(\partial x\delta r - \partial r\delta x) + \text{etc.}$$

ubi est uti supra invenimus

$$\partial x\delta p = \partial\delta y - p\partial\delta x, \quad \partial x\delta q = \partial\delta p - q\partial\delta x,$$
$$\partial x\delta r = \partial\delta q - r\partial\delta x, \quad \text{etc.}$$

quibus valoribus substitutis ob $\partial y = p\partial x$ obtinetur

$$\delta\int V\partial x = V\delta x + \int N\partial x\,(\delta y - p\delta x) + \int P\partial\,.\,(\delta y - p\delta x)$$
$$+ \int Q\partial\,.\,(\delta p - q\delta x) + \int R\partial\,.\,(\delta q - r\delta x) + \text{etc.}$$

Ad hanc expressionem ulterius reducendam, notetur esse

$$\delta p - q\delta x = \frac{\partial\delta y - p\partial\delta x - \partial p\delta x}{\partial x} = \frac{\partial\,.\,(\delta y - p\delta x)}{\partial x},$$
$$\delta q - r\delta x = \frac{\partial\delta p - q\partial\delta x - \partial q\delta x}{\partial x} = \frac{\partial\,.\,(\delta p - q\delta x)}{\partial x},$$
$$\delta r - s\delta x = \frac{\partial\delta q - r\partial\delta x - \partial r\delta x}{\partial x} = \frac{\partial\,.\,(\delta q - r\delta x)}{\partial x},$$

etc.

quo pacto quaevis formula ad praecedentem reducitur; unde si brevitas gratia ponamus $\delta y - p\delta x = \omega$, erit ut sequitur

$$\delta y - p\delta x = \omega,$$
$$\delta p - q\delta x = \frac{1}{\partial x}\partial\omega,$$
$$\delta q - r\delta x = \frac{1}{\partial x}\partial\,.\,\frac{\partial\omega}{\partial x},$$
$$\delta r - s\delta x = \frac{1}{\partial x}\partial\,.\,\frac{1}{\partial x}\partial\,.\,\frac{\partial\omega}{\partial x},$$

etc.

sicque variationibus litterarum derivatarum p, q, r, etc. ex calculo exclusis adipiscimur variationem quaesitam

$$\delta\int V\delta x = V\delta x + \int N\partial x \,.\, \omega + \int P\partial\omega + \int Q\partial \,.\, \frac{\partial\omega}{\partial x} + \int R\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x}$$
$$+ \int S\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x} + \int T\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x} + \text{etc.}$$

cujus formae lex progressionis est manifesta, cujuscunque gradus differentialia in formulam V ingrediantur.

Corollarium 1.

80. Hujus igitur variationis pars prima $V\delta x$ a signo integrationis est immunis, atque adeo solam variationem δx involvit, reliquae vero partes utramque perpetuo eodem modo junctam et in littera

$$\omega = \delta y - p\delta x,$$

comprehensam continet.

Corollarium 2.

81. Secunda pars

$$\int N\partial x \,.\, \omega = \int N\omega\partial x$$

commodius exprimi nequit, tertia vero $\int P\partial\omega$ commodius ita exprimi videtur, ut sit

$$\int P\partial\omega = P\omega - \int\omega\partial P,$$

ac post signum integrale jam ipsa quantitas ω reperiatur.

Corollarium 3.

82. Quarta pars $\int Q\partial \,.\, \frac{\partial\omega}{\partial x}$ simili modo reducitur ad

$$Q\frac{\partial\omega}{\partial x} - \int\partial Q \,.\, \frac{\partial\omega}{\partial x},$$

hocque membrum posterius, cum sit $\int\frac{\partial Q}{\partial x} \,.\, \partial\omega$, porro praebet

$$\frac{\partial Q}{\partial x}\omega - \int\omega\partial \,.\, \frac{\partial Q}{\partial x},$$

ita ut tertia pars resolvatur in haec membra

$$Q \cdot \frac{\partial\omega}{\partial x} - \frac{\partial Q}{\partial x} \cdot \omega + \int \omega \partial \frac{\partial Q}{\partial x}.$$

Corollarium 4.

83. Quinta pars

$$\int R \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x}$$

reducitur primo ad

$$R \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \int \frac{\partial R}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x},$$

tum vero posterius membrum ad

$$\frac{\partial R}{\partial x} \cdot \frac{\partial\omega}{\partial x} - \int \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \partial\omega,$$

hocque tandem ulterius ad

$$\frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \omega - \int \omega \partial \frac{1}{\partial x} \cdot \partial \frac{\partial R}{\partial x};$$

ita ut haec quinta pars jam ita exprimatur

$$R \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \frac{\partial R}{\partial x} \cdot \frac{\partial\omega}{\partial x} + \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \omega - \int \omega \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x},$$

Corollarium 5.

84. Simili modo sexta pars

$$\int S \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x}$$

ita reperitur expressa

$$S \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \frac{\partial S}{\partial x} \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} + \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial\omega}{\partial x}$$
$$- \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x} \cdot \omega + \int \omega \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x}.$$

Problema 7.

85. Positis

$$\partial y = p \partial x, \quad \partial p = q \partial x, \quad \partial q = r \partial x, \quad \partial r = s \partial x, \text{ etc.}$$

si V fuerit functio quaecunque quantitatum $x, y, p, q, r, s,$ etc. ita ut sit

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

formulae integralis $\int V\partial x$ variationem ex utriusque variabilis x et y variatione natam ita exprimere, ut post signum integrale nulla occurrunt variationum differentialia.

Solutio.

In corollariis praecedentis problematis jam omnia ita sunt ad hunc scopum praeparata, ut nihil aliud opus sit, nisi transformationes singularum partium in ordinem redigantur, quo pacto duplicis generis partes obtinentur; uno continente formulas integrales, quas quidem omnes in eandem summam colligere licet, altero partes absolutas quas ita in membra distribuemus, ut secundum ipsas variationes δx et δy earumque differentialia cujusque gradus procedant. Posita autem brevitatis gratia formula $\delta y - p\delta x = \omega$ variatio quaesita ita se habebit

$$\begin{aligned}\delta\int V\partial x = &\int \omega\,\partial x\left(N - \frac{\partial P}{\partial x} + \frac{1}{\partial x}\partial.\frac{\partial Q}{\partial x} - \frac{1}{\partial x}\partial.\frac{1}{\partial x}\partial.\frac{\partial R}{\partial x} + \frac{1}{\partial x}\partial.\frac{1}{\partial x}\partial.\frac{1}{\partial x}\partial.\frac{\partial S}{\partial x} - \text{etc.}\right)\\
&+ V\delta x + \omega\left(P - \frac{\partial Q}{\partial x} + \frac{1}{\partial x}\partial.\frac{\partial R}{\partial x} - \frac{1}{\partial x}\partial.\frac{1}{\partial x}\partial.\frac{\partial S}{\partial x} + \text{etc.}\right)\\
&+ \frac{\partial\omega}{\partial x}\left(Q - \frac{\partial R}{\partial x} + \frac{1}{\partial x}\partial.\frac{\partial S}{\partial x} - \text{etc.}\right)\\
&+ \frac{1}{\partial x}\partial.\frac{\partial\omega}{\partial x}\left(R - \frac{\partial S}{\partial x} + \text{etc.}\right)\\
&+ \frac{1}{\partial x}\partial.\frac{1}{\partial x}\partial.\frac{\partial\omega}{\partial x}(S - \text{etc.}) + \text{etc.}\end{aligned}$$

cujus formae indoles ex sola inspectione statim est manifesta, ut uberiori illustratione non sit opus.

Corollarium 3.

86. Haec expressio multo simplicior redditur, si elementum ∂x capiatur constans, quo quidem amplitudo expressionis nequaquam restringitur, tum enim fiet

$$\delta\int V\partial x = \int\omega\partial x\,(N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.})$$
$$+ V\delta x + \omega\,(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \frac{\partial^3 S}{\partial x^3} + \text{etc.})$$
$$+ \frac{\partial\omega}{\partial x}(Q - \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} - \text{etc.})$$
$$+ \frac{\partial\partial\omega}{\partial x^2}(R - \frac{\partial S}{\partial x} + \text{etc.})$$
$$+ \frac{\partial^3\omega}{\partial x^3}(S - \text{etc.}) + \text{etc.}$$

Corollarium 2.

87. Si quaestio sit de linea curva, prima pars integralis valorem per totam curvam ab initio usque ad terminum, ubi coordinatae x et y subsistunt, congregat, simul omnes variationes in singulis curvae punctis factas complectens, dum reliquae partes absolutae tantum per variationes in extremitate curvae factas definiuntur.

Corollarium 3.

88. Si curvam coordinatis x et y definitam ut datam spectemus, aliaque curva ab ea infinit eparum discrepans consideretur, dum in singulis punctis utrique coordinatae variationes quaecunque tribuantur, expressio inventa indicat, quantum formulae integralis $\int V\partial x$ valor ex' curva variata collectus superat ejusdem valorem ex ipsa curva data desumtum.

Corollarium 4.

89. Cum sit $\omega = \delta y - p\delta x$, patet hanc quantitatem ω evanescere, si in singulis punctis variationes δx et δy ita accipiantur, ut sit

$$\delta y : \delta x = p : 1 = \partial y : \partial x.$$

Hoc igitur casu curva variata plane non discrepat a data, ac tota variatio formulae $\int V\partial x$ reducitur ad $V\delta x$.

Scholion 1.

90. Variatio haec pro formula integrali $\int V\partial x$ inventa statim sappeditat regulam, quam olim tradidi pro curva invenienda in qua valor ejusdem formulae integralis sit maximus vel minimus. Illa enim regula postulat, ut haec expressio

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

nihilo aequalis statuatur. Hic autem statim evidens est, ad id, ut variatio formulae $\int V\partial x$ evanescat, quemadmodum natura maximorum et minimorum exigit, ante omnia requiri, ut prima pars signo integrali contenta evanescat, ex quo fit

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0.$$

Praeterea vero etiam partes absolutas nihilo aequari oportet, in quo applicatio ad utrumque curvae terminum continetur. Ipsa enim curvae natura per illam aequationem exprimitur, quae cum ob differentialia altioris gradus totidem integrationes totidemque constantes arbitrarias assumat, harum constantium determinationi illae partes absolutae inserviunt, ut tam in initio quam in fine quaesita curva certis conditionibus respondeat, veluti ad datas lineas curvas terminetur. Ac si aequatio illa fuerit differentialis ordinis quarti vel adeo altioris, partium quoque absolutarum numerus augetur, quibus effici potest, ut curva quaesita non solum utrinque ad datas lineas terminetur, sed ibidem quoque certa directio, quin etiam si ad altiora differentialia assurgat, certa curvaminis lex praescribi queat. Semper autem applicationem faciendo pulcherrime evenire solet, ut ipsa quaestionum indoles ejusmodi conditiones involvat, quibus per partes absolutas commodissime satisfieri possit.

Scholion 2.

91. Quanta autem mysteria in hac forma, quam pro variatione formulae integralis $\int V \partial x$ invenimus, lateant, in ejus applicatione ad maxima et minima multo luculentius declarare licet, hic tantum observo, partem integralem necessario in istam variationem ingredi. Cum enim rem in latissimo sensu simus complexi, ut in singulis curvae punctis utrique variabili x et y variationes quascunque nulla plane lege inter se connexas tribuerimus, fieri omnino nequit, ut variatio toti curvae conveniens non simul ab omnibus variationibus intermediis pendeat, quippe quibus aliter constitutis necesse est, ut inde totius curvae variatio mutationem perpetuamur. Atque in hoc variatio formularum integralium potissimum dissidet a variatione ejusmodi exgressionum, quales in superiori capite consideravimus, quae unice a variatione ultimis elementis tributa pendet. Ex quo luculenter sequitur, si forte quantitas V ita fuerit comparata, ut formula differentialis $V \partial x$ integrationem admittat, nulla stabilita relatione inter variabiles x et y sique integralis $\int V \partial x$ sit functio absoluta quantitatum x, y, p, q, r, etc. tum etiam ejus variationem tantum a variatione extremorum elementorum pendere posse, sicque partem variationis integralem plane in nihilum abire debere, ex quo sequens insigne Theorema colligitur.

Theorema 3.

92. *Posito* $\partial y = p \partial x$, $\partial p = q \partial x$, $\partial q = r \partial x$, $\partial r = s \partial x$, *etc. si V fuerit ejusmodi functio ipsarum* x, y, p, q, r, s, *etc. ut posito ejus differentiali*

$$\partial V = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + S \partial s + \text{etc.}$$

fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

sumto elemento ∂x constante, tum formula differentialis $V\partial x$ per se erit integrabilis, nulla stabilita relatione inter variabiles x et y; ac vicissim.

Demonstratio.

Si fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

tum formulae integralis $\int V\partial x$ variatio nullam implicat formulam integralem, ideoque pro quovis situ coordinatarum x et y a solis variationibus, quae ipsis in extremo termino tribuuntur, pendet, quod fieri neutiquam posset, si formula $V\partial x$ integrationem respueret, propterea quod tum variatio insuper ab omnibus variationibus intermediis simul necessario penderet; unde sequitur quoties aequatio illa locum habet, toties formulam $V\partial x$ integrationem admittere; ita ut $\int V\partial x$ futura sit certa ac definita functio quantitatum x, y, p, q, r, s, etc. Vicissim autem quoties formula differentialis $V\partial x$ integrationem admittit, ejusque propterea integrale $\int V\partial x$ est vera functio quantitatum x, y, p, q, r, s, etc. toties quoque ejus variatio tantum ab extremis variationibus ipsarum x et y pendet, neque variationes intermediae jam ullo modo afficere possunt. Ex quo necesse est ut variationis pars integralis supra inventa evanescat, id quod fieri nequit, nisi fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

sicque Theorema propositum etiam inversum veritati est consentaneum.

Corollarium 1.

93. En ergo insigne criterium, cujus ope formula differentialis duarum variabilium, cujuscunque gradus differentialia in eam ingrediantur, dijudicari potest, utrum sit integrabilis nec ne? Multo

latius ergo patet illo criterio satis noto, quo formularum differentialium primi gradus integrabilitas dignosci solet.

Corollarium 2.

94. Primo ergo si quantitas V sit tantum functio ipsarum x et y nullam differentialium rationem involvens, ut sit

$$\partial V = M\partial x + N\partial y,$$

tum formula differentialis $V\partial x$ integrationem non admittit, nisi sit $N = 0$, hoc est nisi V sit functio ipsius x tantum, quod quidem per se est perspicuum.

Corollarium 3.

95. Proposita autem hujusmodi formula differentiali $v\partial x + u\partial y$, ea cum forma $V\partial x$ ob $\partial y = p\partial x$ comparata, dat $V = u + pu$, ideoque

$$M = \left(\frac{\partial v}{\partial x}\right) + p\left(\frac{\partial u}{\partial x}\right), \quad N = \left(\frac{\partial v}{\partial y}\right) + p\left(\frac{\partial u}{\partial y}\right),$$

et $P = u$, quandoquidem quantitates v et u nulla differentialia implicare sumuntur, Erit ergo

$$\partial P = \partial u = \partial x \left(\frac{\partial u}{\partial x}\right) + \partial y \left(\frac{\partial u}{\partial y}\right).$$

Quare cum criterium integrabilitatis postulet ut sit

$$N - \frac{\partial P}{\partial x} = 0,$$

erit pro hoc casu

$$\left(\frac{\partial v}{\partial y}\right) + p\left(\frac{\partial u}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right) - p\left(\frac{\partial u}{\partial y}\right) = 0,$$

seu $\left(\frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\right)$,

quod est criterium jam vulgo cognitum.

Scholion 1.

96. Demonstratio hujus Theorematis omnino est singularis,

cum ex doctrina variationum sit petita, quae tamen ab hoc argumento prorsus est aliena; vix vero alia via patet ad ejus demonstrationem pertingendi. Tum vero hic accuratior cognitio functionum diligenter est animadvertenda, qua ostendimus, formulam integralem $\int V\partial x$ neutiquam ut functionem quantitatum x, y, p, q, r, etc. spectari posse, nisi revera integrationem admittat. Natura enim functionum semper hanc proprietatem habet adjunctam, ut statim atque quantitatibus, quae eam ingrediuntur, valores determinati tribuuntur, ipsa functio ex iis formata determinatum adipiscatur valorem; veluti haec functio xy, si ponamus $x = 2$ et $y = 3$, valorem accipit $= 6$. Longe secus autem evenit in formula integrali $\int y\partial x$, cujus valor pro casu $x = 2$ et $y = 3$ neutiquam assignari potest, nisi inter y et x certa quaedam relatio statuatur; tum autem ea formula abit in functionem unicae variabilis. Formularum ergo integralium, quae integrari nequeunt, natura sollicite a natura functionum distingui debet, cum functiones, statim atque quantitatibus variabilibus, ex quibus conflantur, determinati valores tribuuntur, ipsae quoque determinatos valores recipiant, etiamsi variabiles nullo modo a se invicem pendeant; quod minime evenit in formulis integralibus, quippe quarum determinatio omnes plane valores intermedios simul includit. Imprimis autem huic discrimini universa doctrina de maximis et minimis, ad quam hic attendimus, innititur, ubi formulas, quibus maximi minimive proprietates conciliari debet, necessario ejusmodi integrales esse oportet, quae per se integrationem non admittant.

Scholion 2.

97. Ad majorem Theorematis illustrationem ejusmodi formulam integralem $\int V\partial x$ consideremus, quae per se sit integrabilis, ponamusque exempli gratia

$$\int V\partial x = \frac{x\partial y}{y\partial x} = \frac{xp}{y},$$

ita ut sit

$$V = \frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y},$$

atque ideo haec formula differentialis

$$\left(\frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}\right) \partial x,$$

sit absolute integrabilis; ac videamus, an Theorema nostrum hanc integrabilitatem declaret? Quantitatem ergo V differentiemus, et differentiali cum forma

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q$$

comparato, obtinebimus

$$M = \frac{-pp}{yy} + \frac{q}{y}, \quad N = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

$$P = \frac{1}{y} - \frac{2xp}{yy} \text{ et } Q = \frac{x}{y}.$$

Cum nunc secundum Theorema fieri debeat

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} = 0,$$

primo colligimus differentiando

$$\frac{\partial P}{\partial x} = \frac{-3p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \text{ et } \frac{\partial Q}{\partial x} = \frac{1}{y} - \frac{xp}{yy},$$

tum vero

$$\frac{\partial\partial Q}{\partial x^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Ergo

$$\frac{\partial P}{\partial x} - \frac{\partial\partial Q}{\partial x^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

cui valori quantitas N utique est aequalis.

Scholion 3.

98. Caeterum quando formula differentialis $V\partial x$ integrationem per se admittit, ideoque posito

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

secundum Theorema est

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

hinc alia insignia consectaria deducuntur. Primo enim cum per ∂x multiplicando et integrando fiat

$$\int N\partial x - P + \frac{\partial Q}{\partial x} - \frac{\partial\partial R}{\partial x^2} + \frac{\partial^3 S}{\partial x^3} - \text{etc.} = A,$$

patet etiam formulam $N\partial x$ absolute esse integrabilem. Deinde cum hinc porro fiat

$$\int\partial x\,(\int N\partial x - P) + Q - \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} - \text{etc.} = Ax + B.$$

etiam formula

$$\partial x\,(\int N\partial x - P),$$

integrationem admittit. Postea etiam simili modo integrabilis erit haec forma

$$\partial x\,[\int\partial x\,(\int N\partial x - P) + Q],$$

tum vero etiam haec

$$\partial x\,[\int\partial x\,(\int\partial x\,(\int N\partial x - P) + Q) - R],$$

et ita porro. Unde sequens Theorema non minus notatu dignum et in praxi utilissimum colligimus.

Theorema 4.

99. *Positis* $\partial y = p\partial x$, $\partial p = q\partial x$, $\partial q = r\partial x$, $\partial r = s\partial x$, *etc. si* V *ejusmodi fuerit functio ipsarum* $x, y, p, q, r, s,$ *etc. ut formula differentialis* $V\partial x$ *per se sit integrabilis, tum posito*

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

etiam sequentes formulas differentiales per se integrationem admittent:

I. Formula $N\partial x$ erit per se integrabilis;

tum posito $P - \int N\partial x = \mathfrak{P}$,

II. Formula $\mathfrak{P}\partial x$ erit per se integrabilis; porro posito $Q - \int \mathfrak{P}\partial x = \mathfrak{Q}$,

III. Formula $\mathfrak{Q}\partial x$ erit per se integrabilis; deinde posito $R - \int \mathfrak{Q}\partial x = \mathfrak{R}$,

IV. Formula $\mathfrak{R}\partial x$ erit per se integrabilis; ulterius posito $S - \int \mathfrak{R}\partial x = \mathfrak{S}$,

V. Formula $\mathfrak{S}\partial x$ erit per se integrabilis; et ita porro.

Demonstratio.

Hujus Theorematis veritas jam in praecedente §. est evicta, unde simul patet, si omnes hae formulae integrationem admittant, etiam principalem $V\partial x$ absolute fore integrabilem.

Corollarium 1.

100. Cum V sit functio quantitatum

$$x,\ y,\ p = \frac{\partial y}{\partial x},\quad q = \frac{\partial p}{\partial x},\quad r = \frac{\partial q}{\partial x},\ \text{etc.}$$

quantitates per differentiationem inde derivatae M, N, P, Q, R, etc. etiam ita exhiberi possunt, ut sit

$$M = \left(\frac{\partial V}{\partial x}\right);\quad N = \left(\frac{\partial V}{\partial y}\right);\quad P = \left(\frac{\partial V}{\partial p}\right),\quad Q = \left(\frac{\partial V}{\partial q}\right),\ \text{etc.}$$

unde ob primam formulam patet, si fuerit formula $V\partial x$ integrabilis, tum etiam formulam $\left(\frac{\partial V}{\partial y}\right)\partial x$ fore integrabilem.

Corollarium 2.

101. Deinde ergo quoque ob eandem rationem formula haec $\left(\frac{\partial\partial V}{\partial y^2}\right)\partial x$, hincque porro istae

$$\left(\frac{\partial^3 V}{\partial y^3}\right)\partial x,\quad \left(\frac{\partial^4 V}{\partial y^4}\right)\partial x,\ \text{etc.}$$

omnes per se integrationem admittent.

Corollarium 8.

102. Quia tot tantum litterae P, Q, R, etc. adsunt, quoti gradus differentialia in formula $V\partial x$ reperiuntur, et sequentes omnes evanescunt, litterae germanicae inde derivatae $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$, etc. tandem evanescere vel in functiones solius quantitatis x abire debent, quia alioquin sequentes integrabilitates locum habere non possent.

Exemplum.

103. *Sit V ejusmodi functio, ut fiat*

$$\int V\partial x = \frac{y\,(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{x\partial x\partial\partial y}.$$

Factis substitutionibus

$$\partial y = p\partial x,\quad \partial p = q\partial x,\quad \partial q = r\partial x,\quad \text{etc.}$$

pro hoc exemplo functio V ita exprimetur

$$V = \frac{p\,(1+pp)^{\frac{3}{2}}}{xq} - \frac{y\,(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3yp\sqrt{(1+pp)}}{x} - \frac{yr\,(1+pp)^{\frac{3}{2}}}{xqq},$$

unde per differentiationem elicuimus sequentes valores

$$N = \frac{-(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r\,(1+pp)^{\frac{3}{2}}}{xqq},$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y\,(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

$$Q = \frac{-p\,(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y\,(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr\,(1+pp)^{\frac{3}{2}}}{xq^3},$$

$$R = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Jam igitur primo integrabilem esse oportet formulam $N\partial x$ seu

$$-\frac{\partial x(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\partial x\sqrt{(1+pp)}}{x} - \frac{\partial q(1+pp)^{\frac{3}{2}}}{xqq},$$

unde statim patet integrale hoc fore

$$\int N\partial x = \frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Jam pro secunda formula hinc nanciscimur

$$\mathfrak{P} = P - \int N\partial x = \frac{3pp\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

ita ut integranda sit haec formula

$$\mathfrak{P}\partial x = \frac{3p\partial y\sqrt{(1+pp)}}{xq} - \frac{3yp\partial x\sqrt{(1+pp)}}{xxq} + \frac{3y\partial x(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3yp\partial q\sqrt{(1+pp)}}{xqq},$$

cujus integrale, vel saltem ejus pars ex postremo membro manifesto colligitur $\frac{3yp\sqrt{(1+pp)}}{qx}$, cujus differentiale cum totam formulam exhauriat erit

$$\int \mathfrak{P}\partial x = \frac{3yp\sqrt{(1+pp)}}{xq}.$$

Nunc pro tertia formula habebimus

$$\mathfrak{Q} = Q - \int \mathfrak{P} \partial x = \frac{-p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3yp\sqrt{(1+pp)}}{xq},$$

unde per ∂x multiplicando, ob $\partial x = \frac{\partial p}{q}$ in ultimo membro fit

$$\mathfrak{Q}\partial x = \frac{-\partial y(1+pp)^{\frac{3}{2}}}{x\,qq} + \frac{y\partial x(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2y\partial q(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3yp\partial p\sqrt{(1+pp)}}{xqq},$$

cujus penultimum membrum declarat integrale

$$\int \mathfrak{Q}\partial x = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Quarta porro formula ita erit comparata

$$\mathfrak{R} = R - \int \mathfrak{Q}\partial x = 0,$$

unde perspicuum est, non solum hanc $\mathfrak{R}\partial x$ sed etiam sequentes omnes fore integrabiles.

