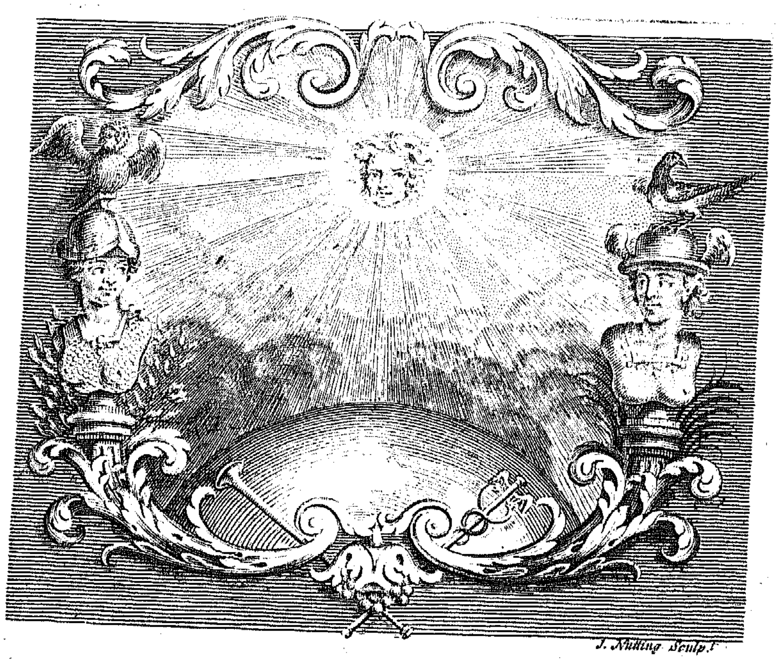
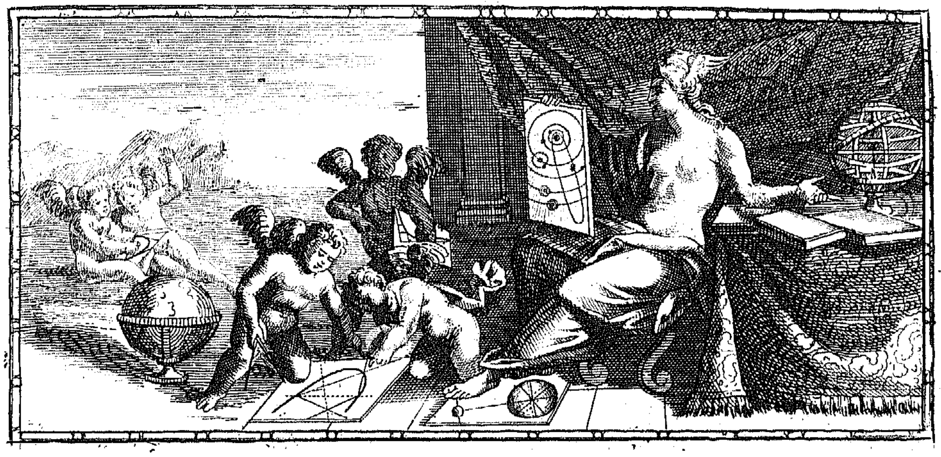
ANALYSIS
Per Quantitatum
SERIES, FLUXIONES,
AC
DIFFERENTIAS:
CUM
Enumeratione Linearum
TERTII ORDINIS.



LONDINI:
Ex Officina PEARSONIANA. *Anno* M.DCC.XI.



Præfatio Editoris.

Tractatus aliquot Mathematicos Viri Illustrissimi D. Newtoni *in lucem edimus, quorum primus & ultimus nunc primum prodeunt, reliqui vero vel à se vel ab aliis ante hac publici juris facti sunt. Hæc autem Editio casui, sed tamen non sine ipsius consensu prius impetrato, ortum acceptum refert.*

Etenim secundus jam agitur annus ex quo scriniæ D. Collinsii (*qui, uti notum est, amplissimum cum sui sæculi Mathematicis commercium habuit*) *meas in manus inciderunt; & in illis plurima reperi à cunctis fere totius* Europæ *eruditis ipsi communicata; & inter ea non pauca, quæ à Viro Cl* D. Newtono *scripta fuerunt; quæ cum tantæ molis essent, ut simul Tractatum breviusculum possent conficere, cæpi de iis*

 edendis

edendis cogitare. Quum autem animadvertissem scripta ejus quæ jam in lucem prodierunt ferme idem cum hisce argumentum habere, haud operæ pretium me facturum, si typis mandarem, existimavi.

Unus tamen erat brevis de Curvarum Quadratura Tractatus adeo luculenter & concinne scriptus, atque ita accommodatus ad instituendos eos, qui nondum totam istam Methodum perspectam habeant, ut abstinere non potuerim, quominus Auctoris licentiam eundem edendi peterem. Quam Ille non solum summa cum humanitate concessit; sed & insuper veniam dedit reliqua ipsius colligendi, quæ ad idem argumentum spectabant.