Scholion.

104. Theoremata haec eo pulchriora videntur, quod eorum demonstratio ejusmodi principio innititur, cujus ratio hinc prorsus est aliena; propterea quod in his veritatibus nullum amplius vestigium variationum apparet; ex quo nullum est dubium quin demonstratio etiam ex alio fonte magis naturali hauriri queat.

CAPUT IV.

DE
VARIATIONE FORMULARUM INTEGRALIUM COMPLICATARUM DUAS VARIABILES INVOLVENTIUM.

Problema 8.

105.

Posito $v = \int \mathfrak{V} \partial x$, existente $\mathfrak{V}$ functione quacunque binarum variabilium x, y earumque differentialium

$$\partial y = p \partial x, \quad \partial p = q \partial x, \quad \partial q = r \partial x, \quad \text{etc.}$$

si V denotet functionem quamcunque ipsius v, investigare variationem formulae integralis complicatae $\int V \partial x$.

Solutio.

Quia quantitas v ipsa est formula integralis $\int \mathfrak{V} \partial x$, formula $\int V \partial x$ est utique complicata. Cum igitur functio V solam quantitatem v involvere ponatur, statuamus $\partial V = L \partial v$, tum vero pro functione $\mathfrak{V}$ sit ejus differentiale

$$\partial \mathfrak{V} = \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \mathfrak{R} \partial r + \text{etc.}$$

His positis cum variatio quaesita sit

$$\delta \int V \partial x = \int \delta (V \partial x) = \int (\delta V \partial x + V \partial \delta x),$$

et per reductionem supra adhibitam

$$\delta \int V \partial x = V \delta x + \int (\partial x \delta V - \partial V \delta x).$$

Cum autem per hypothesin sit $\partial V = L \partial v$, erit etiam pro variatione $\delta V = L \delta v$, verum ob $v = \int \mathfrak{V} \partial x$ erit primo $\partial v = \mathfrak{V} \partial x$, ideoque $\partial V = L \mathfrak{V} \partial x$, tum vero

$$\delta v = \delta\int \mathfrak{V}\partial x = \mathfrak{V}\delta x + \int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x),$$

ac propterea

$$\delta V = L\mathfrak{V}\delta x = L\int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x),$$

hincque

$$\delta\int V\partial x = V\delta x$$
$$+\int[L\mathfrak{V}\partial x\delta x + L\partial x\int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x) - L\mathfrak{V}\partial x\,\delta x],$$

seu $\delta\int V\partial x = V\delta x - \int L\partial x\int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x)$.

Ex praecedente autem capite patet esse

$$\begin{aligned}\int(\partial x\delta\mathfrak{V}-\partial\mathfrak{V}\delta x) = \delta\int\mathfrak{V}\partial x-\mathfrak{V}\partial x = \int\omega\partial x\Big(\mathfrak{N}-\frac{\partial\mathfrak{P}}{\partial x}+\frac{\partial\partial\mathfrak{Q}}{\partial x^2}-\frac{\partial^3\mathfrak{R}}{\partial x^3}+\frac{\partial^4\mathfrak{S}}{\partial x^4}-\text{etc.}\Big)&\\ +\ \omega\ \Big(\mathfrak{P}-\frac{\partial\mathfrak{Q}}{\partial x}+\frac{\partial\partial\mathfrak{R}}{\partial x^2}-\frac{\partial^3\mathfrak{S}}{\partial x^3}+\text{etc.}\Big)&\\ +\frac{\partial\omega}{\partial x}\Big(\mathfrak{Q}-\frac{\partial\mathfrak{R}}{\partial x}+\frac{\partial\partial\mathfrak{S}}{\partial x^2}-\text{etc.}\Big)&\\ +\frac{\partial\partial\omega}{\partial x^2}\Big(\mathfrak{R}-\frac{\partial\mathfrak{S}}{\partial x}+\text{etc.}\Big)&\\ \text{etc.}&\end{aligned}$$

sumto elemento ∂x constante et posito brevitatis ergo $\omega = \delta y - p\delta x$. Verum cum hinc substitutio molestias pariat, praestabit ex primo fonte rem repetere; cum igitur ex differentiali et variatione quantitatis $\mathfrak{V}$ fiat

$$\begin{aligned}\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x = \partial x\,(\mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.})&\\ - \delta x\,(\mathfrak{M}\partial x + \mathfrak{N}\partial y + \mathfrak{P}\partial p + \mathfrak{Q}\partial q + \mathfrak{R}\partial r + \text{etc.})&\end{aligned}$$

ob $\partial y = p\partial x$, $\partial p = q\partial x$, $\partial q = r\partial x$, $\partial r = s\partial x$, etc. erit

$$\begin{aligned}\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x = \mathfrak{N}\partial x\,(\delta y - p\delta x) + \mathfrak{P}\partial x\,(\delta p - q\delta x)&\\ + \mathfrak{Q}\partial x\,(\delta q - r\delta x) + \text{etc.}&\end{aligned}$$

Verum ob ∂x constans, ex §. 79. fit

$$\delta y - p\delta x = \omega,\quad \delta p - q\delta x = \frac{\partial\omega}{\partial x},\quad \delta q - r\delta x = \frac{\partial\partial\omega}{\partial x^2},$$
$$\delta r - s\delta x = \frac{\partial^3\omega}{\partial x^3},\quad \text{etc.}$$

sicque habebitur

$$\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x = \mathfrak{N} \omega \partial x + \mathfrak{P} \partial \omega + \mathfrak{Q} \frac{\partial \partial \omega}{\partial x} + \mathfrak{R} \frac{\partial^3 \omega}{\partial x^2} + \mathfrak{S} \frac{\partial^4 \omega}{\partial x^3} + \text{etc.}$$

cujus quidem integrale praebet superiorem expressionem. Ponatur nunc integrale $\int L \partial x = I$, eritque

$$\delta \int V \partial x = V \delta x + I \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) - \int I (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x).$$

Nunc vero facile colligitur fore

$$\begin{aligned} \int I (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) = \int \omega \partial x & \left(I \mathfrak{N} - \frac{\partial . I \mathfrak{P}}{\partial x} + \frac{\partial \partial . I \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 . I \mathfrak{R}}{\partial x^3} + \text{etc.} \right) \\ + \omega & \left(I \mathfrak{P} - \frac{\partial . I \mathfrak{Q}}{\partial x} + \frac{\partial \partial . I \mathfrak{R}}{\partial x^2} - \text{etc.} \right) \\ + \frac{\partial \omega}{\partial x} & \left(I \mathfrak{Q} - \frac{\partial . I \mathfrak{R}}{\partial x} + \text{etc.} \right) \text{ etc.} \end{aligned}$$

unde facta substitutione concluditur variatio quaesita

$$\begin{aligned} \delta \int V \partial x = V \delta x + I \int \omega \partial x & \left(\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial \partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \text{etc.} \right) \\ - \int \omega \partial x & \left(I \mathfrak{N} - \frac{\partial . I \mathfrak{P}}{\partial x} + \frac{\partial \partial . I \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 I \mathfrak{R}}{\partial x^3} + \text{etc.} \right) \\ + I \omega & \left(\mathfrak{P} - \frac{\partial \mathfrak{Q}}{\partial x} + \frac{\partial \partial \mathfrak{R}}{\partial x^2} - \frac{\partial^3 \mathfrak{S}}{\partial x^3} + \text{etc.} \right) \\ - \omega & \left(I \mathfrak{P} - \frac{\partial . I \mathfrak{Q}}{\partial x} + \frac{\partial \partial . I \mathfrak{R}}{\partial x^2} - \frac{\partial^3 . I \mathfrak{S}}{\partial x^3} + \text{etc.} \right) \\ + \frac{I \partial \omega}{\partial x} & \left(\mathfrak{Q} - \frac{\partial \mathfrak{R}}{\partial x} + \frac{\partial \partial \mathfrak{S}}{\partial x^2} - \text{etc.} \right) \\ - \frac{\partial \omega}{\partial x} & \left(I \mathfrak{Q} - \frac{\partial . I \mathfrak{R}}{\partial x} + \frac{\partial \partial . I \mathfrak{S}}{\partial x^2} - \text{etc.} \right) \\ + \frac{I \partial \partial \omega}{\partial x^2} & \left(\mathfrak{R} - \frac{\partial \mathfrak{S}}{\partial x} + \text{etc.} \right) \\ - \frac{\partial \partial \omega}{\partial x^2} & \left(I \mathfrak{R} - \frac{\partial . I \mathfrak{S}}{\partial x} + \text{etc.} \right) \\ + \frac{I \partial^3 \omega}{\partial x^3} & \left(\mathfrak{S} - \text{etc.} \right) \\ - \frac{\partial^3 \omega}{\partial x^3} & \left(I \mathfrak{S} - \text{etc.} \right) + \text{etc.} \end{aligned}$$

Si hic partes binae priores differentiatae iterum integrentur, reliquarum facta reductione, impetrabimus loco ∂I valorem $L \partial x$ restituendo

$$\delta\int V\partial x = V\delta x + \int L\partial x\int\omega\partial x\left(\mathfrak{N} - \frac{\partial\mathfrak{P}}{\partial x} + \frac{\partial\partial\mathfrak{Q}}{\partial x^2} - \frac{\partial^3\mathfrak{R}}{\partial x^3} + \text{etc.}\right)$$
$$+ \int\omega\partial x\left(L\mathfrak{P} - \frac{L\partial\mathfrak{Q} - \partial L\mathfrak{Q}}{\partial x} + \frac{L\partial\partial\mathfrak{R} + \partial.L\partial\mathfrak{R} + \partial\partial.L\mathfrak{R}}{\partial x^2} - \text{etc.}\right)$$
$$+ \omega\left(L\mathfrak{Q} - \frac{L\partial\mathfrak{R} - \partial.L\mathfrak{R}}{\partial x} + \frac{L\partial\partial\mathfrak{S} + \partial.L\partial\mathfrak{S} + \partial\partial.L\mathfrak{S}}{\partial x^2} - \text{etc.}\right)$$
$$+ \frac{\partial\omega}{\partial x}\left(L\mathfrak{R} - \frac{L\partial\mathfrak{S} - \partial.L\mathfrak{S}}{\partial x} + \text{etc.}\right)$$
$$+ \frac{\partial\partial\omega}{\partial x^2}\left(L\mathfrak{S} - \text{etc.}\right) + \text{etc.}$$

quae forma videtur simplicissima et ad usum maxime accomodata.

Corollarium 1.

106. Si ejusmodi relatio inter x et y quaeratur, ut integrale $\int V\partial x$ maximum minimumve evadat, variationis partes integrales nihilo aequari oportet, quod in genere fieri nequit, sed ad terminum, quousque integrale $\int V\partial x$ extenditur, spectari oportet, pro quo si ponamus fieri $I = \int L\partial x = A$, ex priori forma colligimus hanc aequationem

$$0 = (A - I)\mathfrak{N} - \frac{\partial.(A - I)\mathfrak{P}}{\partial x} + \frac{\partial\partial.(A - I)\mathfrak{Q}}{\partial x^2} - \frac{\partial^3.(A - I)\mathfrak{R}}{\partial x^3} + \text{etc.}$$

Corollarium 2.

107. Quomodocunque autem haec aequatio pro quovis casu oblato tractetur, semper tandem eo est deveniendum ut formula integralis $I = \int L\partial x$ per differentiationem exturbari debeat, qua operatione simul quantitatem A inde extrudi evidens est; sicque aequatio resultans non amplius a termino integrationis pendebit.

Corollarium 3.

108. Quod si in genere pro variatione formulae integralis $\int V\partial x$ invenienda, valorem $\int L\partial x = I$ toti integrali respondentem ponamus $= A$, variatio quaesita ita exprimetur

$$\delta\int V\partial x = V\delta x + \int\omega\partial x\left[(A-I)\mathfrak{N} - \frac{\partial.(A-I)\mathfrak{P}}{\partial x} + \frac{\partial\partial.(A-I)\mathfrak{Q}}{\partial x^2} - \frac{\partial^3.(A-I)\mathfrak{R}}{\partial x^3} + \text{etc.}\right]$$
$$+\ \omega\left(L\mathfrak{Q} - \frac{L\partial\mathfrak{R} - \partial.L\mathfrak{R}}{\partial x} + \frac{L\partial\partial\mathfrak{S} + \partial.L\partial\mathfrak{S} + \partial\partial.L\mathfrak{S}}{\partial x^2} - \text{etc.}\right)$$
$$+\ \frac{\partial\omega}{\partial x}\left(L\mathfrak{R} - \frac{L\partial\mathfrak{S} - \partial.L\mathfrak{S}}{\partial x} + \text{etc.}\right)$$
$$+\ \frac{\partial\partial\omega}{\partial x^2}\left(L\mathfrak{S} - \text{etc.}\right) + \text{etc.}$$

ubi $A - I$ est valor formulae $\int L\partial x$ a termino integrationis extremo ad quemvis locum indefinitum medium retro sumtus.

Scholion.

109. In solutione hujus problematis compendium se obtulit, quo etiam analysis in superiori capite adhibita non mediocriter contrahi potest. Cum enim ibi (§. 79.) pervenissemus ad

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x), \text{ ob}$$
$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc. et}$$
$$\delta V = M\delta x + N\delta y + P\delta q + Q\delta r + R\delta s + \text{etc.}$$

erit

$$\partial V = \partial x\,(M + Np + Pq + Qr + Rs + \text{etc.}),$$

hincque colligitur

$$\partial x\delta V - \partial V\delta x$$
$$= \partial x\,[N(\delta y - p\delta x) + P(\delta p - q\delta x) + Q(\delta q - r\delta x) + \text{etc.}].$$

Jam si brevitatis gratia ponatur $\delta y - p\delta x = \omega$, erit differentiando

$$\delta(p\partial x) - q\partial x\delta x - p\delta\partial x = \partial\omega; \text{ at}$$
$$\delta(p\partial x) = p\partial\delta x + \delta p\,\partial x, \text{ ergo}$$
$$\delta p - q\delta x = \frac{\partial\omega}{\partial x}.$$

Simili modo hanc formulam differentiando ob

$$\partial p = q\partial x \text{ et } \partial q = r\partial x \text{ fit}$$
$$q\partial\delta x + \delta q\partial x - q\partial\delta x - \partial q\delta x = \partial x(\delta q - r\delta x) = \partial\cdot\frac{\partial\omega}{\partial x},$$

unde perspicuum est

posito $\delta y - p\delta x = \omega$,

fore $\delta p - q\delta x = \frac{\partial\omega}{\partial x}$,

$\delta q - r\delta x = \frac{1}{\partial x}\partial \cdot \frac{\partial\omega}{\partial x} = \frac{\partial\partial\omega}{\partial x^2}$, sumto ∂x constante,

$\delta r - s\delta x = \frac{1}{\partial x}\partial \cdot \frac{1}{\partial x}\partial \cdot \frac{\partial\omega}{\partial x} = \frac{\partial^3\omega}{\partial x^3}$,

etc.

Quocirca erit

$$\partial x \delta V - \partial V \delta x$$

$$= \partial x \left(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \frac{S\partial^4\omega}{\partial x^4} + \text{etc.}\right),$$

siquidem differentiale ∂x constans accipiatur.

Problema 9.

110. Si fuerit $v = \int \mathfrak{V}\partial x$, existente

$$\partial\mathfrak{V} = \mathfrak{M}\partial x + \mathfrak{N}\partial y + \mathfrak{P}\partial p + \mathfrak{Q}\partial q + \mathfrak{R}\partial r + \text{etc.}$$

tum vero sit V functio quaecunque non solum quantitates

$$x,\ y,\ p = \frac{\partial y}{\partial x},\quad q = \frac{\partial p}{\partial x},\quad r = \frac{\partial q}{\partial x},\ \text{etc.}$$

sed etiam ipsam formulam integralem $v = \int \mathfrak{V}\partial x$ implicans, investigare variationem formulae integralis complicatae $\int V\partial x$.

Solutio.

Quoniam V est functio quantitatum v, x, y, p, q, r, etc. sumatur ejus differentiale quod sit

$$\partial V = L\partial v + M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

ac habebitur variatio ipsius V ita expressa

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

tum vero notetur, ob

$$\partial v = \mathfrak{V}\partial x,\quad \partial y = p\partial x,\quad \partial p = q\partial x,\ \text{etc. esse}$$

$$\partial V = \partial x\,(L\mathfrak{V} + M + Np + Pq + Qr + Rs + \text{etc.}) \quad \text{et}$$

$$\delta\mathfrak{V} = \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

Praeterea habemus

$$\delta v = \int(\mathfrak{V}\partial\delta x + \partial x\delta\mathfrak{V}) = \mathfrak{V}\delta x + \int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x),$$

unde posito $\delta y - p\delta x = \omega$, erit per ante inventa

$$\delta v = \mathfrak{V}\delta x + \int\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.}),$$

ubi commoditatis ergo sumsimus ∂x constans.

His praeparatis cum variatio quaesita sit

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x),$$

ut reductione supra inventa uti possimus, ponamus

$$\partial V = L\partial v + \partial W,$$

ut sit

$$\delta V = L\delta v + \delta W \quad \text{et}$$

$$\delta W = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

Quocirca nanciscemur hanc formam

$$\delta\int V\partial x = V\delta x + \int(L\partial x\delta v - L\partial v\delta x) + \int(\partial x\delta W - \partial W\delta x),$$

ubi est

$$\partial x\delta W - \partial W\delta x = \partial x\,(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}).$$

Tum vero est

$$\partial x\delta v - \partial v\delta x = \partial x\int\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$

ob $\partial v\delta x = \mathfrak{V}\partial x\delta x$. Quibus substitutis colligitur variatio quaesita

$$\delta\int V\partial x = V\delta x + \int L\partial x\int\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$
$$+ \int\partial x\,(N + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}).$$

Quo jam hanc formam ulterius reducamus, ponamus integrale $\int L\partial x = I$ ita sumtum, ut pro initio, unde integrale $\int V\partial x$ capi-tur, evanescat, pro fine autem ubi integrale $\int V\partial x$ terminatur, fiat $I = A$, sicque fiet

$$\delta\int V\partial x = V\delta x + A\int\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$

$$- \int I\,\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$

$$+ \int\partial x\,(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial v^3} + \text{etc.})$$

ad quam formam contrahendam statuamus

$$N + (A - I)\,\mathfrak{N} = N',$$
$$P + (A - I)\,\mathfrak{P} = P',$$
$$Q + (A - I)\,\mathfrak{Q} = Q',$$
$$R + (A - I)\,\mathfrak{R} = R',$$
etc.

ut prodeat forma illi, quam supra tractavimus, similis

$$\delta\int V\partial x = V\delta x + \int\partial x\,(N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.}),$$

ubi ergo si post signum integrale differentialia ipsius ω elimenentur, perveniemus secundum §. 86. ad hanc expressionem

$$\delta\int V\partial x = \int\omega\partial x\,(N' - \frac{\partial P'}{\partial x} + \frac{\partial\partial Q'}{\partial x^2} - \frac{\partial^3 R'}{\partial x^3} + \frac{\partial^4 S'}{\partial x^4} - \text{etc.})$$

$$+ V\delta x + \omega\,(P' - \frac{\partial Q'}{\partial x} + \frac{\partial\partial R'}{\partial x^2} - \frac{\partial^3 S'}{\partial x^3} + \text{etc.})$$

$$+ \text{Const.} + \frac{\partial\omega}{\partial x}\,(Q' - \frac{\partial R'}{\partial x} + \frac{\partial\partial S'}{\partial x^2} - \text{etc.})$$

$$+ \frac{\partial\partial\omega}{\partial x^2}\,(R' - \frac{\partial S'}{\partial x} + \text{etc.})$$

$$+ \frac{\partial^3\omega}{\partial x^3}\,(S' - \text{etc.}) + \text{etc.}$$

Constanti autem per integrationem invectae ejusmodi valor tribui debet, ut pro initio integrationis formulae $\int V\partial x$ partes absolutae ad nihilum redigantur, siquidem prima pars integralis ita sumatur, ut pro eodem initio evanescat; tum vero universam expressionem ad finem integrationis, produci oportet pro quo jam posuimus fieri

$$\int L\,\partial x = I = A.$$

Corollarium 1.

111. In parte integrali variabilitas per totam integrationis extensionem debet comprehendi, in partibus autem absolutis, sufficit respexisse ad initium ac finem integrationis, pro utroque autem termino conditiones variationis praescriptae suppeditant valores ∂x, ω, $\frac{\partial\omega}{\partial x}$, $\frac{\partial\partial\omega}{\partial x^2}$, etc. Ac postquam ex conditionibus initii constans rite fuerit determinata, tum superest, ut singula membra ad finem integrationis accommodentur.

Corollarium 2.

112. Pro initio igitur integrationis ubi $I = 0$, erit primo

$$N' = N + A\mathfrak{N}, \quad P' = P + A\mathfrak{P}, \quad Q' = Q + A\mathfrak{Q}; \quad R' = R + A\mathfrak{R}, \text{ etc.}$$

pro differentialibus vero ob $\partial I = L\partial x$ erit

$$\frac{\partial N'}{\partial x} = \frac{\partial N}{\partial x} + \frac{A\partial\mathfrak{N}}{\partial x} - L\mathfrak{N},$$

et ita de reliquis; similique modo pro differentialibus secundis

$$\frac{\partial\partial N'}{\partial x^2} = \frac{\partial\partial N}{\partial x^2} + \frac{A\partial\partial\mathfrak{N}}{\partial x^2} - \frac{2L\partial\mathfrak{N}}{\partial x} - \frac{\mathfrak{N}\partial L}{\partial x}.$$

Corollarium 3.

113. Pro fine autem integrationis, ubi $I = A$ fit

$$N' = N, \quad P' = P, \quad Q' = Q, \quad R' = R, \text{ etc.}$$

valores vero differentiales ita se habebunt

$$\frac{\partial N'}{\partial x} = \frac{\partial N}{\partial x} + L\mathfrak{N}, \quad \frac{\partial P'}{\partial x} = \frac{\partial P}{\partial x} - L\mathfrak{P}, \quad \frac{\partial Q'}{\partial x} = \frac{\partial Q}{\partial x} - L\mathfrak{Q}, \text{ etc.}$$

secundi vero gradus hoc modo

$$\frac{\partial\partial N'}{\partial x^2} = \frac{\partial\partial N}{\partial x^2} - \frac{2L\partial N}{\partial x} - \frac{\mathfrak{N}\partial L}{\partial x},$$

$$\frac{\partial\partial P'}{\partial x^2} = \frac{\partial\partial P}{\partial x^2} - \frac{2L\partial\mathfrak{P}}{\partial x} - \frac{\mathfrak{P}\partial L}{\partial x},$$

et ita porro.

Scholion 1.

114. Quamquam natura variationum atque etiam quaestionum eo pertinentium jam satis est explicata, tamen hujus argumenti tam dignitas quam novitas ampliorem illustrationem requirere videntur, cum ne superfluum quidem foret eadem saepius inculcari. Cum igitur ante geometria et hujus calculi applicatione ad maxima et minima usi simus, ad hanc doctrinam magis explanandam, hic rem generalius pro sola Analysi contemplabimur. Primo igitur spectatur relatio quaecunque inter binas variabiles x et y, sive ea sit cognita, sive demum definienda, indeque formata consideratur formula integralis quaecunque $\int V \partial x$, quae intra certos terminos comprehensa, seu integratione a dato initio ad datum finem extensa, utique certum quendam valorem recipere debet. Tum illa relatio inter x et y, quaecunque fuerit, quomodocunque infinite parum immutetur, ut singulis x, variationibus quibuscunque δx auctis, jam respondeant eaedem y, variationibus quoque quibuscunque δy auctae, ubi quidem observandum est, tam in initio quam in fine rationum harum variationum per conditiones quaestionum dari, in medio autem istas variationes ita generaliter assumi, ut nulla plane lege inter se connectantur. Tum ex hac relatione variata ejusdem formulae integralis $\int V \partial x$ ab eodem initio ad eundem finem expansus, seu intra eosdem terminos contentus, definiri concipitur, ac tota jam quaestio in hoc versatur, ut hujus postremi valoris variati excessus supra priorem illum valorem formulae $\int V \partial x$ investigetur. Qui excessus cum per $\delta \int V \partial x$, quae forma ipsa est variatio formulae $\int V \partial x$, indicetur, hujus quaestionis solutionem hactenus dedimus ita late patentem, ut omnes casus quibus quantitas V est functio quaecunque non solum ipsarum x, y, p, q, r, s, etc. sed etiam insuper formulam quandam integralem $v = \int \mathfrak{V} \partial x$ utcunque involvens, in se complectatur.

Scholion 2.

115. Quod in praecedente capite tacite assumsimus de quantitate constante variationi inventae adjicienda, quippe quam pars integralis variationis sponte involvit, hoc in istius problematis solutione accuratius exponere est visum. Cum scilicet in hujusmodi quaestionibus, quae ad formulas integrales reducuntur, perpetuo ad terminos integrationis sit respiciendum, siquidem integrale nihil aliud est nisi summa elementorum a termino dato seu initio ad alium terminum seu finem continuatorum, haec consideratio prorsus essentialis est omni integrationi, sine qua idea valoris integralis ne consistere quidem potest. Quamobrem constitutis integrationis terminis initio scilicet et fine, statim ac variationis pars integralis ita est accepta ut pro initio evadat nulla, tum ejusmodi constantem adjici oportet, ut etiam partes absolutae pro eodem initio destruantur, sicque universa variationis expressio ad nihilum redigatur. Quod cum fuerit factum, ad finem integrationis demum progredi licet, ut hoc pacto vera variatio formulae integralis positae ab initio ad finem extensae obtineatur. Haec autem variationum doctrina ad duplicis generis quaestiones accommodari potest; dum in altero relatio inter variabiles x et y data assumitur, et formulae integralis itidem datae $\int V \partial x$ variatio investigatur, postquam per totam integrationis extensionem variabilibus x et y variationes quaecunque fuerint tributae, in altero autem genere ipsa illa variabilium x et y relatio quaeritur, ut formulae integralis $\int V \partial x$ variatio certa proprietate sit praedita; quemadmodum si ea formula maximum minimumve valorem recipere debeat, hanc variationem in nihilum abire necesse est. Ubi iterum duo casus se offerunt, prout maximum minimumve locum habere debet, vel quaecunque variationes ipsis x et y tribuantur, vel si tantum hae variationes certae cuidam legi adstringantur. Ex quo manifestum est, hanc Theoriam multo latius patere, quam quidem ea adhuc in usum est vocata.

Problema 10.

116. Si functio V praeter binas variabiles x, y, cum suis valoribus differentialibus

$$p = \frac{\partial y}{\partial x}, \quad q = \frac{\partial p}{\partial x}, \quad r = \frac{\partial q}{\partial x}, \quad \text{etc.}$$

etiam duas pluresve formulas integrales

$$v = \int \mathfrak{V} \partial x, \quad v' = \int \mathfrak{V}' \partial x, \quad \text{etc.}$$

involvat, ut sit

$$\partial \mathfrak{V} = \mathfrak{M}\, \partial x + \mathfrak{N}\, \partial y + \mathfrak{P}\, \partial p + \mathfrak{Q}\, \partial q + \mathfrak{R}\, \partial r + \text{etc.}$$
$$\partial \mathfrak{V}' = \mathfrak{M}' \partial x + \mathfrak{N}' \partial y + \mathfrak{P}' \partial p + \mathfrak{Q}' \partial q + \mathfrak{R}' \partial r + \text{etc.}$$

atque differentiali sumto

$$\partial \mathrm{V} = \mathrm{L} \partial v + \mathrm{L}' \partial v' + \mathrm{M} \partial x + \mathrm{N} \partial y + \mathrm{P} \partial p + \mathrm{Q} \partial q + \text{etc.}$$

invenire variationem formulae integralis $\int \mathrm{V} \partial x$.

Solutio.

Si hujus problematis solutio eodem modo instituatur ac praecedentis, mox patebit, calculum a geminata formula integrali

$$v = \int \mathfrak{V} \partial x \quad \text{et} \quad v' = \int \mathfrak{V}' \partial x$$

non turbari, neque etiam si plures ejusmodi involverentur. Quare tota solutio tandem huc redibit, ut constitutis integrationis terminis, primo integralia

$$\int \mathrm{L} \partial x = \mathrm{I} \quad \text{et} \quad \int \mathrm{L}' \partial x = \mathrm{I}'$$

ita sint capienda, ut pro initio integrationis evanescant, tum vero pro fine integrationis fiat $\mathrm{I} = \mathrm{A}$ et $\mathrm{I}' = \mathrm{A}'$; quibus quantitatibus inventis statuatur porro

$$\mathrm{N} + (\mathrm{A} - \mathrm{I})\, \mathfrak{N} + (\mathrm{A} - \mathrm{I}')\, \mathfrak{N}' = \mathrm{N}',$$
$$\mathrm{P} + (\mathrm{A} - \mathrm{I})\, \mathfrak{P} + (\mathrm{A}' - \mathrm{I}')\, \mathfrak{P}' = \mathrm{P}',$$

$$Q + (A - I)\,\mathfrak{Q} + (A' - I')\,\mathfrak{Q}' = Q',$$
$$R + (A - I)\,\mathfrak{R} + (A' - I')\,\mathfrak{R}' = R',$$
etc.

eritque variatio quaesita, dum utrique variabili x et y variationes quaecunque tribuuntur, ex praecedentis solutione:

$$\begin{aligned}\delta\int V\partial x = \int\omega\partial x\,(N' - \tfrac{\partial P'}{\partial x} + \tfrac{\partial\partial Q'}{\partial x^2} - \tfrac{\partial^3 R'}{\partial x^3} + \tfrac{\partial^4 S'}{\partial x^4} - \text{etc.})\\ + V\delta x + \omega\,(P' - \tfrac{\partial Q'}{\partial x} + \tfrac{\partial\partial R'}{\partial x^2} - \tfrac{\partial^3 S'}{\partial x^3} + \text{etc.})\\ + \text{Const.} + \tfrac{\partial\omega}{\partial x}\,(Q' - \tfrac{\partial R'}{\partial x} + \tfrac{\partial\partial S'}{\partial x^2} - \text{etc.})\\ + \tfrac{\partial\partial\omega}{\partial x^2}\,(R' - \tfrac{\partial S'}{\partial x} + \text{etc.})\\ + \tfrac{\partial^3\omega}{\partial x^3}\,(S' - \text{etc.}) + \text{etc.}\end{aligned}$$

ubi commoditatis gratia elementum ∂x constans est assumtum.

Corollarium.

117. Si ergo etiam plures hujusmodi formulae integrales $\int\mathfrak{V}\partial x$ in functionem V quomodocunque ingrediantur, expressio variationis quaesitae inde non mutatur, sed tantum quantitates N′, P′, Q′, R′, etc. ex iis rite definiri convenit.

Scholion.

118. Etsi formulae integrales

$$I = \int L\partial x, \quad I' = \int L'\partial x,$$

binas variabiles involvunt, ideoque valores fixos recipere non posse videntur, tamen perpendendum est, in omnibus hujusmodi quaestionibus semper certam quandam relationem inter binas variabiles x et y supponi, sive ea absolute detur, sive demum per calculum definiri debeat. Hac igitur ipsa relatione jam in usum vocata, ut quantitas y instar functionis ipsius x spectari possit, formulae illae integrales utique determinatos valores sortientur.

Problema 11.

119. Si functio $\mathfrak{V}$ praeter variabiles x et y, earumque valores differentiales p, q, r, s, etc. ipsam quoque formulam integralem $u = \int \mathfrak{v}\partial x$ involvat, ut ejus differentiale sit

$$\partial\mathfrak{V} = \mathfrak{L}\partial u + \mathfrak{M}\partial x + \mathfrak{N}\partial y + \mathfrak{P}\partial p + \mathfrak{Q}\partial q + \mathfrak{R}\partial r + \text{etc.}$$

existente

$$\partial\mathfrak{v} = \mathfrak{m}\partial x + \mathfrak{n}\partial y + \mathfrak{p}\partial p + \mathfrak{q}\partial q + \mathfrak{r}\partial r + \text{etc.}$$

tum vero sit V functio quaecunque ipsarum x, y, p, q, r, etc. insuperque formulae integralis $v = \int \mathfrak{V}\partial x$, ut sit

$$\partial V = L\partial v + M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

invenire variationem formulae integralis $\int V\partial x$.

Solutio.

Ex problemate 9. statim invenimus variationem formulae integralis $\int \mathfrak{V}\partial x = v$; constitutis enim integrationis terminis sumtoque integrali $\int \mathfrak{L}\partial x = \mathfrak{J}$, ita ut evanescente pro integrationis initio, pro fine fiat $\mathfrak{J} = \mathfrak{A}$, tum fiat brevitatis gratia

$$\mathfrak{N} + (\mathfrak{A} - \mathfrak{J})\,\mathfrak{n} = \mathfrak{N}', \quad \mathfrak{P} + (\mathfrak{A} - \mathfrak{J})\,\mathfrak{p} = \mathfrak{P}',$$
$$\mathfrak{Q} + (\mathfrak{A} - \mathfrak{J})\,\mathfrak{q} = \mathfrak{Q}', \text{ etc.}$$

erit ex illius problematis solutione

$$\delta v = \mathfrak{V}\delta x + \int \partial x \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'\partial\omega}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\omega}{\partial x^3} + \text{etc.}\right)$$

posito $\omega = \delta y - p\delta x$ et sumto ∂x constante.

Jam vero cum quaeratur $\delta\int V\partial x$, ob

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x),$$

posito brevitatis ergo

$$\partial V = L\partial v + \partial W \quad \text{et} \quad \delta V = L\delta v + \delta W,$$

ut sit

$$\partial W = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

erit ut ibidem vidimus

$$\delta \int V\partial x = V\delta x + \int (L\partial x\delta v - L\partial v\delta x)$$
$$+ \int \partial x \left(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}\right),$$

ubi si loco ∂v et δv valores modo inventi substituantur, erit

$$\partial x\delta v - \partial v\delta x = \partial x \int \partial x \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'\partial\omega}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$

Nunc ponatur $\int L\partial x = I$, integrali ita sumto ut evanescat in integrationis initio, in fine autem fiat $I = A$, et habebimus

$$\int L(\partial x\delta v - \partial v\delta x) = \int (A - I)\, \partial x \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'\partial\omega}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$

Restituantur pro $\mathfrak{N}'$, $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{R}'$, etc. valores supra assumti, et ad calculum contrahendum ponatur

$$N + (A - I)\,\mathfrak{N} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{n} = N',$$
$$P + (A - I)\,\mathfrak{P} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{p} = P',$$
$$Q + (A - I)\,\mathfrak{Q} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{q} = Q',$$
$$R + (A - I)\,\mathfrak{R} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{r} = R',$$
$$\text{etc.}$$

ac manifestum est, fore variationem quaesitam

$$\delta \int V\partial x = V\delta x + \int \partial x \left(N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.}\right),$$

quae forma porro evolvitur in eandem expressionem, quam sub finem problematis 9. (§. 110.) exhibuimus, quam ergo hic denuo opponere foret superfluum.

Corollarium 1.

120. Hic ergo formula integralis $\int V\partial x$, cujus variationem assignavimus ita est comparata, ut non solum functio V formulam

integralem $\int \mathfrak{V} \partial x$ involvat, sed etiam haec functio $\mathfrak{V}$ aliam formulam integralem $\int \mathfrak{v} \partial x$ in se complectatur; ubi quidem functio $\mathfrak{v}$ nullam amplius formulam integralem implicat.

Corollarium 2.

121. Sin autem et haec functio $\mathfrak{v}$ insuper formulam integralem in se involvat, jam satis perspicuum est, quomodo tum solutionem institui oporteat; siquidem tum valores N', P', Q', R', etc. partes insuper recipient, a postrema formula integrali pendentes.

Scholion.

122. Quomodocunque ergo formula integralis $\int V \partial x$ fuerit complicata, praecepta hactenus exposita omnino sufficiunt ad ejus variationem investigandam, etiamsi forte complicatio fuerit infinita. Cum igitur omnes expressiones binas variabiles implicantes, quarum variationes unquam sint investigandae, vel a formulis integralibus sint liberae, vel unam pluresve in se complectantur, easque vel simplices vel complicatas utcunque, huic Calculi variationum parti, quae circa duas variabiles versatur, abunde satisfactum videtur, ut vix quicquam amplius desiderari queat. Quamobrem ad formulas trium variabilium progrediamur ac primo quidem tales, quarum relatio per geminam aequationem definiri ponitur, ut binae variabiles tanquam functiones tertiae spectari queant, sive haec duplex relatio sit cognita, sive ex ipsa variationis indole investiganda.

CAPUT V.

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM, ET DUPLICEM RELATIONEM IMPLICANTIUM.

Problema 12.

123.

Proposita formula quacunque ternas variabiles x, y, z cum suis differentialibus cujuscunque gradus involvente, ejus variationem definire ex variationibus omnium trium variabilium oriundam.

Solutio.

Sit W formula ista proposita, cujus primo quaeratur valor variatus $W + \delta W$, qui oritur si loco x, y, z scribantur ipsarum valores variati

$$x + \delta x, \quad y + \delta y, \quad z + \delta z,$$

similiterque pro earum differentialibus

$$\partial x + \partial\delta x, \quad \partial y + \partial\delta y, \quad \partial z + \partial\delta z,$$

et ita porro: a quo si ipsa formula W auferatur, remanebit ejus variatio δW. Ex quo intelligitur hanc variationem per consuetam differentiationem obtineri si modo loco signi differentiationis ∂, signum variationes capi oporteat, perinde esse, in quonam loco inter differentiationis signa signum variationis δ collocetur, quemadmodum supra demonstravimus; unde signum variationis perpetuo in postremo loco poni poterit, quod cum ad formulas integrales progrediemur, commodissimum videtur, sicut ex iis quae hactenus de formulis integralibus binas variabiles involventibus sunt tradita, satis est manifestum.

Corollarium 1.

124. Quoniam z perinde ac y tanquam functio ipsius x spectari potest, si ponatur

$$\frac{\partial y}{\partial x} = p \quad \text{et} \quad \frac{\partial z}{\partial x} = \mathfrak{p}, \quad \text{erit}$$

$$\delta p = \frac{\partial \delta y - p \partial \delta x}{\partial x} \quad \text{et} \quad \delta \mathfrak{p} = \frac{\partial \delta z - \mathfrak{p} \partial \delta x}{\partial x},$$

similique modo formulae hinc derivatae a superioribus non discrepant.

Corollarium 2.

125. Ponamus

$$\delta y - p \delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p} \delta x = \mathfrak{w},$$

eritque

$$\partial \delta y - p \partial \delta x - q \partial x \delta x = \partial \omega \quad \text{et} \quad \partial \delta z - \mathfrak{p} \partial \delta x - \mathfrak{q} \partial x \delta x = \partial \mathfrak{w},$$

si scilicet statuamus

$$\frac{\partial p}{\partial x} = q \quad \text{et} \quad \frac{\partial \mathfrak{p}}{\partial x} = \mathfrak{q},$$

unde patet fore

$$\delta p - q \delta x = \frac{\partial \omega}{\partial x} \quad \text{et} \quad \delta \mathfrak{p} - \mathfrak{q} \delta x = \frac{\partial \mathfrak{w}}{\partial x}.$$

Corollarium 3.

126. Si ulterius statuamus

$$\frac{\partial q}{\partial x} = r; \quad \frac{\partial \mathfrak{q}}{\partial x} = \mathfrak{r}; \quad \frac{\partial r}{\partial x} = s; \quad \frac{\partial \mathfrak{r}}{\partial x} = \mathfrak{s} \quad \text{etc.}$$

erit simili modo sumto ∂x constante

$$\delta q - r\delta x = \frac{\partial\partial\omega}{\partial x^2}; \quad \delta \mathfrak{q} - \mathfrak{r}\delta x = \frac{\partial\partial\mathfrak{w}}{\partial x^2},$$

$$\delta r - s\delta x = \frac{\partial^3\omega}{\partial x^3}; \quad \delta \mathfrak{r} - \mathfrak{s}\delta x = \frac{\partial^3\mathfrak{w}}{\partial x^3},$$

sicque deinceps.

Scholion 1.

127. Sive ergo formula varianda habuerit valorem finitum sive infinitum, sive evanescentem, ope horum praeceptorum ejus variatio perinde ac supra inveniri potest, neque enim haec praecepta a superioribus discrepant, nisi quod hic duplicis generis valores differentiales, alteri litteris latinis, p, p, r, s etc. alteri germanicis $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. indicati, introduci debeant, cujus rei ratio in eo est sita, quod hic utraque variabilis y et z tanquam functio ipsius x spectari potest. Sin autem unica aequatio inter ternas coordinatas daretur, vel quaereretur, litterae hic introductae p et $\mathfrak{p}$ nullos habiturae essent valores certos, cum salva illa aequatione fractiones $\frac{\partial y}{\partial x}$ et $\frac{\partial z}{\partial x}$ omnes omnino valores recipere possent. Omissis autem his litteris, ipsisque differentialibus in calculo relictis, etiam pro hoc casu regula in solutione exposita variationem declarabit.

Scholion 2.

128. Supra jam notavi, hunc casum trium variabilium, quarum relatio gemina aequatione definitur, sollicite esse distinguendum ab eo, ubi relatio unica aequatione definiri assumitur. Discrimen hoc ex Geometria clarissime illustratur, ubi ternae variabiles vicem ternarum coordinatarum gerunt; totidem autem in calculo adhiberi oportet non solum quando quaestio circa superficies versatur, sed etiam quando lineae curvae non in eodem plano sitae sunt explorandae. Atque hoc quidem casu posteriori determinatio lineae curvae duas aequationes inter ternas coordinatas postulat, ita ut binae quaevis tanquam functiones tertiae spectari possint.

Superficiei autem natura jam unica aequatione inter ternas coordinatas definitur, ita ut unaquaeque tanquam functio binarum reliquarum spectari queat, unde ingens discrimen in ipsa tractatione oritur. Praesens igitur caput inservire poterit ejusmodi lineis curvis indagandis quae non in eodem plano sitae maximi minimive quapiam gaudeant proprietate.

Problema 13.

129. Si V fuerit functio quaecunque trium variabilium x, y, z, earum insuper differentialia cujusque ordinis implicans, eaeque variabiles variationes quascunque recipiant, invenire variationem formulae integralis $\int V\partial x$.

Solutio.

Quaecunque differentialia in functionem V ingrediantur, ea his factis substitutionibus

$$\partial y = p\partial x;\ \partial p = q\partial x;\ \partial q = r\partial x;\ \partial r = s\partial x \text{ etc.}$$
$$\partial z = \mathfrak{p}\partial x;\ \partial\mathfrak{p} = \mathfrak{q}\partial x;\ \partial\mathfrak{q} = \mathfrak{r}\partial x;\ \partial\mathfrak{r} = \mathfrak{s}\partial x \text{ etc.}$$

tollentur, et quantitas V erit functio quantitatum finitarum x, y, z, p, q, r, s etc. $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. Ejus ergo differentiale hujusmodi habebit formam

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$
$$+ \mathfrak{N}\partial z + \mathfrak{P}\partial\mathfrak{p} + \mathfrak{Q}\partial\mathfrak{q} + \mathfrak{R}\partial\mathfrak{r} + \mathfrak{S}\partial\mathfrak{s} + \text{etc.}$$

unde mutatis signis differentiationis ∂ in δ, simul habebitur variatio δV. Ex supra autem demonstratis etiam pro hoc casu trium variabilium habebitur

$$\delta\int V\partial x = \int(V\,\partial\delta x + \partial x\delta V) = V\delta x + \int(\partial x\delta V - \partial V\delta x).$$

At facta substitutione fiet

$$\frac{\partial x\delta V - \partial V\delta x}{\partial x} = M\,\delta x + N\,\delta y + P\delta p + Q\delta q + R\,\delta r + \text{etc.}$$
$$+ \mathfrak{N}\delta z + \mathfrak{P}\delta\mathfrak{p} + \mathfrak{Q}\delta\mathfrak{q} + \mathfrak{R}\delta\mathfrak{r} + \text{etc.}$$

$$-M\delta x - Np\delta x - Pq\delta x - Qr\delta x - Rs\delta x - \text{etc.}$$
$$-\mathfrak{N}\mathfrak{p}\delta x - \mathfrak{P}\mathfrak{q}\delta x - \mathfrak{Q}\mathfrak{r}\delta x - \mathfrak{R}\mathfrak{s}\delta x - \text{etc.}$$

Quodsi jam brevitas gratia statuamus

$$\delta y - p\delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p}\delta x = \mathfrak{w}$$

sumto elemento ∂x constante, ex §§. 125. et 126. erit

$$\delta p - q\delta x = \frac{\partial\omega}{\partial x}; \quad \delta\mathfrak{p} - \mathfrak{q}\delta x = \frac{\partial\mathfrak{w}}{\partial x};$$
$$\delta q - r\delta x = \frac{\partial\partial\omega}{\partial x^2}; \quad \delta\mathfrak{q} - \mathfrak{r}\delta x = \frac{\partial\partial\mathfrak{w}}{\partial x^2};$$
$$\delta r - s\delta x = \frac{\partial^3\omega}{\partial x^3}; \quad \delta\mathfrak{r} - \mathfrak{s}\delta x = \frac{\partial^3\mathfrak{w}}{\partial x^3};$$

etc.

unde variatio quaesita hoc modo commode exprimetur

$$\delta\int V\partial x = V\delta x + \int\partial x \left\{ \begin{array}{l} N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}\mathfrak{w} + \frac{\mathfrak{P}\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}\partial\partial\mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}\partial^3\mathfrak{w}}{\partial x^3} + \text{etc.} \end{array} \right\}$$

quae ut supra ad hanc formam reducitur

$$\begin{aligned} \delta\int V\partial x = &+ \int\omega\partial x \left(N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.}\right) \\ &+ \int\mathfrak{w}\partial x \left(\mathfrak{N} - \frac{\partial\mathfrak{P}}{\partial x} + \frac{\partial\partial\mathfrak{Q}}{\partial x^2} - \frac{\partial^3\mathfrak{R}}{\partial x^3} + \frac{\partial^4\mathfrak{S}}{\partial x^4} - \text{etc.}\right) \\ &+ V\delta x + \omega\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \frac{\partial^3 S}{\partial x^3} + \text{etc.}\right) \\ &+ \text{Const.} + \mathfrak{w}\left(\mathfrak{P} - \frac{\partial\mathfrak{Q}}{\partial x} + \frac{\partial\partial\mathfrak{R}}{\partial x^2} - \frac{\partial^3\mathfrak{S}}{\partial x^3} + \text{etc.}\right) \\ &+ \frac{\partial\omega}{\partial x}\left(Q - \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} - \text{etc.}\right) \\ &+ \frac{\partial\mathfrak{w}}{\partial x}\left(\mathfrak{Q} - \frac{\partial\mathfrak{R}}{\partial x} + \frac{\partial\partial\mathfrak{S}}{\partial x^2} - \text{etc.}\right) \\ &+ \frac{\partial\partial\omega}{\partial x^2}\left(R - \frac{\partial S}{\partial x} + \text{etc.}\right) \\ &+ \frac{\partial\partial\mathfrak{w}}{\partial x^2}\left(\mathfrak{R} - \frac{\partial\mathfrak{S}}{\partial x} + \text{etc.}\right) \\ &+ \frac{\partial^3\omega}{\partial x^3}\left(S - \text{etc.}\right) \\ &+ \frac{\partial^3\mathfrak{w}}{\partial x^3}\left(\mathfrak{S} - \text{etc.}\right) + \text{etc.} \end{aligned}$$

cujus indoles ex superioribus satis est manifesta, eademque circa constantis additionem sunt observanda.

Corollarium 1.

130. In hac solutione ambae variabiles y et z tanquam functiones ipsius x spectantur, sive jam sint cognitae, sive demum ex variationis indole definiendae. Neque etiam formula integralis $\int V\partial x$ certum esset habitura valorem, nisi tam y quam z per x determinari conciperetur.

Corollarium 2.

131. Si formula $V\partial x$ per se sit integrabilis, nulla assumta relatione inter ternas variabiles, variatio integralis $\int V\partial x$ nullas quoque formulas integrales involvere potest; ideoque necesse est, ut tum sit

$$\text{et } N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

$$\text{et } \mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \frac{\partial^4 \mathfrak{S}}{\partial x^4} - \text{etc.} = 0.$$

Corollarium 3.

132. Vicissim etiam si hae duae aequationes locum habeant, hoc certum erit criterium, formulam differentialem $V\partial x$ per se integrationem admittere, nulla inter variabiles stabilita relatione.

Exemplum.

133. *Quo hoc criterium magis illustremus, sumamus ejusmodi formulam per se integrabilem, sitque*

$$\int V\partial x = \frac{z\partial y}{x\partial z} = \frac{pz}{x\mathfrak{p}},$$

unde fit

$$V = \frac{-pz}{xx\mathfrak{p}} + \frac{p}{x} + \frac{zq}{x\mathfrak{p}} - \frac{zp\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}},$$

Ex cujus differentiatione colligimus $N = 0$, et

$$P = \frac{-z}{xx\mathfrak{p}} + \frac{1}{x} - \frac{z\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}; \quad Q = \frac{z}{x\mathfrak{p}}; \text{ porro}$$

$$\mathfrak{N} = \frac{-p}{xx\mathfrak{p}} + \frac{q}{x\mathfrak{p}} - \frac{pq}{x\mathfrak{p}\mathfrak{p}},$$

$$\mathfrak{P} = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zpq}{x\mathfrak{p}^3}, \quad \text{et} \quad \mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}.$$

Jam pro prima aequatione ob $N = 0$ fieri oportet

$$-\frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} = 0, \quad \text{seu} \quad P - \frac{\partial Q}{\partial x} = \text{Const.}$$

cujus veritas ex differentiatione ipsius Q statim fit perspicua.

Pro altera aequatione

$$\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} = 0,$$

quia hinc est

$$\int \mathfrak{N}\partial x = \mathfrak{P} - \frac{\partial \mathfrak{Q}}{\partial x},$$

primo necesse est ut integrabilis existat haec formula

$$\mathfrak{N}\partial x = \frac{-p\partial x}{xx\mathfrak{p}} + \frac{q\,\partial x}{x\mathfrak{p}} - \frac{p\,\partial x}{x\mathfrak{p}\mathfrak{p}},$$

unde ob $q\partial x = \partial p$ manifesto fit

$$\int \mathfrak{N}\partial x = \frac{p}{x\mathfrak{p}}.$$

Superest ergo ut sit

$$\frac{\partial \mathfrak{Q}}{\partial x} = \mathfrak{P} - \int \mathfrak{N}\partial x = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zpq}{x\mathfrak{p}^3} - \frac{p}{x\mathfrak{p}}.$$

Verum differentiando $\mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}$, utrinque perfecta aequalitas resultat.

Scholion 1.

134. Quodsi ergo quaestio huc redeat, ut formulae integrali $\int V\partial x$ valor maximus minimusve sit conciliandus, tum ante omnia in ejus variatione ambas partes integrales idque seorsim nihilo aequari oportet, propterea quod utcunque variationes constituantur; variatio $\delta\int V\partial x$ semper debeat evanescere, unde duae emergunt aequationes istae

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0 \quad \text{et}$$

$$\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \frac{\partial^4 \mathfrak{S}}{\partial x^4} - \text{etc.} = 0,$$

quibus duplex relatio inter ternas variabiles x, y, z ita exprimitur, ut deinceps tam y quam z recte tanquam functio ipsius x spectari possit. Quando autem hae aequationes sunt differentiales idque altioris gradus, totidem utrinque constantes arbitrariae per integrationes in calculum invehuntur, quoti gradus utraque fuerit differentialis. Has vero constantes deinceps ita definiri oportet, ut conditionibus tam pro initio quam pro fine integrationis formulae $\int V\partial x$ praescriptis satisfiat, quod negotium eo redit, ut praeterea variationis partes absolutae ad nihilum redigantur. Primo scilicet constans ita definiri debet, ut conditionibus pro initio praescriptis satisfiat, ubi quidem ex quaestionis indole particulae

$$\omega, \mathfrak{w}, \frac{\partial\omega}{\partial x}, \frac{\partial\mathfrak{w}}{\partial x}, \frac{\partial\partial\omega}{\partial x^2}, \frac{\partial\partial\mathfrak{w}}{\partial x^2} \text{ etc.}$$

definitos valores sortiri solent. Tum vero cum idem circa finem integrationis usu veniat, ex singulis constantes per integrationem ingressae determinabuntur.

Scholion 2.

135. Plurimum conducet hic observasse, membra, quibus variatio $\delta\int V\partial x$ exprimitur, sponte in duas classes dispesci, in quarum altera litterae tantum eae conspiciuntur, quae ad variabilitatem ipsius y, seu ad ejus habitum respectu x referuntur, idque ita ac si quantitas z constans esset assumta, altera vero classis similes literas a variabilitate ipsius z tantum pendentes continet, quasi quantitas y esset constans. Ex quo colligere licet, si etiam quarta variabilis v accedat, quae ut functio ipsius x quoque spectari queat, tum ad illas duas classes tertiam insuper esse adjiciendam, quae similia membra a variabilitate solius v pendentia complectatur. Quocirca solutio hic data spectari potest, quasi ad quotcunque va-

riabiles extendatur, dummodo tot inter eas aequationes dari concipiantur, ut omnes pro functionibus unius haberi queant. Etsi ergo hoc caput tantum tres variabiles prae se fert, tamen ad quodcunque pertinere est intelligendum, si modo ejusmodi conditiones proponantur, ut tandem per unam reliquae omnes determinentur. Talem autem conditionem formulae integrales hujus formae $\int V\partial x$ necessario involvunt; quotcunque enim variabiles in quantitatem V ingrediantur, expressio $\int V\partial x$ certum valorem definitum omnino obtinere nequit, nisi omnes variabiles tanquam functiones unius x spectari queant. Longe aliter autem est comparata ratio earum formularum integralium, quae ad duas pluresve variabiles a se invicem minime pendentes referuntur.

Problema 14.

136. Si functio V praeter tres variabiles x, y, z, earumque differentialia cujuscunque gradus, insuper involvat formulam integralem $v = \int \mathfrak{V}\partial x$, ubi $\mathfrak{V}$ sit functio quaecunque earundem variabilium x, y, z, cum suis differentialibus, investigare variationem formulae integralis $\int V\partial x$.

Solutio.

Ut species saltem differentialium e calculo tollatur, ponamus ut ante

$$\partial y = p\partial x,\quad \partial p = q\partial x,\quad \partial q = r\partial x,\quad \partial r = s\partial x,\quad \text{etc.}$$
$$\partial z = \mathfrak{p}\partial x,\quad \partial \mathfrak{p} = \mathfrak{q}\partial x,\quad \partial \mathfrak{q} = \mathfrak{r}\partial x,\quad \partial \mathfrak{r} = \mathfrak{s}\partial x,\quad \text{etc.}$$

ac functione V differentiata prodeat

$$\partial V = L\partial v + M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$
$$+ \mathfrak{N}\partial z + \mathfrak{P}\partial \mathfrak{p} + \mathfrak{Q}\partial \mathfrak{q} + \mathfrak{R}\partial \mathfrak{r} + \text{etc.}$$

tum vero ob $\partial v = \mathfrak{V}\partial x$ sit

$$\partial \mathfrak{V} = M'\partial x + N'\partial y + P'\partial p + Q'\partial q + R'\partial r + \text{etc.}$$
$$+ \mathfrak{N}'\partial z + \mathfrak{P}'\partial \mathfrak{p} + \mathfrak{Q}'\partial \mathfrak{q} + \mathfrak{R}'\partial \mathfrak{r} + \text{etc.}$$

ubi ob defectum litterarum iisdem accentu distinctis utor. Hinc autem simul earundem quantitatum V et $\mathfrak{V}$ variationes habentur. Jam cum quaeratur variatio $\delta\int V\partial x$, habebimus primo quidem ut ante

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x),$$

ubi cum valor ipsius V non discrepet a praecedente, nisi quod hic ad ejus differentiale ∂V accedat pars $L\partial v = L\mathfrak{V}\partial x$, et ad variationem δV haec pars $L\partial v = L\,\delta\int\mathfrak{V}\partial x$; etiam variatio quaesita $\delta\int V\partial x$ forma ante inventa exprimetur, si modo ad eam adjiciatur hoc membrum

$$\int L(\partial x\delta\int\mathfrak{V}\partial x - \mathfrak{V}\partial x\delta x) = \int L\partial x(\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x).$$

Quia vero formula integralis $\int\mathfrak{V}\partial x$ eadem est quae in problemate praecedente est tractata, si ut ibi fecimus, statuamus

$$\delta y - p\delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

elemento ∂x constante assumto habebimus

$$\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x = \int\partial x \begin{cases} N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\mathfrak{w}}{\partial x^3} + \text{etc.} \end{cases}$$

Ponamus jam integrale $\int L\partial x = I$, si scilicet ita capiatur, ut pro initio integrationis evanescat, tum vero pro termino finali integrationis fiat $I = A$, quo facto pro tota integrationis extensione erit

$$\int L\partial x(\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x) = \int(A - I)\partial x \begin{cases} N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\mathfrak{w}}{\partial x^2} + \text{etc.} \end{cases}$$

Nunc igitur introducamus sequentes abbreviationes

$$\begin{aligned} N + (A - I)N' &= N^\circ, & \mathfrak{N} + (A - I)\mathfrak{N}' &= \mathfrak{N}^\circ, \\ P + (A - I)P' &= P^\circ, & \mathfrak{P} + (A - I)\mathfrak{P}' &= \mathfrak{P}^\circ, \\ Q + (A - I)Q' &= Q^\circ, & \mathfrak{Q} + (A - I)\mathfrak{Q}' &= \mathfrak{Q}^\circ, \\ R + (A - I)R' &= R^\circ, & \mathfrak{R} + (A - I)\mathfrak{R}' &= \mathfrak{R}^\circ, \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

atque manifestum est variationem quaesitam ita expressam iri

$$\delta \int V \partial x = V \delta x + \int \partial x \left\{ \begin{array}{l} N^\circ \omega + \frac{P^\circ \partial \omega}{\partial x} + \frac{Q^\circ \partial \partial \omega}{\partial x^2} + \frac{R^\circ \partial^3 \omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}^\circ \mathfrak{w} + \frac{\mathfrak{P}^\circ \partial \mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}^\circ \partial \partial \mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}^\circ \partial^3 \mathfrak{w}}{\partial x^3} + \text{etc.} \end{array} \right.$$

quae etiam ut ante evolvitur in hanc formam

$$\begin{aligned} \delta \int V \partial x = & + \int \omega \partial x \left(N^\circ - \frac{\partial P^\circ}{\partial x} + \frac{\partial \partial Q^\circ}{\partial x^2} - \frac{\partial^3 R^\circ}{\partial x^3} + \frac{\partial^4 S^\circ}{\partial x^4} - \text{etc.}\right) \\ & + \int \mathfrak{w} \partial x \left(\mathfrak{N}^\circ - \frac{\partial \mathfrak{P}^\circ}{\partial x} + \frac{\partial \partial \mathfrak{Q}^\circ}{\partial x^2} - \frac{\partial^3 \mathfrak{R}^\circ}{\partial x^3} + \frac{\partial^4 \mathfrak{S}^\circ}{\partial x^4} - \text{etc.}\right) \\ & + V \delta x \qquad + \omega \left(P^\circ - \frac{\partial Q^\circ}{\partial x} + \frac{\partial \partial R^\circ}{\partial x^2} - \frac{\partial^3 S^\circ}{\partial x^3} + \text{etc.}\right) \\ & + \text{Const.} \qquad + \mathfrak{w} \left(\mathfrak{P}^\circ - \frac{\partial \mathfrak{Q}^\circ}{\partial x} + \frac{\partial \partial \mathfrak{R}^\circ}{\partial x^2} - \frac{\partial^3 \mathfrak{S}^\circ}{\partial x^3} + \text{etc.}\right) \\ & \qquad + \frac{\partial \omega}{\partial x} \left(Q^\circ - \frac{\partial R^\circ}{\partial x} + \frac{\partial \partial S^\circ}{\partial x^2} - \text{etc.}\right) \\ & \qquad + \frac{\partial \mathfrak{w}}{\partial x} \left(\mathfrak{Q}^\circ - \frac{\partial \mathfrak{R}^\circ}{\partial x} + \frac{\partial \partial \mathfrak{S}^\circ}{\partial x^2} - \text{etc.}\right) \\ & \qquad + \frac{\partial \partial \omega}{\partial x^2} \left(R^\circ - \frac{\partial S^\circ}{\partial x} + \text{etc.}\right) \\ & \qquad + \frac{\partial \partial \mathfrak{w}}{\partial x^2} \left(\mathfrak{R}^\circ - \frac{\partial \mathfrak{S}^\circ}{\partial x} + \text{etc.}\right) \\ & \qquad + \frac{\partial^3 \omega}{\partial x^3} \left(S^\circ - \text{etc.}\right) \\ & \qquad + \frac{\partial^3 \mathfrak{w}}{\partial x^3} \left(\mathfrak{S}^\circ - \text{etc.}\right) + \text{etc.} \end{aligned}$$

ubi neminem offendat signum nihili litteris suffixum, siquidem non exponentem denotat, sed tantum ad has litteras ab iisdem nude positis distinguendas adhibetur.

Corollarium 1.

137. Si igitur formula integralis $\int V \partial x$ habere debeat valorem maximum vel minimum, variationis inventae bina membra priora statim nihilo aequalia statui oportet, unde duae resultant aequationes differentiales, quibus indefinita relatio utriusque variabilis y et z ad x definitur.

Corollarium 2.

138. Etiamsi hic conditionum, quae forte pro initio et fine integrationis proponantur, nondum ratio habetur, tamen ea jam occulte in calculum ingreditur, quia litterae I et A terminos integrationis respiciunt. Interim tamen eae in ipsa aequationum differentialium tractatione iterum ex calculo expelluntur; dum enim formula integralis $\int L\partial x = I$ eliditur, simul quantitas constans A egreditur.

Corollarium 3.

139. Expeditis autem aequationibus his duabus differentialibus, idque generalissime, ut totidem constantes arbitrariae in calculum invehantur, quot integrationes institui oportuit, tum demum ad conditiones utriusque termini integrationis formulae $\int V\partial x$ est attendendum, quandoquidem hinc ex reliquis variationis membris absolutis illae constantes determinari debent.

Scholion.

140. Solutio hujus problematis ita est comparata ut jam satis sit perspicuum, quemadmodum etiam formulas magis complicatas, veluti si functio V plures formulas integrales involvat, vel si quoque 𝔙 formulas novas integrales complectatur, expediri conveniat. Quin etiam nunc est manifestum, si hujusmodi formulae integrales plures tribus variabiles contineant, quomodo tum variationes inveniri oporteat, atque adeo non solum taediosum sed etiam superfluum foret si copiosius hoc argumentum persequi vellem. Ad partem igitur hujus doctrinae alteram multo abstrusiorem progredior, ubi etiam relationibus inter variabiles constitutis duae pluresve a se invicem minime pendentes in calculo relinquuntur.

CAPUT VI.

DE

VARIATIONE FORMULARUM DIFFERENTIALIUM TRES VARIABILES INVOLVENTIUM, QUARUM RELATIO UNICA AEQUATIONE CONTINETUR.

Problema 15.

141.

Proposita aequatione inter tres variabiles x, y et z, quibus variationes quaecunque δx, δy, δz tribuuntur, definire variationes formularum differentialium primi gradus

$$p = \left(\frac{\partial z}{\partial x}\right) \text{ et } p' = \left(\frac{\partial z}{\partial y}\right).$$

Solutio.

Cum unica aequatio inter tres variabiles dari ponitur, quaelibet earum tanquam functio binarum reliquarum spectari potest. Erit ergo z functio ipsarum x et y, et meminisse hic oportet expressionem $\left(\frac{\partial z}{\partial x}\right) = p$ denotare rationem differentialium ipsarum z et x, si in aequatione illa data hae solae ut variabiles tractentur, tertia y pro constante habita, quod idem de altera formula $\left(\frac{\partial z}{\partial y}\right) = p'$ est tenendum. Simili modo ipsae quoque variationes δx, δy, δz ut functiones infinite parvae binarum variabilium x et y spectari possunt, quoniam si etiam a tertia z penderent, haec ipsa est functio ipsarum x et y; unde simul intelligitur quid istae formulae

$$\left(\frac{\partial \delta z}{\partial x}\right), \left(\frac{\partial \delta z}{\partial y}\right), \text{ item } \left(\frac{\partial \delta x}{\partial x}\right), \left(\frac{\partial \delta x}{\partial y}\right) \text{ et } \left(\frac{\partial \delta y}{\partial x}\right), \left(\frac{\partial \delta y}{\partial y}\right),$$

significent. Cum igitur valor variatus formulae

$$\left(\frac{\partial z}{\partial x}\right) = p \quad \text{sit} \quad p + \delta p = \left(\frac{\partial\,(z + \delta z)}{\partial\,(x + \delta x)}\right),$$

si scilicet hic variabilis y constans sumatur, erit hac conditione observata

$$p + \delta p = \left(\frac{\partial z + \partial \delta z}{\partial x + \partial \delta x}\right) = \left(\frac{\partial z}{\partial x} + \frac{\partial \delta z}{\partial x} - \frac{\partial z \partial \delta x}{\partial x^2}\right),$$

propterea quod variationes δx et δz prae x et z evanescunt. Hinc ergo ob $\left(\frac{\partial z}{\partial x}\right) = p$ habebitur variatio quaesita

$$\delta p = \left(\frac{\partial \delta z}{\partial x}\right) - \left(\frac{\partial z}{\partial x} \cdot \frac{\partial \delta x}{\partial x}\right) = \left(\frac{\partial \delta z}{\partial x}\right) - p\left(\frac{\partial \delta x}{\partial x}\right),$$

quarum formularum significatus, cum tam δz quam δx sint functiones ipsarum x et y, hicque y constans habeatur, per se est manifestus. Simili autem modo reperietur fore

$$\delta p' = \left(\frac{\partial \delta z}{\partial y}\right) - p'\left(\frac{\partial \delta y}{\partial y}\right),$$

ubi jam variabilis x pro constante habetur.

Corollarium 1.

142. Hic omnia ad binas variabiles x et y sunt perducta, atque ut earum functiones spectantur, non solum tertia z, sed etiam omnes tres variationes δx, δy, δz: manifestum autem est, has tres variabiles pro lubitu inter se permutari posse.

Corollarium 2.

143. Sufficit autem his binis formulis pro differentialibus primi gradus uti, quoniam reliquas ad has reducere licet, siquidem sit

$$\left(\frac{\partial x}{\partial z}\right) = \frac{1}{p}, \quad \left(\frac{\partial y}{\partial z}\right) = \frac{1}{p'}, \quad \text{et}$$

$$\left(\frac{\partial y}{\partial x}\right) = \frac{-p}{p'}, \quad \text{et} \quad \left(\frac{\partial x}{\partial y}\right) = \frac{-p'}{p},$$

ubi p et p' sunt functiones binarum x et y.

Corollarium 3.

144. Inventis ergo variationibus harum duarum formularum

$$p = \left(\frac{\partial z}{\partial x}\right) \quad \text{et} \quad p' = \left(\frac{\partial z}{\partial y}\right),$$

reliquarum formularum modo memoratarum variationes hinc facile reperientur. Erit enim

$$\delta\left(\frac{\partial x}{\partial z}\right) = -\frac{\delta p}{pp} = -\frac{1}{pp}\left(\frac{\partial \delta z}{\partial x}\right) + \frac{1}{p}\left(\frac{\partial \delta x}{\partial x}\right),$$

$$\delta\left(\frac{\partial y}{\partial z}\right) = -\frac{\delta p'}{p'p'} = -\frac{1}{p'p'}\left(\frac{\partial \delta z}{\partial y}\right) + \frac{1}{p'}\left(\frac{\partial \delta y}{\partial y}\right),$$

$$\delta\left(\frac{\partial y}{\partial x}\right) = -\frac{\delta p}{p'} + \frac{p\delta p'}{p'p'} = -\frac{1}{p'}\left(\frac{\partial \delta z}{\partial x}\right) + \frac{p}{p'}\left(\frac{\partial \delta x}{\partial x}\right) + \frac{p}{p'p'}\left(\frac{\partial \delta z}{\partial y}\right) - \frac{p}{p'}\left(\frac{\partial \delta y}{\partial y}\right).$$

Scholion 1.

145. Hic ante omnia observo, formulas differentiales certum valorem habere non posse, nisi duo differentialia ita inter se comparentur, ut tertia variabilis, si tres habeantur, seu reliquae omnes, si plures adsint, constantes accipiantur. Ita hoc casu quo inter tres variabiles x, y et z unica aequatio datur, vel saltem dari concipitur, formula $\frac{\partial z}{\partial x}$ nullum plane habet significatum, nisi tertia variabilis y constans sumatur, quam conditionem vinculis includendo hanc formulam innuere consueverunt, etiamsi ea tuto omitti possent, quoniam alioquin ne ullus quidem significatus adesset. Quod quo magis perspicuum reddatur, quaecunque aequatio inter ternas variabiles x, y, z proponatur, ex ea valor ipsius z elici concipiatur, ut z aequetur certae functioni ipsarum x et y, ejusque sumto differentiali prodeat $\partial z = p\partial x + p'\partial y$, ubi iterum p et p' certae erunt functiones ipsarum x et y, idque tales ut sit $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right)$. Sumta nunc y constante fit $\partial z = p\partial x$ seu $p = \left(\frac{\partial z}{\partial x}\right)$, sumta autem x constante prodit $p' = \left(\frac{\partial z}{\partial y}\right)$. Tum vero etiam ma-

nifestum est, sumta z constante fore $\frac{\partial y}{\partial x} = \frac{-p}{p'}$, hujusmodi autem formulas excludi conveniet, quando tam z quam variationes δx, δy, et δz ut functiones ipsarum x et y repraesentamus.

Scholion 2.

Fig. 4. 146. Ex Geometria hoc argumentum multo clarius illustrare licet. Denotent enim tres nostrae variabiles x, y, z ternas coordinatas AX, XY, YZ, inter quas aequatio proposita certam quandam superficiem assignabit, in qua ordinata $YZ = z$ terminabitur, quae utique tanquam certa functio binarum reliquarum $AX = x$ et $XY = y$ spectari potest, ita ut sumtis pro lubitu his binis x et y, tertia $YZ = z$ ex aequatione proposita determinetur. Quodsi jam alia superficies quaecunque concipiatur ab ista infinite parum discrepans, eaque ita cum hac comparetur, ut ejus punctum quodvis z cum propositae puncto Z conferatur, ita tamen ut intervallum Zz sit semper infinite parvum, variationes ita repraesentabuntur, ut sit

$$\delta x = Ax - AX = Xx, \quad \delta y = xy - XY \quad \text{et}$$
$$\delta z = yz - YZ,$$

et cum hae variationes prorsus arbitrio nostro permittantur, neque ullo modo a se invicem pendeant, eae etiam tanquam functiones binarum x et y spectari possunt, idque ita ut nulla a reliquis pendeat, sed unaquaeque pro arbitrio fingi queat. Quin etiam hinc intelligitur, quoniam superficies proxima a proposita diversa esse debet, neutiquam fore

$$\delta z = p\delta x + p'\delta y,$$

siquidem pro superficie proposita fuerit

$$\partial z = p\partial x + p'\partial y,$$

alioquin punctum z foret in eadem superficie, ex quo omnino ternas functiones ipsarum x et y pro variationibus δx, δy et δz ita comparatas esse oportet, ut non sit

$$\delta z = p\delta x + p'\delta y$$

sed potius ab hoc valore quomodocunque discrepet; ubi quidem imprimis notandum est, has functiones ita late patere, ut discontinuae non excludantur, atque adeo pro lubitu variationes tantum in unico puncto vel saltem exiguo spatio constitui queant. Ne autem hic ulli dubio locus relinquatur, probe notandum est, ex eo quod ponimus z ejusmodi functionem ipsarum x et y, ut sit

$$\partial z = p\partial x + p'\partial y,$$

minime sequi fore quoque

$$\delta z = p\delta x + p'\delta y,$$

quemadmodum supra assumsimus, propterea quod hic ipsi z propriam tribuimus variationem neutiquam pendentem a variationibus ipsarum x et y.

Problema 16.

147. Proposita aequatione inter tres variabiles x, y, z, quibus variationes quaecunque δx, δy, δz tribuuntur, investigare variationes formularum differentialium secundi gradus

$$q = \left(\frac{\partial\partial z}{\partial x^2}\right), \quad q' = \left(\frac{\partial\partial z}{\partial x\partial y}\right) \quad \text{et} \quad q'' = \left(\frac{\partial\partial z}{\partial y^2}\right).$$

Solutio.

Hic iterum z spectatur ut functio ipsarum x et y, quarum etiam sunt functiones ternae variationes δx, δy, δz, nullo modo a se invicem pendentes. Quoniam in praecedente problemate posuimus

$$p = \left(\frac{\partial z}{\partial x}\right) \quad \text{et} \quad p' = \left(\frac{\partial z}{\partial y}\right),$$

his formulis in subsidium vocatis habebimus

$$q = \left(\frac{\partial p}{\partial x}\right), \quad q' = \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right), \quad \text{et} \quad q'' = \left(\frac{\partial p'}{\partial y}\right);$$

hicque ratio variationum δp et $\delta p'$ est habenda, quas invenimus

$$\delta p = \left(\frac{\partial \delta z}{\partial x}\right) - p\left(\frac{\partial \delta x}{\partial x}\right) \quad \text{et} \quad \delta p' = \left(\frac{\partial \delta z}{\partial y}\right) - p'\left(\frac{\partial \delta y}{\partial y}\right).$$

Simili ergo modo calculum subducendo reperiemus primo

$$\delta q = \left(\frac{\partial \delta p}{\partial x}\right) - q\left(\frac{\partial \delta x}{\partial x}\right);$$

ubi $\left(\frac{\partial \delta p}{\partial x}\right)$ invenitur si valor δp differentietur posita y constante, ac differentiale per ∂x dividatur, unde oritur

$$\left(\frac{\partial \delta p}{\partial x}\right) = \left(\frac{\partial\partial \delta z}{\partial x^2}\right) - q\left(\frac{\partial \delta x}{\partial x}\right) - p\left(\frac{\partial\partial \delta x}{\partial x^2}\right), \quad \text{ob} \quad q = \left(\frac{\partial p}{\partial x}\right),$$

unde concludimus

$$\delta q = \left(\frac{\partial\partial \delta z}{\partial x^2}\right) - 2q\left(\frac{\partial \delta x}{\partial x}\right) - p\left(\frac{\partial\partial \delta x}{\partial x^2}\right).$$

Eodem modo ob $q' = \left(\frac{\partial p}{\partial y}\right)$, erit

$$\delta q' = \left(\frac{\partial \delta p}{\partial y}\right) - q'\left(\frac{\partial \delta y}{\partial y}\right), \quad \text{at}$$

$$\left(\frac{\partial \delta p}{\partial y}\right) = \left(\frac{\partial\partial \delta z}{\partial x \partial y}\right) - q'\left(\frac{\partial \delta x}{\partial x}\right) - p\left(\frac{\partial\partial \delta x}{\partial x \partial y}\right),$$

ideoque

$$\delta q' = \left(\frac{\partial\partial \delta z}{\partial x \partial y}\right) - q'\left(\frac{\partial \delta x}{\partial x}\right) - q'\left(\frac{\partial \delta y}{\partial y}\right) - p\left(\frac{\partial\partial \delta x}{\partial x \partial y}\right).$$

Alter autem valor $q' = \left(\frac{\partial p'}{\partial x}\right)$ simili modo tractatus praebet

$$\delta q' = \left(\frac{\partial\partial \delta z}{\partial x \partial y}\right) - q'\left(\frac{\partial \delta x}{\partial x}\right) - q'\left(\frac{\partial \delta y}{\partial y}\right) - p'\left(\frac{\partial\partial \delta y}{\partial x \partial y}\right),$$

cujus valoris ab illo discrepantia incommodum involvit mox accuratius examinandum. Ex tertia autem formula $q'' = \left(\frac{\partial p'}{\partial y}\right)$ elicitur

$$\delta q'' = \left(\frac{\partial\partial \delta z}{\partial y^2}\right) - 2q''\left(\frac{\partial \delta y}{\partial y}\right) - p'\left(\frac{\partial\partial \delta y}{\partial y^2}\right).$$

Scholion 1.

148. In originem discrepantiae variationis $\delta q'$ ex gemino valore

$$q' = \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right)$$

natae inquisiturus, observo in his formulis variationem exprimentibus, vel quantitatem x vel quantitatem y pro constanti haberi, prout denominator cujuscunque membri declarat. Verum si quantitatem x constantem manere sumimus, utcunque interea altera y mutabilis existit, natura rei postulat, ut etiam variationes ipsius x nullam mutationem subeant, quod autem secus evenit, si variatio δx quoque a quantitate y pendeat, quod idem de altera variabili y, dum constans ponitur, est tenendum. Ex quo manifestum est, si variationes δx et δy simul ab ambabus variabilibus x et y pendere sumantur, id ipsi hypothesi, qua alterutra perpetuo constans ponitur, adversari. Quamobrem hoc incommodum aliter vitari nequit, nisi statuamus, variationem ipsius x prorsus non ab altera variabili y, neque hujus variationem δy ab altera x pendere. Sin autem δx per solam x, et δy per solam y determinatur, ut sit

$$\text{et } \left(\frac{\partial \delta x}{\partial y}\right) = 0 \quad \text{et} \quad \left(\frac{\partial \delta y}{\partial x}\right) = 0,$$

erit etiam

$$\left(\frac{\partial\partial \delta x}{\partial x \partial y}\right) = 0 \quad \text{et} \quad \left(\frac{\partial\partial \delta y}{\partial x \partial y}\right) = 0$$

sicque ambo illi valores discrepantes pro $\delta q'$ inventi ad consensum perducuntur.

Scholion 2.

149. Omnibus autem dubiis in hac investigatione felicissime occurremus, si soli quantitati z variationes tribuamus, binis reliquis x et y plane invariatis relictis, ita ut sit tam $\delta x = 0$ quam $\delta y = 0$, quo pacto non solum calculo consulitur, sed etiam usus hujus calculi variationum vix restringitur. Quodsi enim superficiem quamcunque cum alia sibi proxima comparamus, nihil impedit, quominus singula proposita superficiei puncta ad ea proxima puncta

referamus, quibus eaedem binae coordinatae x et y respondeant, solaque tertia z variationem patiatur. Quin etiam haec suppositio, cum ad formulas integrales progrediemur, eo magis est necessaria, quandoquidem semper totum negotium ad ejusmodi formulas integrales perducitur, quae duplicem integrationem requirunt, in quarum altera sola x in altera vero sola y ut variabilis tractatur; nisi ergo harum variationes nullae statuantur, maxima incommoda inde in calculum inveherentur; qui cum per se plerumque sit difficillimus, minime consultum videtur, ut ex hac parte difficultates multiplicentur. Quamobrem hanc tractationem ita sum expediturus, ut in posterum perpetuo binis variabilibus x et y nullas plane variationes tribuam, solamque tertiam z variatione quacunque δz augeri assumam, ubi quidem δz ut functionem quamcunque ipsarum x et y sive continuam sive discontinuam sum spectaturus.

Problema 17.

150. Si z fuerit functio quaecunque ipsarum x et y, eique tribuatur variatio δz pariter utcunque ab x et y pendens. investigare variationes formularum omnium differentialium cujuscunque ordinis.

Solutio.

Pro differentialibus primi gradus habentur hae duae formulae

$$p = \left(\frac{\partial z}{\partial x}\right) \quad \text{et} \quad p' = \left(\frac{\partial z}{\partial y}\right),$$

quarum variationes cum x et y nullam variationem pati concipiantur, ex supra inventis ita se habebunt

$$\delta p = \left(\frac{\partial \delta z}{\partial x}\right) \quad \text{et} \quad \delta p' = \left(\frac{\partial \delta z}{\partial y}\right).$$

Pro differentialibus secundi ordinis hae tres formulae habentur

$$q = \left(\frac{\partial \partial z}{\partial x^2}\right), \quad q' = \left(\frac{\partial \partial z}{\partial x \partial y}\right) \quad \text{et} \quad q'' = \left(\frac{\partial \partial z}{\partial y^2}\right),$$

ita ut sit

$$q = \left(\frac{\partial p}{\partial x}\right), \quad q' = \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right) \text{ et } q'' = \left(\frac{\partial p'}{\partial y}\right),$$

quarum variationes ex praecedente problemate ob $\delta x = 0$ et $\delta y = 0$ sunt

$$\delta q = \left(\frac{\partial\partial\delta z}{\partial x^2}\right), \quad \delta q' = \left(\frac{\partial\partial\delta z}{\partial x\partial y}\right), \quad \delta q'' = \left(\frac{\partial\partial\delta z}{\partial y^2}\right).$$

Simili modo si ad differentialia tertii ordinis ascendamus, hae quatuor formulae occurrunt

$$r = \left(\frac{\partial^3 z}{\partial x^3}\right), \quad r' = \left(\frac{\partial^3 z}{\partial x^2\partial y}\right), \quad r'' = \left(\frac{\partial^3 z}{\partial x\partial y^2}\right), \quad r''' = \left(\frac{\partial^3 z}{\partial y^3}\right),$$

quarum variationes ita expressum iri manifestum est

$$\delta r = \left(\frac{\partial^3\delta z}{\partial x^3}\right), \quad \delta r' = \left(\frac{\partial^3\delta z}{\partial x^2\partial y}\right), \quad \delta r'' = \left(\frac{\partial^3\delta z}{\partial x\partial y^2}\right), \quad \delta r''' = \left(\frac{\partial^3\delta z}{\partial y^3}\right),$$

unde per se patet, quomodo variationes formularum differentialium superiorum ordinum sint exprimendae.

Corollarium 1.

151. Hinc jam manifestum est, fore in genere pro formula differentiali cujuscunque ordinis $\left(\dfrac{\partial^{\mu+\nu} z}{\partial x^\mu \partial y^\nu}\right)$ ejus variationem $= \left(\dfrac{\partial^{\mu+\nu}\delta z}{\partial x^\mu \partial y^\nu}\right)$, in qua forma superiores omnes continentur.

Corollarium 2.

152. Deinde etiam perspicuum est, introducendis loco differentialium primi ordinis litteris p, p', secundi ordinis litteris q, q', q'', tertii ordinis litteris r, r', r'', r''', quarti ordinis litteris s, s', s'', s''', s'''', etc. speciem differentialium tolli, quemadmodum etiam supra hujusmodi litteris speciem differentialium sustulimus.

Scholion.

153. Quoniam binae variabiles x et y prorsus a se invicem non pendent, ita ut altera adeo eundem valorem retinere queat, dum altera per omnes valores possibiles variatur, evidens est, hujusmodi formulam differentialem $\frac{\partial y}{\partial x}$, quippe quae nullum plane significatum certum esset habitura, in calculo nunquam locum invenire posse. Contra vero cum quantitas z sit functio ipsarum x et y, hae formulae $(\frac{\partial z}{\partial x})$, $(\frac{\partial z}{\partial y})$ et reliquae omnes quas supra sum contemplatus, definitos habent significatus, neque ullae aliae in calculum ingredi possunt. Deinde quia semper quaestiones huc pertinentes eo reducere licet, ut z tanquam functio binarum x et y spectari possit, ejusmodi formulae $(\frac{\partial y}{\partial x})$, ubi quantitas z esset pro constanti habita, hinc prorsus excluduntur, neque ullae aliae praeter supra memoratas in calculo admitti sunt censendae, sicque omnes expressiones a formulis integralibus liberae praeter ipsas variabiles x, y, z alias formulas differentiales non implicabunt praeter eas, quarum variationes hic sunt indicatae.

Problema 18.

154. Si z sit functio ipsarum x et y, eique tribuatur variatio δz utcunque ab x et y pendens, tum vero fuerit V quantitas quomodocunque ex tribus variabilibus x, y, z earumque differentialibus cujuscunque ordinis composita, ejus variationem δV investigare.

Solutio.

Ut in expressione V species differentialium tollantur, ponamus ut hactenus fecimus

$$p = \left(\frac{\partial z}{\partial x}\right), \quad p' = \left(\frac{\partial z}{\partial y}\right),$$

$$q = \left(\frac{\partial\partial z}{\partial x^2}\right), \quad q' = \left(\frac{\partial\partial z}{\partial x\partial y}\right), \quad q'' = \left(\frac{\partial\partial z}{\partial y^2}\right),$$

$$r = \left(\frac{\partial^3 z}{\partial x^3}\right), \quad r' = \left(\frac{\partial^3 z}{\partial x^2\partial y}\right), \quad r'' = \left(\frac{\partial^3 z}{\partial x\partial y^2}\right), \quad r''' = \left(\frac{\partial^3 z}{\partial y^3}\right),$$

etc.

quarum formularum variationes a variatione ipsius z oriundas ita definimus, ut posita evidentiae gratia ista variatione $\delta z = \omega$, quam ut functionem quamcunque binarum variabilium x et y spectari oportet, sit

$$\delta p = \left(\frac{\partial \omega}{\partial x}\right), \quad \delta p' = \left(\frac{\partial \omega}{\partial y}\right),$$

$$\delta q = \left(\frac{\partial\partial \omega}{\partial x^2}\right), \quad \delta q' = \left(\frac{\partial\partial \omega}{\partial x\partial y}\right), \quad \delta q'' = \left(\frac{\partial\partial \omega}{\partial y^2}\right),$$

$$\delta r = \left(\frac{\partial^3 \omega}{\partial x^3}\right), \quad \delta r' = \left(\frac{\partial^3 \omega}{\partial x^2\partial y}\right), \quad \delta r'' = \left(\frac{\partial^3 \omega}{\partial x\partial y^2}\right), \quad \delta r''' = \left(\frac{\partial^3 \omega}{\partial y^3}\right),$$

etc.

Illis autem factis substitutionibus expressio proposita V fiet functio harum quantitatum x, y, z, p, p', q, q', q'', r, r', r'', r''', etc. Ejus ergo differentiale talem induet formam

$$\begin{aligned}\partial V = L\partial x + M\partial y + N\partial z &+ P\,\partial p + Q\,\partial q + R\,\partial r \\ &+ P'\partial p' + Q'\,\partial q' + R'\,\partial r' \\ &\qquad\quad + Q''\partial q'' + R''\partial r'' \\ &\qquad\qquad\qquad + R'''\partial r'''\end{aligned}$$

etc.

Quoniam nunc formula V eatenus tantum variationem recipit, quatenus quantitates, ex quibus componitur, variantur, binae autem x et y immunes statuuntur, ejus variatio quam quaerimus erit

$$\begin{aligned}\delta V = N\delta z &+ P\,\delta p + Q\,\delta q + R\,\delta r \\ &+ P'\delta p' + Q'\,\delta q' + R'\,\delta r' \\ &\qquad\quad + Q''\delta q'' + R''\,\delta r'' \\ &\qquad\qquad\qquad + R'''\delta r'''\end{aligned}$$

etc.

ac si loco variationis δz scribamus ω, habebimus variationes inventas substituendo

$$\delta V = N\omega + P\left(\frac{\partial\omega}{\partial x}\right) + Q\left(\frac{\partial\partial\omega}{\partial x^2}\right) + R\left(\frac{\partial^3\omega}{\partial x^3}\right)$$
$$+ P'\left(\frac{\partial\omega}{\partial y}\right) + Q'\left(\frac{\partial\partial\omega}{\partial x\partial y}\right) + R'\left(\frac{\partial^3\omega}{\partial x^2\partial y}\right)$$
$$+ Q''\left(\frac{\partial\partial\omega}{\partial y^2}\right) + R''\left(\frac{\partial^3\omega}{\partial x\partial y^2}\right)$$
$$+ R'''\left(\frac{\partial^3\omega}{\partial y^3}\right)$$

etc.

cujus formatio, si forte etiam differentialia altiorum graduum ingrediantur, per se est manifesta.

Corollarium 1.

155. Cum ω spectetur ut functio binarum variabilium x et y, singularum partium, quae variationem δV constituunt, significatus est determinatus, atque haec variatio perfecte definita est censenda.

Corollarium 2.

156. Quomodocunque autem expressio V differentialibus sit referta, quandoquidem valorem certum indicare est censenda, substitutionibus adhibitis semper a specie differentialium liberari debet.

Corollarium 3.

Fig. 6. 157. Si nostrae tres variabiles ad superficiem referantur, ut sint ejus coordinatae $AX = x$, $XY = y$, $YZ = z$, sola ordinata $YZ = z$ ubique incrementum infinite parvum $Zz = \delta z = \omega$ accipere intelligitur, ita ut puncta z cadant in aliam superficiem ab illa infinite parum discrepantem.

Scholion.

158. Dubio hic occurri debet inde oriundo, quod quantitatem z ut functionem binarum x et y spectandam esse diximus: quoniam enim ipsis x et y nullas variationes tribuimus, si in expressione V loco z ejus valor in x et y substitueretur, ea ipsa in meram functionem ipsarum x et y abiret, neque propterea ullam variationem esset receptura. Verum notandum est, tametsi z ut functio ipsarum x et y consideratur, eam tamen plerumque esse incognitam, quando scilicet ejus naturam demum ex conditione variationis erui oportet; sin autem jam ab initio esset data, tamen dum variatio quaeritur, functionem hanc z quasi incognitam spectari convenit, minimeque ejus loco valorem per x et y expressum substitui licet, antequam variatio, quippe quae a sola z pendet, penitus fuerit explorata.

CAPUT VII.

DE

VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM, QUARUM UNA UT FUNCTIO BINARUM RELIQUARUM SPECTATUR.

Problema 19.

159.

Formularum integralium huc pertinentium naturam evolvere, ac rationem qua earum variationes investigari conveniat, explicare.

Solutio.

Cum tres habeantur variabiles x, y et z, quarum una z ut functio binarum reliquarum x et y est spectanda, etiamsi in ipsa variationis investigatione ratio hujus functionis pro incognita haberi debet, formulae integrales quae in hoc calculi genere occurrunt, plurimum discrepant ab iis, quae circa binas tantum variabiles proponi solent. Quemadmodum enim talis forma integralis $\int V\partial x$, ubi V duas variabiles x et y implicare censetur, quarum y ab x pendere concipitur, quasi summa omnium valorum elementarium $V\partial x$ per omnes valores ipsius x collectorum considerari potest; ita quando tres variabiles x, y et z habentur, quarum haec z a binis x et y simul pendere concipitur, integralia huc pertinentia collectionem omnium elementorum ad omnes valores tam ipsius x, quam ipsius y relatorum involvunt, ideoque duplicem integrationem alteram per omnes valores ipsius x, alteram vero ipsius y elementa congregantem requirunt. Ex quo hujusmodi integralia tali forma $\int\int V\partial x\partial y$ contineri debent, qua scilicet duplex integratio innuatur; cujus evolutio ita institui solet, ut primo altera variabilis y ut constans spectetur, et formulae $\int V\partial x$ valor per terminos integrationis extensus quaeratur; in quo cum jam x obtineat valorem vel datum vel ab y pendentem, hoc integrale $\int V\partial x$ in functionem ipsius y tantum abibit, qua in ∂y ducta superest ut integrale $\int\partial y\int V\partial x$ investigetur, quae ergo forma $\int\partial y\int\partial V\delta x$ hoc modo tractata illi $\int\int V\partial x\partial y$ aequivalere est censenda. Ac si ordine inverso primo quantitas x constans accipiatur, et integrale $\int V\partial y$ per terminos praescriptos extendatur, id deinceps ut functio ipsius x spectari et integrale quaesitum $\int\partial x\int V\partial y$ inveniri poterit. Perinde autem est utro modo valorem integralis formulae duplicatae $\int\int V\partial x\partial y$ utamur.

Cum igitur in hoc genere aliae formulae integrales nisi hujusmodi $\int\int V\partial x\partial y$ occurrere nequeant, totum negotium huc redit, ut quemadmodum hujusmodi formae variationem inveniri oporteat, ostendamus. Quoniam autem quantitates x et y variationis expertes assumimus, ex iis quae initio sunt demonstrata facile colligitur fore

$$\delta\int\int V\partial x\partial y = \int\int\delta V\partial x\partial y,$$

ubi δV variationem ipsius V denotat; hicque integratione pariter duplici est opus, prorsus ut modo ante innuimus.

Corollarium 1.

160. Si ponamus integrale $\int\int V\partial x\partial y = W$, cum sit $\int\partial\int x V\partial y = W$, erit per solam x differentiando

$$\int V\partial y = \left(\frac{\partial W}{\partial x}\right),$$

hincque porro per y differentiando $V = (\frac{\partial\partial W}{\partial x \partial y})$; unde patet integrale W ita comparatum esse, ut fiat $V = (\frac{\partial\partial W}{\partial x \partial y})$.

Corollarium 2.

161. Cum duplex integratio sit instituenda, utraque quantitas arbitraria introducitur; altera autem integratio loco constantis functionem quamcunque ipsius x quae sit X, altera autem functionem quamcunque ipsius y, quae sit Y invehit, ita ut completum integrale sit

$$\int\int V \partial x \partial y = W + X + Y.$$

Corollarium 3.

162. Hoc etiam per ipsam resolutionem confirmatur, fit enim primo

$$\int V \partial y = (\frac{\partial W}{\partial x}) + (\frac{\partial X}{\partial x}), \text{ ob } (\frac{\partial Y}{\partial x}) = 0.$$

Tum vero fit $V = (\frac{\partial\partial W}{\partial x \partial y})$, quia neque X neque $\frac{\partial X}{\partial x}$ ab y pendet. Quare si fuerit $(\frac{\partial\partial W}{\partial x \partial y}) = V$, erit integrale completum

$$\int\int V \partial x \partial y = W + X + Y.$$

Scholion 1.

163. Omnino autem necessarium est, ut indoles hujusmodi formularum integralium duplicatarum $\int\int V \partial x \partial y$ accuratius examini subjicietur, quod commodissime per Theoriam superficierum praestari poterit. Sint ergo ut hactenus x et y binae coordinatae or-
Fig. 7. thogonales in basi assumtae, $AX = x$, $XY = y$, cui in Y normaliter insistat tertia ordinata $YZ = z$ ad superficiem usque porrecta. Si jam binae illae coordinatae x et y suis differentialibus crescant $XX' = \partial x$ et $YY' = \partial y$, inde basi oritur parallelogrammum elementare $YxyY' = \partial x \partial y$, cui elementum formulae integralis conve-

nit. Ita si de soliditate a superficie inclusa sit quaestio, ejus elementum erit $= z\partial x \partial y$, ideoque tota soliditas $= \int\int z\partial x \partial y$; si superficies ipsa quaeratur, posito $\partial z = p\partial x + p'\partial y$, erit ejus elementum huic rectangulo $\partial x \partial y$ imminens

$$= \partial x \partial y \sqrt{(1 + pp + p'p')},$$

ideoque ipsa superficies

$$= \int\int \partial x \partial y \sqrt{(1 + pp + p'p')},$$

ex quo generatim intelligitur ratio formulae integralis duplicatae $\int\int V \partial x \partial y$. Quod si jam talis formulae valor quaeratur, qui dato spatio in basi veluti ADYX respondeat, primo sumta x constante investigetur integrale simplex $\int V\partial y$, ac tum ipsi y assignetur magnitudo XY ad curvam DY porrecta, quae ex hujus curvae natura aequabitur certae functioni ipsius x. Sic igitur $\partial x \int V \partial y$ exprimet formulae propositae elementum rectangulo $XYxX' = y\partial x$ conveniens, cujus integrale denuo sumtum $\int \partial x \int V \partial y$ et ex sola variabili x constans, tandem dabit valorem toti spatio ADYX respondentem, siquidem utraque integratio adjectione constantis rite determinetur.

Scholion 1.

164. Ita se habere debet evolutio hujusmodi formularum integralium duplicatarum, si ad figuram in basi datam veluti ADYX fuerit accommodanda; sin autem utramque integrationem indefinite expedire velimus, ut primo sumta x constante quaeramus integrale $\int V\partial y$, quod rectangulo elementari $XYyX' = y\partial x$ convenire est intelligendum, siquidem in ∂x ducatur, deinde vero in integratione formulae $\int \partial x \int V \partial y$ quantitatem $y = XY$ eandem manere concipiamus, sola x pro variabili sumta, tum valor prodibit rectangulo indefinito $APYX = xy$ respondens, si quidem constantes per

utramque integrationem ingressae debite definiantur. At si spatii istius reliqui termini praeter lineas XY et PY ut indefiniti spectentur, integrale $\int\int V \partial x \partial y$ recipiet binas functiones $X + Y$ indefinitas, illam ipsius x, hanc vero ipsius y. Quodsi ergo ad calculum maximorum et minimorum haec deinceps accommodare velimus, quoniam maximi minimive proprietas, quae in spatium quodpiam datum ADYX competere debet, simulquoque cuivis spatio indefinito veluti APYX conveniat necesse est, duplicem illam integrationem modo hic exposito indefinito administrari conveniet.

Problema 20.

165. Si V sit formula quaecunque ex ternis variabilibus x, y, z earumque differentialibus composita, invenire variationem formulae integralis duplicatae $\int\int V \partial x \partial y$, dum quantitati z, quae ut functio binarum x et y spectetur, variationes quaecunque tribuuntur.

Solutio.

Ad speciem differentialium tollendam statuamus

$$p = \left(\frac{\partial z}{\partial x}\right), \quad p' = \left(\frac{\partial z}{\partial y}\right),$$

$$q = \left(\frac{\partial p}{\partial x}\right), \quad q' = \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right), \quad q'' = \left(\frac{\partial p'}{\partial y}\right),$$

$$r = \left(\frac{\partial q}{\partial x}\right), \quad r' = \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial q'}{\partial x}\right), \quad r'' = \left(\frac{\partial q'}{\partial x}\right) = \left(\frac{\partial q''}{\partial x}\right), \quad r''' = \left(\frac{\partial q''}{\partial y}\right),$$

ut V fiat functio quantitatum finitarum x, y, z, p, p', q, q', q'', r, r', r'', r''', etc. Tum ponatur ejus differentiale

$$\begin{aligned} \partial V = L\partial x + M\partial y + N\partial z + P\,\partial p &+ Q\,\partial q + R\,\partial r \\ P'\partial p' &+ Q'\,\partial q' + R'\,\partial r' \\ &+ Q''\partial q'' + R''\partial r'' \\ &\qquad\qquad + R'''\partial r''' \\ &\text{etc.} \end{aligned}$$

ex quo cum simul ejus variatio δV innotescat, ex problemate praecedente colligitur variatio quaesita

$$\delta \int\int V \partial x \partial y = \int\int \partial x \partial y \left\{ \begin{array}{l} N\delta z + P\,\delta p + Q\,\delta q + R\,\delta r + \text{etc.} \\ \quad + P'\delta p' + Q'\,\delta q' + R'\,\delta r' \\ \quad\quad + Q''\delta q'' + R''\,\delta r'' \\ \quad\quad\quad + R'''\delta r''' \\ \quad\quad \text{etc.} \end{array} \right.$$

Quodsi jam uti §. 154. fecimus, ponamus variationem $\delta z = \omega$, quam ut functionem quamcunque binarum variabilium x et y spectare licet, indidem istam variationem concludimus fore

$$\delta \int\int V \partial x \partial y = \int\int \partial x \partial y \left\{ \begin{array}{l} N\omega + P\left(\frac{\partial \omega}{\partial x}\right) + Q\left(\frac{\partial \partial \omega}{\partial x^2}\right) + R\left(\frac{\partial^3 \omega}{\partial x^3}\right) + \text{etc.} \\ \quad + P'\left(\frac{\partial \omega}{\partial y}\right) + Q'\left(\frac{\partial \partial \omega}{\partial x \partial y}\right) + R'\left(\frac{\partial^3 \omega}{\partial x^2 \partial y}\right) \\ \quad\quad + Q''\left(\frac{\partial \partial \omega}{\partial y^2}\right) + R''\left(\frac{\partial^3 \omega}{\partial x \partial y^2}\right) \\ \quad\quad\quad + R'''\left(\frac{\partial^3 \omega}{\partial y^3}\right) \\ \quad\quad \text{etc.} \end{array} \right.$$

Corollarium 1.

166. Si ergo utriusque functionis z et $\delta z = \omega$ indoles, seu ratio compositionis ex binis variabilibus x et y esset data, tum per praecepta ante exposita variatio formulae integralis duplicatae $\int\int V \partial x \partial y$ assignari posset; quomodocunque quantitas V ex variabilibus x, y, z earumque differentialibus fuerit conflata.

Corollarium 2.

167. Totum scilicet negotium redibit ad evolutionem formulae integralis duplicatae inventae, quae cum pluribus constet partibus, singulas partes per duplicem integrationem, uti ante explicatum, tractari conveniet.

Scholion.

168. Quando autem ratio functionis z non constat, eaque demum ex conditione variationis elici debet, ita ut ipsa variatio $\delta z = \omega$ nullam plane determinationem patiatur, quemadmodum fit si formula $\int\int V \partial x \partial y$ valorem maximum minimumve obtinere debeat; tum omnino necessarium est, ut singula variationis inventae $\delta\int\int V\partial x\partial y$ membra ita reducantur, ut ubique post signum integrationis duplicatum non valores differentiales variationis $\delta z = \omega$ sed haec ipsa variatio occurrat; cujusmodi reductione jam supra in formulis binas tantum variabiles involventibus sumus usi. Talis autem reductio, cum pro formulis integralibus duplicatis minus sit consueta, accuratiorem explicationem postulat. Quem in finem observo, hujusmodi reductione perveniri ad formulas simpliciter integrales, in quibus alterutra tantum quantitatum x et y pro variabili habeatur, altera ut constante spectata, ad quod indicandum, ne signa praeter necessitatem multiplicentur, talis forma $\int T\partial x$ denotabit integralae formulae differentialis $T\partial x$, dum quantitas y pro constanti habetur; similique modo intelligendum est in hac forma $\int T\partial y$ solam quantitatem y ut variabilem considerari, quod eo magis perspicuum est, cum hac conditione omissa, hae formulae nullum plane significatum essent habiturae. Neque ergo in posterum opus est declarari, si T ambas variabiles x et y complectatur, utra earum in formulis integralibus simplicibus $\int T\partial x$ vel $\int T\partial y$, sive variabilis accipiatur, cum ea sola, cujus differentiale exprimitur, pro variabili sit habenda. In formulis autem duplicatis $\int\int V\partial x\partial y$ perpetuo tenendum est, alteram integrationem ad solius x, alteram vero ad solius y variabilitatem adstringi, perindeque esse, utra integratio prior instituatur.

Problema 21.

169. Variationem formulae integralis duplicatae $\int\int V\partial x\partial y$,

in praecedente problematae inventam, ita transformare, ut post signum integrale duplicatum ubique ipsa variatio $\delta z = \omega$ occurrat, exturbatis ejus differentialibus.

Solutio.

Quo haec transformatio latius pateat, sint T et v functiones quaecunque binarum variabilium x et y, et consideretur haec formula duplicata $\int\int T\,\partial x\,\partial y\,(\frac{\partial v}{\partial x})$, quae separata integrationum varietate ita repraesentetur $\int\partial y\int T\,\partial x\,(\frac{\partial v}{\partial x})$, ut in integratione $\int T\partial x\,(\frac{\partial v}{\partial x})$ sola quantitas x ut variabilis spectetur. Tum autem erit $\partial x(\frac{\partial v}{\partial x}) = \partial v$, quia y pro constante habetur, ideoque fiet

$$\int T\partial v = Tv - \int v\partial T,$$

ubi cum in differentiali ∂T solius variabilis x ratio habetur, ad hoc declarandum loco ∂T scribi convenit $\partial x\,(\frac{\partial T}{\partial x})$, ita ut sit

$$\int T\partial x\,(\frac{\partial v}{\partial x}) = Tv - \int v\partial x\,(\frac{\partial T}{\partial x}),$$

hincque nostra formula ita prodeat reducta

$$\int\int T\partial x\partial y\,(\frac{\partial v}{\partial x}) = \int Tv\partial y - \int\int v\partial x\partial y\,(\frac{\partial T}{\partial x}).$$

Simili modo permutatis variabilibus consequemur

$$\int\int T\partial x\partial y\,(\frac{\partial v}{\partial y}) = \int Tv\partial x - \int\int v\partial x\partial y\,(\frac{\partial T}{\partial y}).$$

Hoc jam quasi lemmate praemisso, variationis in praecedente problemate inventae reductio ita se habebit

$$\int\int P\,\partial x\partial y\,(\frac{\partial \omega}{\partial x}) = \int P\,\omega\partial y - \int\int \omega\partial x\partial y\,(\frac{\partial P}{\partial x}),$$

$$\int\int P'\partial x\partial y\,(\frac{\partial \omega}{\partial y}) = \int P'\omega\partial x - \int\int \omega\partial x\partial y\,(\frac{\partial P'}{\partial y}).$$

Porro pro sequentibus membris sit primo $(\frac{\partial \omega}{\partial x}) = v$, ideoque $(\frac{\partial\partial \omega}{\partial x^2}) = (\frac{\partial v}{\partial x})$, unde colligitur

$$\int\int Q\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\int\partial x\partial y\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\omega}{\partial x}\right),$$

ac postremo membro similiter reducto, fit

$$\int\int Q\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q}{\partial x}\right)+\int\int\omega\partial x\partial y\left(\frac{\partial\partial Q}{\partial x^2}\right).$$

Per eandem substitutionem habebimus $\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\left(\frac{\partial v}{\partial y}\right)$, hincque

$$\int\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)-\int\int\partial x\partial y\left(\frac{\partial\omega}{\partial x}\right)\left(\frac{\partial Q'}{\partial y}\right),\quad\text{seu}$$

$$\int\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q'}{\partial y}\right)+\int\int\omega\partial x\partial y\left(\frac{\partial\partial Q'}{\partial x\partial y}\right),$$

quae forma ob

$$\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)=Q'\omega-\int\omega\partial x\left(\frac{\partial Q'}{\partial x}\right),$$

abit in hanc

$$\int\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=Q'\omega-\int\omega\partial x\left(\frac{\partial Q'}{\partial x}\right)+\int\int\omega\partial x\partial y\left(\frac{\partial\partial Q'}{\partial x\partial y}\right),$$
$$-\int\omega\partial y\left(\frac{\partial Q'}{\partial x}\right)$$

tum vero pro tertia forma hujus ordinis nanciscimur

$$\int\int Q''\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q''\partial x\left(\frac{\partial\omega}{\partial y}\right)-\int\omega\partial x\left(\frac{\partial Q''}{\partial y}\right)+\int\int\omega\partial x\partial y\left(\frac{\partial\partial Q''}{\partial y^2}\right).$$

Porro ob $\left(\frac{\partial^3\omega}{\partial x^3}\right)=\left(\frac{\partial\partial v}{\partial x^2}\right)$, manente $v=\left(\frac{\partial\omega}{\partial x}\right)$, fiet

$$\int\int R\partial x\partial y\left(\frac{\partial\partial v}{\partial x^2}\right)=\int R\partial y\left(\frac{\partial v}{\partial x}\right)-\int v\partial y\left(\frac{\partial R}{\partial x}\right)+\int\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)\quad\text{et}$$

$$\int\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)=\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R}{\partial x^3}\right),$$

ita ut sit

$$\int\int R\partial x\partial y\left(\frac{\partial^3\omega}{\partial x^3}\right)=\int R\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)-\int\partial y\left(\frac{\partial\omega}{\partial x}\right)\left(\frac{\partial R}{\partial x}\right)+\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)$$
$$-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R}{\partial x^3}\right).$$

Deinde ob $\left(\frac{\partial^3\omega}{\partial x^2\partial y}\right)=\left(\frac{\partial\partial v}{\partial x\partial y}\right)$, erit

$$\int\int R'\partial x\partial y\left(\frac{\partial\partial v}{\partial x\partial y}\right)=R'v-\int v\partial x\left(\frac{\partial R'}{\partial x}\right)+\int\int v\partial x\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)$$
$$-\int v\partial y\left(\frac{\partial R'}{\partial y}\right),$$

et quia hic

$$\int\!\!\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x\partial y}\right)=\int\omega\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right),$$

concludimus fore

$$\int\!\!\int R'\partial x\partial y\left(\frac{\partial^3\omega}{\partial x^2\partial y}\right)=R'\left(\frac{\partial\omega}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial x\left(\frac{\partial R'}{\partial x}\right)+\int\omega\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right).$$

Tandem permutandis x et y hinc colligimus

$$\int\!\!\int R''\partial x\partial y\left(\frac{\partial^3\omega}{\partial x\partial y^2}\right)=R''\left(\frac{\partial\omega}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial y\left(\frac{\partial R''}{\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R''}{\partial x\partial y}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R''}{\partial x}\right)-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R'''}{\partial x\partial y^2}\right)\text{ et}$$

$$\int\!\!\int R'''\partial x\partial y\left(\frac{\partial^3\omega}{\partial y^3}\right)=\int R'''\partial x\left(\frac{\partial\partial\omega}{\partial y^2}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R'''}{\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R'''}{\partial y^2}\right)$$
$$-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R'''}{\partial y^3}\right).$$

Quos valores si substituamus, reperimus

$$\delta\int\!\!\int V\partial x\partial y=\int\!\!\int\omega\partial x\partial y\left\{\begin{array}{l}N-\left(\frac{\partial P}{\partial x}\right)+\left(\frac{\partial\partial Q}{\partial x^2}\right)-\left(\frac{\partial^3 R}{\partial x^3}\right)+\text{etc.}\\ -\left(\frac{\partial P'}{\partial y}\right)+\left(\frac{\partial\partial Q'}{\partial x\partial y}\right)-\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right)\\ \qquad\qquad+\left(\frac{\partial\partial Q''}{\partial y^2}\right)-\left(\frac{\partial^3 R''}{\partial x\partial y^2}\right)\\ \qquad\qquad\qquad\qquad-\left(\frac{\partial^3 R'''}{\partial y^3}\right)\end{array}\right.$$

$$+\int P\omega\partial y+\int Q\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q}{\partial x}\right)+Q'\omega$$
$$+\int P'\omega\partial x-\int\omega\partial x\left(\frac{\partial Q'}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q'}{\partial y}\right)$$
$$+\int Q''\partial x\left(\frac{\partial\omega}{\partial y}\right)-\int\omega\partial x\left(\frac{\partial Q''}{\partial y}\right)$$
$$+\int R\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)+R'\left(\frac{\partial\omega}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial x\left(\frac{\partial R'}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)+\int R'''\partial x\left(\frac{\partial\partial\omega}{\partial y^2}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R}{\partial x}\right)+R''\left(\frac{\partial\omega}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R''}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R'''}{\partial y}\right)$$
$$+\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)+\int\omega\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R''}{\partial x\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R'''}{\partial y^2}\right).$$

Corollarium 1.

170. Hujus expressionis pars prima satis est perspicua,

reliquae vero partes commodè ita disponi possunt, ut earum ratio comprehendatur

$$\int \omega \partial y \left\{ \begin{array}{l} P - (\frac{\partial Q}{\partial x}) + (\frac{\partial \partial R}{\partial x^2}) \\ \quad - (\frac{\partial Q'}{\partial y}) + (\frac{\partial \partial R'}{\partial x \partial y}) \text{ etc.} \\ \qquad\qquad + (\frac{\partial \partial R''}{\partial y^2}) \end{array} \right\} + \int \omega \partial x \left\{ \begin{array}{l} P' - (\frac{\partial Q''}{\partial y}) + (\frac{\partial \partial R'''}{\partial y^2}) \\ \quad - (\frac{\partial Q'}{\partial x}) + (\frac{\partial \partial R''}{\partial x \partial y}) \text{ etc.} \\ \qquad\qquad + (\frac{\partial \partial R'}{\partial x^2}) \end{array} \right\}$$

$$+ \int (\frac{\partial \omega}{\partial x}) \partial y \left\{ \begin{array}{l} Q - (\frac{\partial R}{\partial x}) \text{etc.} \\ \quad - (\frac{\partial R'}{\partial y}) \end{array} \right\} + \int (\frac{\partial \omega}{\partial y}) \partial x \left\{ \begin{array}{l} Q'' - (\frac{\partial R''}{\partial y}) \text{ etc.} \\ \quad - (\frac{\partial R'}{\partial x}) \end{array} \right\}$$

$$+ \int (\frac{\partial \partial \omega}{\partial x^2}) \partial y \, (R - \text{etc.}) + \int (\frac{\partial \partial \omega}{\partial y^2}) \partial x \, (R''' - \text{etc.})$$

$$+ \omega \left\{ \begin{array}{l} Q' - (\frac{\partial R'}{\partial x}) \text{ etc.} \\ \quad - (\frac{\partial R''}{\partial y}) \end{array} \right\} \begin{array}{l} + (\frac{\partial \omega}{\partial x}) (R' - \text{etc.}) \\ + (\frac{\partial \omega}{\partial y}) (R'' - \text{etc.}). \end{array}$$

Corollarium 2.

171. Hic levi attentione adhibita mox patebit, quomodo istae partes ulterius continuari debeant, si forte quantitas V differentialia altiorum graduum complectatur.

Corollarium 3.

172. In harum formularum integralium aliis, quae differentiali ∂y sunt affectae, quantitas x constans sumitur, cui tribuitur valor termino integrationis conveniens; aliis vero quae differentiali ∂x sunt affectae, y est constans et termino integrationis aequalis, unde patet in terminis integrationum tam x quam y recipere valorem constantem.

Scholion.

173. Haec ergo variationis formula ad eum casum est accommodata, quo utriusque integrationis termini tribuunt tam ipsi x quam ipsi y valores constantes. Veluti si de superficie fuerit
Fig. 7. quaestio, formula integralis $\int\int V \partial x \partial y$ ad rectangulum APYX in

basi assumtum est referenda; ejusque valor ita definiri debet, ut sumtis $x = 0$ et $y = 0$, qui sunt valores initiales, evanescat, quo facto statui oportet $x = AX$ et $y = AP$, qui sunt valores finales; atque ad eandem legem ipsa variatio inventa est expedienda. Quodsi jam ea quaeratur superficies, in qua formulae $\int\int V\partial x\partial y$ hoc modo definitae valor fiat maximus vel minimus, ante omnia necesse est, ut pars variationis prima duplicem integrationem involvens ad nihilum redigatur, quomodocunque variatio $\delta z = \omega$ accipiatur, unde haec nascetur aequatio

$$\begin{aligned} 0 = N &- \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial\partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \text{etc.} \\ &- \left(\frac{\partial P'}{\partial y}\right) + \left(\frac{\partial\partial Q'}{\partial x\partial y}\right) - \left(\frac{\partial^3 R'}{\partial x^2\partial y}\right) \\ &+ \left(\frac{\partial\partial Q''}{\partial y^2}\right) - \left(\frac{\partial^3 R''}{\partial x\partial y^2}\right) \\ &- \left(\frac{\partial^3 R'''}{\partial y^3}\right) \end{aligned}$$

qua natura superficiei hac indole praeditae exprimetur. Constantes autem per duplicem integrationem ingressae ita determinari debent, ut reliquis variationis partibus satisfiat.

Scholion 2.

174. Quo haec investigatio in se maxime abstrusa exemplo illustretur, ponamus ejusmodi superficiem investigari debere, quae inter omnes alias eandem soliditatem includentes sit minima. Hunc in finem efficiendum est ut haec formula integralis duplicata

$$\int\int \partial x\partial y\, [z + a\sqrt{(1 + pp + p'p')}],$$

maximum minimumve evadat. Cum ergo sit

$$V = z + a\sqrt{(1 + pp + p'p')},\ \text{erit}$$
$$L = 0,\ M = 0,\ N = 1,$$

atque

$$P = \frac{ap}{\sqrt{(1 + pp + p'p')}} \quad\text{et}\quad P' = \frac{ap'}{\sqrt{(1 + pp + p'p')}},$$

ideoque

$$\partial V = N\partial z + P\partial p + P'\partial p',$$

existente

$$\partial z = p\partial x + p'\partial y.$$

Quare superficiei quaesitae natura hac aequatione exprimetur

$$N - \left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P'}{\partial y}\right) = 0\,, \quad \text{seu} \quad 1 = \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P'}{\partial y}\right).$$

Est vero

$$\left(\frac{\partial P}{\partial x}\right) = \frac{a}{(1 + pp + p'p')^{\frac{3}{2}}}\left[(1 + p'p')\left(\frac{\partial p}{\partial x}\right) - pp'\left(\frac{\partial p'}{\partial x}\right)\right].$$

$$\left(\frac{\partial P'}{\partial y}\right) = \frac{a}{(1 + pp + p'p')^{\frac{3}{2}}}\left[(1 + pp)\left(\frac{\partial p'}{\partial y}\right) - pp'\left(\frac{\partial p}{\partial y}\right)\right].$$

ubi notetur esse $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right)$. Ex quo ista obtinetur aequatio

$$\frac{(1 + pp + p'p')^{\frac{3}{2}}}{a} = (1 + p'p')\left(\frac{\partial p}{\partial x}\right) - 2pp'\left(\frac{\partial p}{\partial y}\right) + (1 + pp)\left(\frac{\partial p'}{\partial y}\right),$$

quam autem quomodo tractari oporteat, haud patet, etiamsi facile perspiciatur, in ea aequationem pro superficie sphaerica

$$zz = cc - xx - yy,$$

quin etiam cylindrica $zz = cc - yy$ contineri.

SUPPLEMENTUM

CONTINENS

EVOLUTIONEM CASUUM SINGULARIUM CIRCA INTEGRATIONEM

AEQUATIONUM DIFFERENTIALIUM.

EVOLUTIO

CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM.

1.

Cum adhuc plurimae atque inter se maxime discrepantes methodi sint in medium allatae, aequationes differentiales integrandi, quaestio exoritur summi sane momenti, an non unica detur eaque aequabilis methodus, cujus ope omnes illae diversae aequationes differentiales, quas etiamnum resolvere licuit, integrari queant? nullum enim est dubium quin inventio talis methodi maxima incrementa in universam Analysin esset allatura. Pluribus Geometris quidem separatio binarum variabilium hujusmodi methodum suppeditare est visa, cum omnes aequationum differentialium integrationes vel hac ratione sint integratae, vel eo facile possint revocari. Praeterquam autem quod haec methodus substitutionibus absolvitur, quae plerumque non minorem sagacitatem postulant, quam id ipsum quod quaeritur, ac nonnunquam soli casui deberi videntur, haec methodus etiam neutiquam extenditur ad aequationes differentiales secundi altiorumque graduum; et qui tales aequationes adhuc tractaverunt, longe alia artificia in subsidium vocare sunt coacti. Quamobrem separationem variabilium nequaquam tanquam methodum uniformem ac latissime patentem spectare licet, quae omnes integrationes, quae adhuc successerunt, in se complectatur.

2. Talem autem methodum universalem jam pridem mihi equidem indicasse videor, dum ostendi proposita quacunque aequatione differentiali sive primi sive altioris gradus, semper dari ejusmodi quantitatem, per quam si aequatio multiplicetur, evadat integrabilis, ita ut hoc modo nulla plane substitutione alibi anxie quaerenda sit opus. Ex quo non dubito, hanc methodum aequationes differentiales ope multiplicationum ad integrabilitatem revocandi, tanquam latissime patentem atque naturae maxime convenientem pronunciare; cum nulla integratio adhuc sit expedita, quae hoc modo non facile absolvi possit. Cum scilicet omnis aequatio differentialis primi gradus in hac forma $P\partial x + Q\partial y = 0$ contineatur, denotantibus litteris P et Q functiones quascunque binarum variabilium x et y, semper datur ejusmodi multiplicator M itidem functio quaedam ambarum variabilium x et y, ut facta multiplicatione haec forma $MP\partial x + MQ\partial y$ fiat integrabilis; cujus propterea integrale quantitati constanti arbitrariae aequatum exhibebit aequationem integralem aequationis differentialis propositae $P\partial x + Q\partial y = 0$, quae eadem ratio quoque in aequationibus differentialibus altiorum graduum locum habet. Verum hoc argumentum hic fusius exponere non est animus; sed potius praestantiam hujus methodi prae separatione variabilium etiam ejusmodi casibus quibus id minime videatur, simulque summam ejus utilitatem hic declarare constitui.

3. Quoties scilicet in aequatione differentiali variabiles x et y jam sunt separatae, totum negotium vulgo ut jam confectum spectari solet, quandoquidem hujus aequationis

$$X\partial x + Y\partial y = 0,$$

ubi X denotat functionem solius x et Y solius y, integrale in promtu est

$$\int X \partial x + \int Y \partial y = \text{Const.}$$

Interim tamen saepenumero usu venire potest, ut hoc pacto neutiquam forma integralis simplicissima obtineatur, vel ea demum per plures ambages inde derivari debeat. Veluti ex hac aequatione

$$\frac{\partial x}{x} + \frac{\partial y}{y} = 0,$$

primo elicitur integrale logarithmicum

$$lx + ly = la,$$

unde quidem statim se prodit algebraicum $xy = a$. Verum ex hac forma

$$\frac{\partial x}{aa + xx} + \frac{\partial y}{aa + yy} = 0,$$

integratio solita praebet

$$\text{Ang. tang. } x + \text{Ang. tang. } y = \text{Const.}$$

unde non tam facile forma integralis algebraica $\frac{x + y}{aa - xy} = C$ deducitur. Ac proposita hac forma

$$\frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma xx)}} + \frac{\partial y}{\sqrt{(\alpha + \beta y + \gamma yy)}} = 0,$$

in genere ne patet quidem, utrum utraque pars integralis arcu circulari an logarithmo exprimatur. Interim tamen ejus integrale ita algebraice exhiberi potest

$$CC(x + y)^2 + 2\gamma Cxy + \beta C(x + y) + 2\alpha C + \tfrac{1}{4}\beta\beta - \alpha\gamma = 0,$$

quae certe forma simplicissima nonnisi per plures ambages ex integrali transcendente derivatur.

4. His quidem casibus perspicitur, quomodo reductionem ad formam algebraicam instituí oporteat, sed ante aliquot annos ejusmodi integrationes protuli, in quibus ne hoc quidem ullo modo praestari potest. Veluti si proposita sit haec aequatio

$$\frac{\partial x}{\sqrt{(1 + x^4)}} + \frac{\partial y}{\sqrt{(1 + y^4)}} = 0,$$

integrationem neque per logarithmos neque arcus circulares expedire licet, ut inde deinceps simili ratione aequatio algebraica colligi posset: interim tamen ostendi hujus integrale idque adeo completum hoc modo algebraice exprimi

$$0 = 2C + (CC - 1)(xx + yy) - 2(1 + CC)xy + 2Cxxyy,$$

ubi C denotat constantem per integrationem ingressam. Quin etiam hujus aequationis multo latius patentis

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y + \varepsilon y^4)}} = 0$$

integrale completum est

$$\begin{aligned} 0 = 2\alpha C &+ \beta\beta - \alpha\gamma + 2(\beta C - \alpha\delta(x+y) + (CC - \alpha\varepsilon)(xx+yy)) \\ &+ 2(\gamma C - CC - \alpha\varepsilon - \beta\delta)xy + 2(\delta C - \beta\varepsilon)xy(x+y) \\ &+ (2\varepsilon C + \delta\delta - \gamma\varepsilon)xxyy, \end{aligned}$$

denotante C item constantem quantitatem arbitrariam per integrationem inventam. His igitur casibus perspicuum est separationem variabilium, qua aequationes differentiales sunt praeditae, nihil plane juvare ad integralia earum forma algebraica contenta eruenda, ex quo merito ejusmodi methodus desideratur, cujus beneficio haec integralia statim ex aequationibus differentialibus investigari potuissent, in quo negotio certe omnes ingenii vires tentasse non pigebit.

5. Observavi igitur hunc scopum ope multiplicatorum idoneorum obtineri posse, quibus aequationes differentiales multiplicatae ita integrabiles evadant, ut integralia statim algebraice expressa prodeant. Quod quo clarius perspiciatur ab aequatione primum proposita $\frac{\partial x}{x} + \frac{\partial y}{y} = 0$ exordiar, quae per xy multiplicata statim praebet $y\partial x + x\partial y = C$. Hoc ergo modo sublata separatione aequatio in aliam transformatur, quae integrationem admittit, ex quo intelligitur methodum ope multiplicatorum integrandi id

praestare, quod a separatione variabilium immediate exspectari nequeat. Idem evenit in aequatione $\frac{m\partial x}{x}+\frac{n\partial y}{y}=0$, quae per $x^m y^n$ multiplicata integrale praebet $x^m y^n = C$, dum ex ipsa aequatione proposita statim ad logarithmos fuisset perventum. Simili modo si haec aequatio separata

$$\frac{\partial x}{1+xx}+\frac{\partial y}{1+yy}=0$$

multiplicetur in $\frac{(1+xx)(1+yy)}{(x+y)^2}$, aequatio resultans

$$\frac{\partial x(1+yy)+\partial y(1+xx)}{(x+y)^2}=0$$

integrationem jam sponte admittit, praebetque integrata

$$\frac{-1+xy}{x+y}=\text{Const. seu } \frac{x+y}{1-xy}=a.$$

Hanc vero aequationem

$$\frac{2\partial x}{1+xx}+\frac{\partial y}{1+yy}=0$$

multiplicari convenit in $\frac{(xx+1)^2(1+yy)}{(2xy+xx-1)^2}$, ut prodeat

$$\frac{2\partial x(1+xx)(1+yy)+\partial y(xx+1)^2}{(2xy+xx-1)^2}=0,$$

cujus integrale reperitur

$$\frac{xxy-2x-y}{2xy+xx-1}=\text{Const. seu } \frac{2x+y-xxy}{2xy+xx-1}=a.$$

6. Contra haec exempla, quibus integralia algebraica sine subsidio separationis sunt eruta, objicietur, multiplicatores negotium hoc conficientes ex ipsis integralibus illis transcendentibus, ad quae separatio variabilium immediate perducit esse conclusos, iisque adeo praestantiam methodi per multiplicatores procedentis neutiquam probari. Cui quidem objectioni primum respondeo, priora exempla statim ab inventis integrationis principiis simili modo fuisse expedita, antequam integratio per logarithmos erat explorata, quae

ergo nullum subsidium eo attulisse est censenda. Tum vero quamvis concedam, in posterioribus exemplis integrationem per arcus circulares multiplicatores illos idoneos commode suppeditasse, id tamen in ipsa evolutione minus cernitur, eademque integratio sine dubio inveniri potuisset, antequam constaret formulae $\frac{\partial x}{1+xx}$ integrale esse arcum circuli tangenti x respondentem. Verum aequatio supra allata

$$\frac{\partial x}{\sqrt{(1+x^4)}} + \frac{\partial y}{\sqrt{(1+y^4)}} = 0,$$

cujus integrale completum algebraice exhibere licet, nulli amplius dubio locum relinquit, cum enim neutrius partis integrale ne concessis quidem logarithmis vel arcubus circularibus exhiberi possit, ejusque forma ad genus quantitatum transcendentium etiamnum incognitum sit referenda, haec certe nullum auxilium ad integrale algebraicum inveniendum attulisse censeri potest. Atque hoc multo magis de aequatione illa latius patente in §. 4. proposita est tenendum, quippe cujus integratio omnino singularis ex principiis longe diversissimis a me est eruta.

7. Methodus autem, qua tum sum usus, tantopere est abscondita, ut vix ulla via ad eadem integralia perducens patere videatur, et cum separatio variabilium nihil plane eo contulisset, vix etiam quicquam ab altera methodo ad multiplicatores adstricta sperari posse videbatur, propterea quod tum ipse adhuc in ea opinione versabar, per multiplicatores nihil praestari posse, nisi quatenus separatio variabilium eodem manuducat; quandoquidem quaestio differentialia tantum primi gradus implicaret. Deinceps autem re diligentius considerata perspexi, quoties aequationis cujusque differentialis integrale completum exhibere licet, ex eo vicissim semper ejusmodi multiplicatorem elici posse, per quem si aequatio differentialis multiplicetur, non solum fiat integrabilis, sed etiam integrata id ipsum integrale, quod jam erat cognitum, reproducere de-

beat; ad hoc autem omnino necesse est ut integrale completum sit exploratum, dum ex integralibus particularibus nihil plane pro hoc scopo concludere licet. Si enim proposita sit aequatio differentialis

$$P\partial x + Q\partial y = 0,$$

cujus integrale completum, undecunque sit cognitum, constabit id aequatione, quae praeter binas variabiles x et y et quantitates constantes in ipsa aequatione differentiali contentas insuper quantitatem constantem novam prorsus ab arbitrio nostro pendentem complectetur. Quae si littera C indicetur, eruatur ejus valor ex aequatione integrali, ac reperiatur $C = V$, eritque V certa quaedam functio ipsarum x et y; tum autem hac aequatione differentiata $0 = \partial V$, differentiale ∂V necessario ita formulam differentialem $P\partial x + Q\partial y$ continere debet, ut sit

$$\partial V = M(P\partial x + Q\partial y),$$

ex qua forma multiplicator M, ad hoc integrale $C = V$ perducens, sponte se offert.

8. Quo haec operatio aliquot exemplis illustretur, sumatur primo haec aequatio

$$\frac{m\partial x}{x} + \frac{n\partial y}{y} = 0,$$

cujus integrale completum cum sit $x^m y^n = C$, instituta differentiatione prodit

$$0 = mx^{m-1} y^n \partial x + nx^m y^{n-1} \partial y, \text{ seu}$$
$$0 = x^m y^n \left(\frac{m\partial x}{x} + \frac{n\partial y}{y}\right),$$

unde patet, multiplicatorem ad hoc integrale ducentem esse $x^m y^n$.

Deinde cum hujus aequationis

$$\frac{\partial x}{1 + xx} + \frac{\partial y}{1 + yy} = 0$$

integrale completum sit

$$1 - xy = C(x + y),$$

valor constantis arbitrariae hinc fit $C = \frac{1 - xy}{x + y}$, cujus differentiatio praebet

$$0 = \frac{-\partial x(1 + yy) - \partial y(1 + xx)}{(x + y)^2}, \text{ seu}$$

$$0 = \frac{(1 + xx)(1 + yy)}{(x + y)^2}\left(\frac{\partial x}{1 + xx} + \frac{\partial y}{1 + yy}\right),$$

unde multiplicator quaesitus est $= \frac{(1 + xx)(1 + yy)}{(x + y)^2}$.

Proposita porro sit haec aequatio

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy)}} = 0,$$

cujus integrale completum

$$CC(x - y)^2 - 2C(\alpha + \beta x + \beta y + \gamma xy) + \beta\beta - \alpha\gamma = 0$$

dat primo

$$C = \frac{+\alpha + \beta(x + y) + \gamma xy + \sqrt{\left[\alpha\alpha + 2\alpha\beta(x + y) {+\alpha\gamma(xx + yy) \atop +4\beta\beta xy} {+4\beta\gamma xy(x + y) \atop +\gamma\gamma xxyy}\right]}}{(x - y)^2},$$

seu

$$C = \frac{+\alpha + \beta(x + y) + \gamma xy + \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)}}{(x - y)^2}$$

vel concinnius

$$\frac{\beta\beta - \alpha\gamma}{C} = +\alpha + \beta(x + y) + \gamma xy + \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)},$$

unde differentiando fit

$$0 = +\partial x(\beta + \gamma y) + \partial y(\beta + \gamma x) + \frac{\partial x(\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)}}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{\partial y(\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}}{\sqrt{(\alpha + 2\beta y + \gamma yy)}},$$

hincque colligitur multiplicator quaesitus

$$M = (\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)}$$
$$+ (\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}.$$

9. Simili modo pro aequatione magis complexa

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4)}} = 0,$$

ex ejus integrali completo supra exhibito multiplicator idoneus M investigari poterit, ex quo si statim fuisset cognitus, idem hoc integrale immediate elici potuisset. Verum hic opus multo majus molior, quod autem primo conatu neutiquam ad finem perducere licebit; ex quo satis mihi equidem praestitisse videbor, si saltem primo quasi lineamenta novae atque maxime desiderandae methodi adumbravero, cujus ope, proposita hujusmodi aequatione differentiali, multiplicator idoneus eam reddens integrabilem inveniri queat. Ac primo quidem in hoc negotio plurimum observasse juvabit, si unicus hujusmodi multiplicator innotuerit, ex eo facile infinitos alios idem officium praestantes erui posse. Quodsi enim multiplicator M aequationem differentialem

$$P\partial x + Q\partial y = 0$$

integrabilem reddat, ita ut sit

$$\int M\,(P\partial x + Q\partial y) = V,$$

ideoque aequatio integralis $V = C$, quoniam formula

$$\partial V = M\,(P\partial x + Q\partial y)$$

per functionem quamcunque quantitatis V multiplicata perinde manet integrabilis, perspicuum est hanc formam $Mf:V$, quaecunque functio ipsius V pro $f:V$ accipiatur, semper multiplicatorem idoneum praebere, cum sit

$$(P\partial x + Q\partial y)\,Mf:V = \partial V f:V,$$

ideoque integrabile. Inter infinitos igitur hos multiplicatores idoneos quovis casu eum eligi conveniet, qui negotium facillime conficiat, et integrale si fuerit algebraicum forma simplicissima exhibeat. Etiamsi enim integrale revera sit algebraicum, omnino fieri potest, ut id ne suspicari quidem liceat, nisi multiplicator idoneus in usum vocetur, quemadmodum superiora exempla abunde declarant.

10. Sit ergo aequatio differentialis proposita hujus formae

$$\frac{\partial x}{X} + \frac{\partial y}{Y} = 0,$$

in qua X sit functio x et Y solius y; atque investigari oporteat ejusmodi multiplicatorem M, quo illa aequatio algebraice integrabilis reddatur, siquidem fieri potest: quod cum raro eveniat, vicissim assumta multiplicatoris forma M indagasse juvabit functiones X et Y. Sit primo multiplicator

$$M = \frac{XY}{(\alpha + \beta x + \gamma y)^2},$$

ut integrabilis esse debeat haec forma

$$\frac{Y\partial x + X\partial y}{(\alpha + \beta x + \gamma y)^2} = 0.$$

Hinc sumta y constante colligitur integrale

$$\frac{-Y}{\beta(\alpha + \beta x + \gamma y)} + \Gamma : y,$$

sumta autem x constante prodit

$$\frac{-X}{\gamma(\alpha + \beta x + \gamma y)} + \Delta : x,$$

quam ambas formas inter se aequales esse oportet; unde fit

$$-\gamma Y + \beta\gamma(\alpha + \beta x + \gamma y)\Gamma : y = -\beta X + \beta\gamma(\alpha + \beta x + \gamma y)\Delta : x,$$

seu

$$\beta X - \gamma Y = \beta\gamma(\alpha + \beta x + \gamma y)(\Delta : x - \Gamma : y),$$

sicque patet functiones $\Delta : x$ et $\Gamma : y$ ita comparatas esse debere, ut evoluto posteriori membro termini, qui simul x et y continerent, se mutuo tollant. Ex quo intelligitur fore

$$\Delta : x = m\beta x + \text{Const. et } \Gamma : y = m\gamma y + \text{Const.}$$

$m = \frac{1}{ADhhkk}$, $\alpha = \frac{Bk + Eh}{2}$, $n = \frac{Bk - Eh}{2ADhhkk}$ et $f = \frac{ACkk + DFhh}{2AD\,hhkk}$,

praeterea vero haec conditio requiritur, ut sit

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk},$$

quae si habuerit locum, multiplicator idoneus erit

$$M = \frac{(Axx + Bx + C)(Dyy + Ey + F)}{hk\,[\frac{1}{2}(Bk + Eh) + Akx + Dhy]^2},$$

et aequatio integralis inde resultans erit per hk multiplicando

$$xy - \frac{(Bk - Eh)x}{4Dh} + \frac{(Bk - Eh)y}{4Ak} - \frac{ACkk - DFhh}{2ADhk}$$
$$= G\,[\tfrac{1}{2}(Bk + Eh) + Akx + Dhy],$$

quae immutata constante arbitraria G ad hanc formam revocatur

$$(x + \frac{B}{2A} - GDh)(y + \frac{E}{2D} - GAk) = GGADhk$$
$$+ \frac{(4AC - BB)kk + (4DF - EE)hh}{8ADhk},$$

seu

$$(\frac{2Ax + B}{h} + G)(\frac{2Dy + E}{k} + G) = GG + \frac{4AC - BB}{2hh} + \frac{4DF - EE}{2kk}.$$

12. En ergo Theorema minime spernendum, etiamsi ejus veritas ex aliis principiis satis manifesta esse queat.

Si haec aequatio differentialis

$$\frac{h\partial x}{Axx + Bx + C} + \frac{k\partial y}{Dyy + Ey + F} = 0$$

ita fuerit comparata, ut sit

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk},$$

tum ejus integrale completum erit algebraicum, atque hac aequatione expressum

$$(\frac{2Ax + B}{h})(\frac{2Dy + E}{k}) + G(\frac{2Ax + B}{h} + \frac{2Dy + E}{k}) = \frac{4AC - BB}{2hh} + \frac{4DF - EE}{2kk},$$

sive evolvendo

$$X = (\beta\zeta - \delta\theta)\,xx + (\alpha\zeta + \beta\eta - \gamma\theta - \delta f)\,x + \alpha\eta - \gamma f,$$
$$Y = (\gamma\zeta - \delta\eta)\,yy + (\alpha\zeta + \gamma\theta - \beta\eta - \delta f)\,y + \alpha\theta - \beta f,$$

et aequatio integralis erit

$$\frac{\zeta x + \eta}{\gamma + \delta x} - \frac{X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta xy)} = \text{Const.}$$

quae loco X substituto valore invento abit in hanc formam

$$\frac{\zeta xy + \eta y + \theta x + f}{\alpha + \beta x + \gamma y + \delta xy} = \text{Const.}$$

14. Transferamus haec iterum ad formam

$$\frac{h\partial x}{Axx + Bx + C} + \frac{k\partial y}{Dyy + Ey + F} = 0,$$

ac fieri oportet

$A = h(\beta\zeta - \delta\theta)$,	$D = k(\gamma\zeta - \delta\eta)$,
$B = h(\alpha\zeta + \beta\eta - \gamma\theta - \delta f)$,	$E = k(\alpha\zeta + \gamma\theta - \beta\eta - \delta f)$,
$C = h(\alpha\eta - \gamma f)$,	$F = k(\alpha\theta - \beta f)$.

Primae aequationes praebent

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta h}, \quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k},$$

secundae vero

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk - Eh}{2\delta hk} \quad \text{et} \quad \delta = \frac{2A\gamma k - 2D\beta h}{Bk - Eh},$$

unde ex tertiis colligitur

$$\frac{2Ck(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\gamma}{2}(Bk + Eh) - D\alpha h,$$
$$\frac{2Fh(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\beta}{2}(Bk + Eh) - A\alpha k.$$

Hinc α elidendo fit

$$\frac{2(ACkk - DFhh)(Ak\gamma - Dh\beta)}{Bk - Eh} = \tfrac{1}{2}(Ak\gamma - Dh\beta)(Bk + Eh),$$

unde cum esse nequeat

$$Ak\gamma - Dh\beta = 0,$$

quia alioquin fieret $\delta = 0$, et quantitates θ, η, f infinitae, tum

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

sitque multiplicator eam reddens integrabilem

$$M = P\sqrt{X} + Q\sqrt{Y},$$

ita ut aequatio integrationem admittens sit

$$P\partial x + Q\partial y + \frac{Q\partial x\sqrt{Y}}{\sqrt{X}} + \frac{P\partial y\sqrt{X}}{\sqrt{Y}} = 0,$$

cujus utrumque membrum seorsim integrabile sit oportet. Pro priore ergo erit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$, posterioris vero integrale statuatur $2V\sqrt{XY}$, unde colligitur

$$Q = 2X\left(\frac{\partial V}{\partial x}\right) + V \cdot \frac{\partial X}{\partial x} \text{ et}$$

$$P = 2Y\left(\frac{\partial V}{\partial y}\right) + V \cdot \frac{\partial Y}{\partial y},$$

et ob priorem conditionem

$$2Y\left(\frac{\partial\partial V}{\partial y^2}\right) + \frac{3\partial Y}{\partial y}\left(\frac{\partial V}{\partial y}\right) + V \cdot \frac{\partial\partial Y}{\partial y^2} = 2X\left(\frac{\partial\partial V}{\partial x^2}\right) + \frac{3\partial X}{\partial x}\left(\frac{\partial V}{\partial x}\right) + V \cdot \frac{\partial\partial X}{\partial x^2},$$

ex qua aequatione, si loco V sumserimus certam functionem ipsarum x et y, dispiciendum est, quomodo idonei valores pro functionibus X et Y obtineantur.

16. Demus primo ipsi V valorem constantem puta $V = 1$, ac pervenimus ad hanc conditionem

$$\frac{\partial\partial Y}{\partial y^2} = \frac{\partial\partial X}{\partial x^2},$$

quae aequalitas subsistere nequit, nisi utrumque membrum seorsim aequetur quantitati constanti, quae sit $= 2a$, unde colligemus

$$X = axx + bx + c \text{ et } Y = ayy + dy + e,$$

hincque porro

$$P = \frac{\partial Y}{\partial y} = 2ay + d \text{ et } Q = \frac{\partial X}{\partial x} = 2ax + b,$$

unde aequatio integralis completa colligitur termini ex x et y mixti utrinque aequales fieri non possent. Cum ergo ipsae functiones X et Y ad quartum gradum sint ascensurae, ponamus

$$X = Ax^4 + 2Bx^3 + Cx^2 + 2Dx + E \quad \text{et}$$
$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E}.$$

Facta jam substitutione pro priori parte prodit

$$\begin{array}{lllll}
12\beta\beta A x^4 & + 24\beta\beta B x^3 & + 12\beta\beta C xx & + 24\beta\beta D x & + 12\beta\beta E \\
- 24\beta\beta A & - 36\beta\beta B & - 12\beta\beta C & - 12\beta\beta D & - 12\alpha\beta D \\
+ 12\beta\beta A & - 24\alpha\beta A & - 36\alpha\beta B & - 12\alpha\beta C & + 2\alpha\alpha C \\
 & + 12\beta\beta B & + 2\beta\beta C & + 4\alpha\beta C & \\
 & + 24\alpha\beta A & + 24\alpha\beta B & + 12\alpha\alpha B & \\
 & & + 12\alpha\alpha A & &
\end{array}$$

$$\begin{array}{llll}
- 24\beta\gamma A x^3 y & - 36\beta\gamma B y^2 y & - 12\beta\gamma C xy & - 12\beta\gamma D y \\
+ 24\beta\gamma A & + 24\beta\gamma B & + 4\beta\gamma C & + 4\alpha\gamma C \\
 & + 24\alpha\gamma A & + 24\alpha\gamma B &
\end{array}$$

$$+ 12\gamma\gamma A xxyy + 12\gamma\gamma B xyy + 2\gamma\gamma C yy,$$

qui termini in ordinem disponentur

$$12\gamma\gamma A xxyy + 12\gamma\gamma B xyy + 12\gamma(2\alpha A - \beta B) xxy$$
$$+ 2\gamma\gamma C yy + 8\gamma(3\alpha B - \beta C) xy + 2(6\alpha\alpha A - 6\alpha\beta B + \beta\beta C) xx$$
$$+ 4\gamma(\alpha C - 3\beta D) y + 4(3\alpha\alpha B - 2\alpha\beta C + 3\beta\beta D) x$$
$$+ (2\alpha\alpha C - 6\alpha\beta D + 6\beta\beta E).$$

Simili vero modo altera pars erit

$$12\beta\beta\mathfrak{A} xxyy + 12\beta\beta\mathfrak{B} xxy + 12\beta(2\alpha\mathfrak{A} - \gamma\mathfrak{B}) xyy + 2\beta\beta\mathfrak{C} xx$$
$$+ 8\beta(3\alpha\mathfrak{B} - \gamma\mathfrak{C}) xy + 2(6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C}) yy$$
$$+ 4\beta(\alpha\mathfrak{C} - 3\gamma\mathfrak{D}) x + 4(3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D}) y$$
$$+ 2(\alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}).$$

quare commode statui licebit

$$\mathrm{E} = \frac{\mathrm{B}^4}{16\,\mathrm{A}^3} + \frac{n\,\mathrm{BB}}{4\,\mathrm{AA}} + \frac{m\,\mathrm{B}}{2\,\mathrm{A}\sqrt{\mathrm{A}}} + \frac{l}{\mathrm{A}},$$

$$\mathfrak{E} = \frac{\mathfrak{B}^4}{16\,\mathfrak{A}^3} + \frac{n\,\mathfrak{B}\mathfrak{B}}{4\,\mathfrak{A}\mathfrak{A}} + \frac{m\,\mathfrak{B}}{2\,\mathfrak{A}\sqrt{\mathfrak{A}}} + \frac{l}{\mathfrak{A}}.$$

19. Cum autem sumserimus $\mathrm{V} = \frac{1}{(\alpha+\beta x+\gamma y)^2}$, erit

$$\mathrm{Q} = \frac{-4\beta(\mathrm{A}x^4+2\mathrm{B}x^3+\mathrm{C}xx+2\mathrm{D}x+\mathrm{E})}{(\alpha+\beta x+\gamma y)^3} + \frac{2(2\mathrm{A}x^3+3\mathrm{B}xx+\mathrm{C}x+\mathrm{D})}{(\alpha+\beta x+\gamma y)^2},$$

$$\mathrm{P} = \frac{-4\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3} + \frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{(\alpha+\beta x+\gamma y)^2},$$

sive

$$\mathrm{Q} = \frac{2\gamma y(2\mathrm{A}x^3+3\mathrm{B}xx+\mathrm{C}x+\mathrm{D})+2(2\alpha\mathrm{A}-\beta\mathrm{B})x^3+2(3\alpha\mathrm{B}-\beta\mathrm{C})xx+2(\alpha\mathrm{C}-3\beta\mathrm{D})x+2(\alpha\mathrm{D}-2\beta\mathrm{E})}{(\alpha+\beta x+\gamma y)^3},$$

$$\mathrm{P} = \frac{2\beta x(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^3+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3},$$

unde investigari oportet integrale formulae $\mathrm{P}\partial x + \mathrm{Q}\partial y$, ad quod si deinceps addatur $\frac{2\sqrt{\mathrm{XY}}}{(\alpha+\beta x+\gamma y)^2}$, aggregatum quantitati constanti aequatum exhibebit integrale completum aequationis

$$\frac{\partial x}{\sqrt{\mathrm{X}}} + \frac{\partial y}{\sqrt{\mathrm{Y}}} = 0.$$

Pro illo autem integrali inveniendo, ex prioribus valoribus pro P et Q exhibitis, notetur fore separatim

$$\int \mathrm{Q}\partial y = \frac{2\beta(\mathrm{A}x^4+2\mathrm{B}x^3+\mathrm{C}xx+2\mathrm{D}x+\mathrm{E})}{\gamma(\alpha+\beta x+\gamma y)^2} - \frac{2(2\mathrm{A}x^3+3\mathrm{B}xx+\mathrm{C}x+\mathrm{D})}{\gamma(\alpha+\beta x+\gamma y)} + \Gamma : x,$$

$$\int \mathrm{P}\partial x = \frac{2\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{\beta(\alpha+\beta x+\gamma y)^2} - \frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{\beta(\alpha+\beta x+\gamma y)} + \Delta : y,$$

quae duae expressiones aequales esse debent: quem in finem ponatur

$$\Gamma : x = \frac{2(\mathrm{A}xx+\mathrm{B}x+\mathrm{N})}{\beta\gamma} \quad \text{et} \quad \Delta : y = \frac{2(\mathfrak{A}yy+\mathfrak{B}y+\mathfrak{N})}{\beta\gamma},$$

fietque

$$S + \sqrt{XY} = \text{Const.} \left(\frac{B}{\sqrt{A}} + \frac{\mathfrak{B}}{\sqrt{\mathfrak{A}}} + 2x\sqrt{A} + 2y\sqrt{\mathfrak{A}}\right)^2.$$

Quare dum functiones X et Y conditionibus ante definitis sint praeditae, hoc modo habebitur integrale completum aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0.$$

21. Haec investigatio aliquanto generalius institui potest tribuendo ipsi V talem valorem $\frac{1}{(\alpha + \beta x + \gamma y + \delta xy)^2}$, quo facilius autem calculi molestias superare queamus observo, dummodo variabiles x et y quantitate constante augeantur vel minuantur, eum ad hanc formam $\frac{1}{(a + xy)^2}$ reduci posse: expedito autem calculo restitutio facile instituetur. Considerabo ergo hanc aequationis differentialis formam

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

quam integrabilem reddi assumo ope multiplicatoris $P\sqrt{X} + Q\sqrt{Y}$, ut integrari debeat haec formula

$$P\partial x + Q\partial y + \frac{Q\partial x\sqrt{Y}}{\sqrt{X}} + \frac{P\partial y\sqrt{X}}{\sqrt{Y}} = 0.$$

Statuatur partis posterioris integrale $= 2V\sqrt{XY}$, fietque ut vidimus

$$Q = 2X\left(\frac{\partial V}{\partial x}\right) + V \cdot \frac{\partial X}{\partial x}, \quad \text{et} \quad P = 2Y\left(\frac{\partial V}{\partial y}\right) + V \cdot \frac{\partial Y}{\partial y}.$$

Sit igitur $V = \frac{1}{(a + xy)^2}$, ideoque

$$\left(\frac{\partial V}{\partial x}\right) = \frac{-2y}{(a + xy)^3}, \quad \text{et} \quad \left(\frac{\partial V}{\partial y}\right) = \frac{-2x}{(a + xy)^3},$$

ita ut habeamus

$$Q = \frac{-4Xy}{(a + xy)^3} + \frac{\partial X}{\partial x} \frac{1}{(a + xy)^2}, \quad \text{et}$$
$$P = \frac{-4Yx}{(a + xy)^3} + \frac{\partial Y}{\partial y} \frac{1}{(a + xy)^2}.$$

Nunc autem effici debet ut formula $P\partial x + Q\partial y$ integrationem admittat, hunc in finem duplici modo ejus integrale capiatur, dum

22. Harum formarum coaequatio suppeditat sequentes determinationes

$$\mathfrak{L} = L,\ M = 2\mathfrak{B},\ \mathfrak{M} = 2B,\ N = -2a\mathfrak{A},\ \mathfrak{N} = -2aA,$$
$$\mathfrak{C} = C,\ D = -a\mathfrak{B},\ \mathfrak{D} = -aB,\ E = aa\mathfrak{A},\ \mathfrak{E} = aaA,$$

ita ut habeatur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} + \frac{\partial y}{\sqrt{(\frac{E}{aa}y^4 - \frac{2D}{a}y^3 + Cyy - 2aBy + aaA)}} = 0,$$

cujus integrale completum est

$$\frac{2Bxxy - \frac{2D}{a}xyy - 2aAxx - \frac{2E}{a}yy + 2Cxy - 2aBx + 2Dy + 2\sqrt{XY}}{(a + xy)^2} = \text{Const.}$$

Hic observo si ponamus $y = \frac{-a}{z}$, prodire aequationem initio allatam

$$\frac{\partial x}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} + \frac{\partial z}{\sqrt{(Az^4 + 2Bz^3 + Czz + 2Dz + E)}} = 0,$$

cujus propterea integrale nunc etiam per principia integrationis maxime naturalia assignari potest, cum antea methodo admodum indirecta eo fuissem deductus. Integrale quippe est

$$Axxzz + Bxz(x+z) + Cxz + D(x+z) + E + G(x-z)^2 =$$
$$\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)(Az^4 + 2Bz^3 + Czz + 2Dz + E)},$$

quae ab irrationalitate liberata induit hanc formam

$$GG(x-z)^2 + 2G[Axxzz + Bxz(x+z) + Cxz + D(x+z) + E]$$
$$+ (BB - AC)xxzz - 2ADxz(x+z) - AE(x+z)^2 - 2BDxz$$
$$- 2BE(x+z) + DD - CE = 0,$$

quae aequatio in hanc formam reducta cum superiori convenit

24. Videamus autem quousque problema in genere agressi calculum expedire queamus. Sit igitur proposita aequatio

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

quae per $P\sqrt{Y} + Q\sqrt{Y}$ multiplicata fiat integrabilis, sitque integrale

$$\int(P\partial x + Q\partial y) + \frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y+\delta xy)^2} = \text{Const.}$$

eritque ut vidimus

$$Q = \frac{-4X(\beta+\delta y)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{\partial X}{\partial x(\alpha+\beta x+\gamma y+\delta xy)^2},$$

$$P = \frac{-4Y(\gamma+\delta x)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{\partial Y}{\partial y(\alpha+\beta x+\gamma y+\delta xy)^2},$$

unde colligimus

$$(\gamma+\delta x)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\partial y = 2(\beta\gamma-\alpha\delta)X$$
$$+ [4\delta X - (\gamma+\delta x)\tfrac{\partial X}{\partial x}](\alpha+\beta x+\gamma y+\delta xy)$$
$$+ (\alpha+\beta x+\gamma y+\delta xy)^2\Delta : x,$$

similique modo

$$(\beta+\delta y)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int P\partial x = 2(\beta\gamma-\alpha\delta)Y$$
$$+ [4\delta Y - (\beta+\delta y)\tfrac{\partial Y}{\partial y}](\alpha+\beta x+\gamma y+\delta xy)$$
$$+ (\alpha+\beta x+\gamma y+\delta xy)^2\Gamma : y,$$

quae duae formae ad consensum perduci debent, ita ut prima per $(\gamma+\delta x)^2$, altera vero per $(\beta+\delta y)^2$ divisa eandem functionem exhibeant. Quamobrem necesse est ut prior per $(\gamma+\delta x)^2$, posterior per $(\beta+\delta y)^2$ divisionem admittat, cui ergo requisito ante omnia est satisfaciendum.

25. Evolvamus priorem valorem, partibus ab y pendentibus distinguendis

I. $2(\beta\gamma-\alpha\delta)X + 4\delta(\alpha+\beta x)X - (\alpha+\beta x)(\gamma+\delta x)\frac{\partial X}{\partial x}$
$+ (\alpha+\beta x)^2\Delta : x,$

II. $-y(\gamma+\delta x)[4\delta Y + (\gamma+\delta X)\frac{\partial X}{\partial x} + 2(\alpha+\beta x)\Delta : x],$

III. $+yy(\gamma+\delta x)^2\Delta : x,$

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Quocirca formulae

$$(\alpha + \beta x + \gamma y + \delta xy)^2 \int Q\partial y$$

valor erit

$$\frac{\beta\beta}{\delta\delta}\Delta : x + \frac{2\beta}{\delta} y\Delta : x + yy\Delta : x - \frac{\beta(\alpha+\beta x)}{\delta\delta}\Delta' : x - \frac{(\alpha+\beta x)}{\delta} y\Delta' : x$$
$$+ \frac{(\alpha+\beta x)^2}{2\delta\delta}\Delta'' : x + \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta} y\Delta'' : x$$
$$+ \frac{\beta}{\delta}(\gamma+\delta x)\,S + (\gamma+\delta x)\,yS - \frac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\frac{\partial S}{\partial x}$$
$$- \frac{(\gamma+\delta x)^2}{2\delta} y \cdot \frac{\partial S}{\partial x},$$

seu ita concinnius expressus

$$\frac{(\beta+\delta y)^2}{\delta\delta}\Delta : x - \frac{(\alpha+\beta x)(\beta+\delta y)}{\delta\delta}\Delta' : x + \frac{(\alpha+\beta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Delta'' : x$$
$$+ \frac{(\gamma+\delta x)(\beta+\delta y)}{\delta} S - \frac{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{\partial S}{\partial x},$$

cui alter aequalis fieri debet, qui est

$$\frac{(\gamma+\delta x)^2}{\delta\delta}\Gamma : y - \frac{(\alpha+\gamma y)(\gamma+\delta x)}{\delta\delta}\Gamma' : y + \frac{(\alpha+\gamma y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Gamma'' : y$$
$$+ \frac{(\beta+\delta y)(\gamma+\delta x)}{\delta}\mathfrak{S} - \frac{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\frac{\partial\mathfrak{S}}{\partial y}.$$

26. Quodsi jam ponamus

$$\Delta : x = \delta\delta(Axx + 2Bx + C) \quad \text{et} \quad S = \delta(Dxx + 2Ex + F)$$

item

$$\Gamma : y = \delta\delta(\mathfrak{A}yy + 2\mathfrak{B}y + \mathfrak{C}) \quad \text{et} \quad \mathfrak{S} = \delta(\mathfrak{D}yy + 2\mathfrak{E}y + \mathfrak{F}),$$

reperientur nostrae expressiones ita evolutae

$(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\partial y$	$(\alpha+\beta x+\gamma y+\delta xy)^2\int P\partial x$
$+\delta\delta Axxyy$	$+\delta\delta\mathfrak{A}xxyy$
$+2\delta\delta Bxyy$	$+\delta(\gamma\mathfrak{A}-\beta\mathfrak{D}+\delta\mathfrak{E})xyy$
$+\delta(\beta A-\gamma D+\delta E)xxy$	$+2\delta\delta\mathfrak{B}xxy$
$+\delta\delta Cyy$	$+\delta(\gamma\mathfrak{E}-\alpha\mathfrak{D})yy$
$+\delta(\beta E-\alpha D)xx$	$+\delta\delta\mathfrak{C}xx$
$+[2\beta\delta B+(\beta\gamma-\alpha\delta)A-\gamma\gamma D+\delta\delta F]xy$	$+[2\gamma\delta\mathfrak{B}+(\beta\gamma-\alpha\delta)\mathfrak{A}-\beta\beta\mathfrak{D}+\delta\delta\mathfrak{F}]xy$
$+(\alpha\gamma A-2\alpha\delta B+2\beta\delta C-\gamma\gamma E+\gamma\delta F)y$	$+[\gamma\delta\mathfrak{F}+(\beta\gamma-\alpha\delta)\mathfrak{E}-\alpha\beta\mathfrak{D})y$
$+[\beta\delta F+(\beta\gamma-\alpha\delta)E-\alpha\gamma D]x$	$+(\alpha\beta\mathfrak{A}-2\alpha\delta\mathfrak{B}+2\gamma\delta\mathfrak{C}+\beta\beta\mathfrak{E}+\beta\delta\mathfrak{F})x$
$+\alpha\alpha A-2\alpha\beta B+\beta\beta C-\alpha\gamma E+\beta\gamma F$	$+\alpha\alpha\mathfrak{A}-2\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C}-\alpha\beta\mathfrak{E}+\beta\gamma\mathfrak{F}$

Fingatur prioris partis integrale $= 2R\sqrt{X}$, posterioris vero $= 2S\sqrt{Y}$, ut integrale completum sit

$$R\sqrt{X} + S\sqrt{Y} = \text{Const.}$$

et facta evolutione reperitur

$$P = \frac{R\partial X}{\partial x} + 2X\left(\frac{\partial R}{\partial x}\right); \quad P = \frac{S\partial Y}{\partial y} + 2Y\left(\frac{\partial S}{\partial y}\right);$$

$$Q = 2\left(\frac{\partial R}{\partial y}\right); \qquad Q = 2\left(\frac{\partial S}{\partial x}\right).$$

Cum igitur debeat esse $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$, manifestum est formulam $R\partial x + S\partial y$ integrabilem esse debere. Non autem opus est, ut ea algebraicum habeat integrale, sed sufficit ut integrationis charactere sit praedita.

28. Sumatur enim

$$R = \frac{y}{\alpha + \beta xy + \gamma xxyy} \quad \text{et} \quad S = \frac{x}{\alpha + \beta xy + \gamma xxyy},$$

eritque

$$Q = \frac{2\alpha - 2\gamma xxyy}{(\alpha + \beta xy + \gamma xxyy)^2} \quad \text{et}$$

$$P = \frac{y\partial X}{\partial x(\alpha + \beta xy + \gamma xxyy)} - \frac{2Xyy(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)},$$

simulque

$$P = \frac{x\partial Y}{\partial y(\alpha + \beta xy + \gamma xxyy)} - \frac{2Yxx(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)^2},$$

ita ut habeatur

$$(\alpha + \beta xy + \gamma xxyy)^2 P$$
$$= \frac{y\partial X}{\partial x}(\alpha + \beta xy + \gamma xxyy) - 2yyX(\beta + 2\gamma xy)$$
$$= \frac{x\partial Y}{\partial y}(\alpha + \beta xy + \gamma xxyy) - 2xxY(\beta + 2\gamma xy).$$

Statuatur

$$X = Ax^4 + 2Bx^3 + Cxx + 2Dx + E$$

itemque

$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E},$$

grabilis reddatur, concludi posse; quae conclusio, si integrale tantum fuisset particulare, neutiquam valuisset. Quamobrem integrationum illarum particularium, quas olim simul ex codem principio alieno eram consecutus, longe aliter est ratio comparata, neque adhuc perspicere licet, quomodo methodo quadam directa et naturali ad easdem perveniri queat.

30. Eo magis igitur operae erit pretium, indolem harum integrationum particularium accuratius examinari, quod quidem contemplatione casus simplicissimi fiet. Hujus igitur aequationis differentialis

$$\partial x \sqrt{(1 + xx)} + \partial y \sqrt{(1 + yy)} + ny\partial x + nx\partial y = 0$$

integrale particulare inveneram esse

$$xx + yy + 2xy \sqrt{(1 + nn)} = nn,$$

similiaque integralia innumerabilia etiam inveni pro ejusmodi aequationibus differentialibus, quae neque a logarithmis neque a circuli quadratura pendent: quare haec aequatio ita spectetur, quasi non per logarithmos integrari posset. Hic igitur primo quaeritur, qua via directa hoc integrale particulare ex forma differentiali concludi queat? deinde quomodo aequatio differentialis comparata esse debeat, ut tale integrale particulare exhiberi queat? Ad has ergo quaestiones primum observo, aequationem algebraicam esse integrale completum istius aequationis differentialis

$$\frac{\partial x}{\sqrt{(1 + xx)}} + \frac{\partial y}{\sqrt{(1 + yy)}} = 0,$$

tum vero ex illa sequi

$$x + y \sqrt{(1 + nn)} = n \sqrt{(1 + yy)} \quad \text{et}$$
$$y + x \sqrt{(1 + nn)} = n \sqrt{(1 + xx)},$$

ita ut tam $\sqrt{(1 + xx)}$ quam $\sqrt{(1 + yy)}$ rationaliter per x et y exprimi queat. Cum igitur hinc sit differentiando

$$\frac{x\partial x}{\sqrt{(1+xx)}} = \frac{\partial y + \partial x\sqrt{(1+nn)}}{n} \quad \text{et} \quad \frac{y\partial y}{\sqrt{(1+yy)}} = \frac{\partial x + \partial y\sqrt{(1+nn)}}{n},$$

si harum formarum multipla quaecunque ad illam

$$\frac{\partial x}{\sqrt{(1+xx)}} + \frac{\partial y}{\sqrt{(1+yy)}} = 0$$

addantur, semper prodire aequationem differentialem, cui aequatio algebraica particulariter saltem satisfaciat. In genere ergo hujus aequationis differentialis

$$\frac{\partial x + \mathrm{P}x\partial x}{\sqrt{(1+xx)}} + \frac{\partial y + \mathrm{Q}y\partial y}{\sqrt{(1+yy)}} = \frac{\mathrm{P}\partial y + \mathrm{Q}\partial x + (\mathrm{P}\partial x + \mathrm{Q}\partial y)\sqrt{(1+nn)}}{n}$$

integrale particulare erit

$$xx + yy + 2xy\sqrt{(1+nn)} = nn.$$

Sit jam $\mathrm{P} = x$ et $\mathrm{Q} = y$, ac satisfiet huic aequationi

$$\partial x\sqrt{(1+xx)} + \partial y\sqrt{(1+yy)} = \frac{x\partial y + y\partial x + (x\partial x + y\partial y)\sqrt{(1+nn)}}{n};$$

ex integrali vero fit

$$x\partial x + y\partial y = -(x\partial y + y\partial x)\sqrt{(1+nn)},$$

ita ut habeatur haec aequatio differentialis

$$\partial x\sqrt{(1+xx)} + \partial y\sqrt{(1+yy)} + nx\partial y + ny\partial x = 0,$$

cui ergo integrale supra datum particulariter convenit.

31. Transferamus jam haec ad casus latius patentes, et postquam hujus aequationis

$$\frac{\partial x}{\sqrt{\mathrm{X}}} + \frac{\partial y}{\sqrt{\mathrm{Y}}} = 0$$

inventum fuerit integrale completum, quod sit $\mathrm{W} = \mathrm{Const.}$ notetur hinc semper utrumque valorem radicalem $\sqrt{\mathrm{X}}$ et $\sqrt{\mathrm{Y}}$ per functiones rationales ipsarum x et y definiri. Sit ergo

$$\sqrt{\mathrm{X}} = \mathrm{R} \quad \text{et} \quad \sqrt{\mathrm{Y}} = \mathrm{S},$$

ideoque

$$\frac{\partial \mathrm{X}}{\sqrt{\mathrm{X}}} = 2\partial \mathrm{R} \quad \text{et} \quad \frac{\partial \mathrm{Y}}{\sqrt{\mathrm{Y}}} = 2\partial \mathrm{S}.$$

Sit jam P functio ipsius x et Q ipsius y, hincque conflatur ista aequatio

$$\frac{\partial x + P\partial X}{\sqrt{X}} + \frac{\partial y + Q\partial Y}{\sqrt{Y}} - 2P\partial R - 2Q\partial S = 0,$$

cui aequatio algebraica $W = \text{Const.}$ certe particulariter satisfacit. Hinc si P et Q ita accipiantur, ut formula $P\partial R + Q\partial S$ integrationem admittat, cujus integrale sit $= V$, orietur aequatio transcendens

$$\int \frac{\partial x + P\partial X}{\sqrt{X}} + \int \frac{\partial y + Q\partial Y}{\sqrt{Y}} - 2V = \text{Const.}$$

cui aequationi $W = \text{Const.}$ seu valoribus inde deductis, $\sqrt{X} = R$ vel $\sqrt{Y} = S$ particulariter satisfacit. Tale ergo ratiocinium viam ad hujusmodi integrationes particulares alioquin inventu difficillimas patefacere videtur.

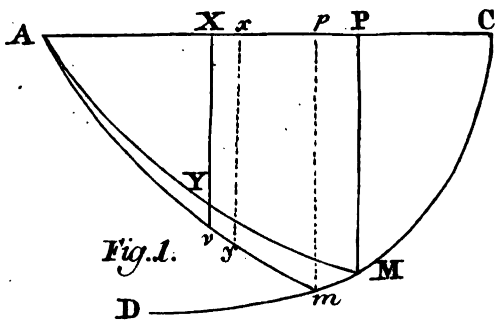

Fig. 1.

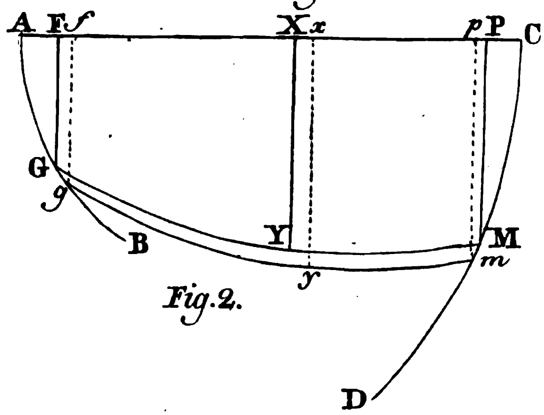

Fig. 2.

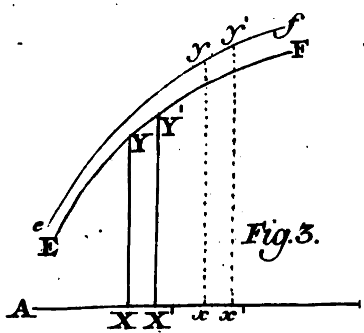

Fig. 3.

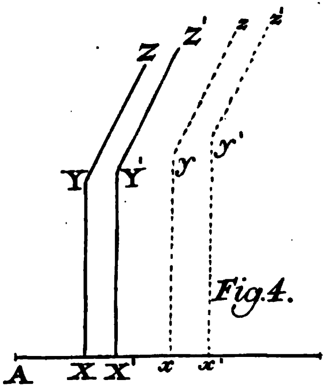

Fig. 4.

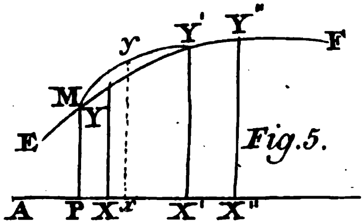

Fig. 5.

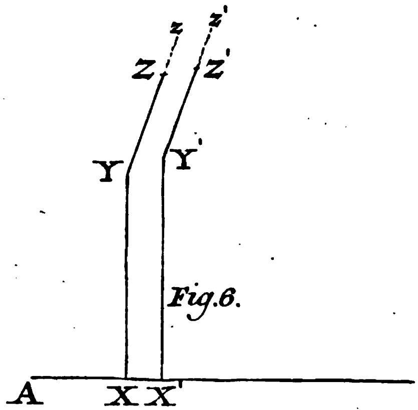

Fig. 6.

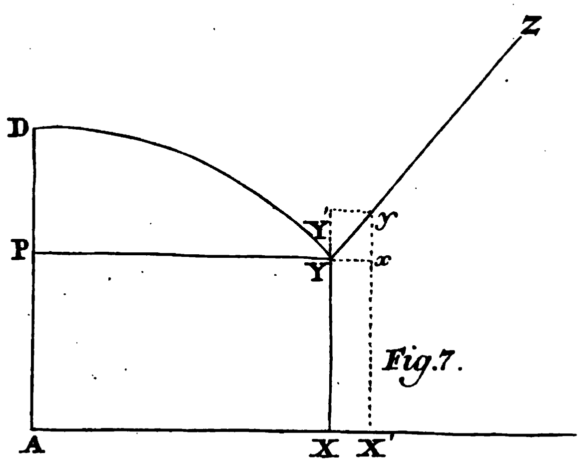

Fig. 7.