Hicce Tractatus, quem D. Collinsii *manu exaratum comperi, inscriptus fuit* De Analysi per Æquationes Infinitas, *& licet neque Auctoris nomen, neque tempus quo scriptus fuerat ullibi comparuerit; multa tamen continere ad D.* Newtoni *Methodos spectantia statim agnovi, utpote Epistolarum autographo illi ad amussim respondentia, quod antea ipse D.* Newtonus *ad D.* Oldenburgum *miserat. Unde suspicabar totum ex eodem fonte emanasse; nihilominus suspenso eram animo, usque dum inventis, inter easdem chartas, Epistolis aliquot D.* Barrovii *ad D.* Collinsium *scriptis; tres reperi ad hunc Tra-*

ctatum

ctatum immediate ſpectantes; ex quibus facile intellexi quomodo D. Collinſius *eum obtinuerat.*

In una Epiſtolarum e Collegio S. S. Tr. data 20 Julii 1669, *verſus finem* D. Barrovius *hæc ſcribit.*

* Amicus quidam apud nos commorans, ‘qui eximio in his rebus pollet ingenio, nudiuſtertius chartas quaſdam mihi tradidit, ‘in quibus Magnitudinum dimenſiones ‘ſupputandi Methodos, *Mercatoris* methodo ‘ſimiles, maxime vero Generales, deſcripſit, ‘ſimulque Æquationes reſolvendi, quæ, ut ‘opinor, tibi placebunt, quas una cum ‘proximis literis ad te mittam.

In hac Epiſtola narrat argumentum chartarum Amici ſui fuiſſe Computationem dimenſionum Magnitudinum, & Æquationum reſolutionem, quod cum Tractatus hujus argumento quadrat.

Verſus finem alterius Epiſtolæ ad D. Collinſium *e Collegio S. S. Tr. datæ ult.* Julii 1669, D. Barrovius *ita ſcribit.*

Mitto

* A Friend of mine here, that hath an excellent Genius to theſe things, brought me the other Day, ſome Papers, wherein he hath ſet down Methods of Calculating the Dimenſions of Magnitudes, like that of Mr. *Mercator*, but very general, as alſo of Reſolving Equations, which I ſuppoſe will pleaſe you, and I ſhall ſend you them by the next.

PRÆFATIO.

† Mitto quas pollicitus eram Amici char-
'tas, quæ uti ſpero haud parum te oblecta-
'bunt. Remittas, quæſo, quum eas quantum
'tibi viſum fuerit perlegeris; id enim poſtu-
'lavit Amicus meus, cum primum eum ro-
'gavi, ut eas tecum communicare mihi lice-
'ret. Quantocyus igitur, obſecro, te eas
'recepiſſe fac me certiorem, quod illis me-
'tuo, quippe qui eas per Veredarium publi-
'cum ad te mittere non dubitaverim, quo
'tibi morem gererem quam citiſſime.

Ex hac Epiſtola conſtat D. Barrovium *dictum Librum miſiſſe, ea lege, ut ſibi remitteretur. Unde manifeſtum eſt, quare Tractatus, quem inveni, D.* Collinſii *manu ſcriptus fuit, autographo nempe ipſi Auctori reſtituto.*

In tertia a D. Barrovio *ad D.* Collinſium *Epiſtola data* 20 Auguſti 1669, *conſtat D.* Newtonum *fuiſſe, Amicum illum, de quo D.* Barrovius *in duabus prioribus Epiſtolis mentionem fecerat, quod his verbis conſignat.*

Amici

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† I ſend you the Papers of my Friend I promis'd, which I preſume will give you much ſatisfaction; I pray, having peruſed them ſo much as you think good, remand them to me, according to his deſire when I ask'd him the Liberty to impart them to you; I pray give me notice of your receiving *them*, with your ſooneſt convenience, that I may be ſatisfied of their reception; becauſe I am afraid of them, venturing them by the Poſt, that I may not longer delay to correſpond with your deſire.

* Amici chartas tibi placuisse gaudeo; 'est illi nomen *Newtonus*, Collegii nostri Socius, & juvenis, (secundus enim, ex quo 'Artium Magistri gradum cepit, jam agitur 'annus,) et qui, eximio quo est acumine, 'permagnos in hac re progressus fecit. Illas, 'si vis, cum Nobili Domino Vicecomite '*Brounkero* communica.

Perspecto jam D. Newtonum *hujus Tractatus Auctorem esse; ab eo sciscitatus sum num penes se adhuc esset autographum, quod quidem ille exquirens invenit, & mihi tradidit, cum exemplari* Collinsiano *ad verbum usque conveniens.*

Etiamsi vero hic Tractatus ad D. Collinsium *missus fuisset mense* Julii 1669, *quod aliquantulum erat posteaquam D.* Mercator *Logarithmotechniam suam in lucem ediderat; manifestum est ex quibusdam aliis Epistolis, (quæ itidem inter D.* Collinsii *chartas fuerunt,) quod antea scriptus esset, imo quod D.* Newtonus *invenisset Methodum investigandi Magnitudinum Dimensiones per Infinitas Series vel aliquot annos antequam D.* Mercator *Librum suum in vulgus edidit; ut liquet ex Epistola a* Collinsio

* I am glad my Friend's Paper gives you so much satisfaction, his Name is Mr. *Newton*, a Fellow of our College, and very young, (being but the second Year Master of Arts,) but of an extraordinary Genius and Proficiency in these things; you may impart the Papers, if you please, to my Lord *Brounker*.

ad D. Jacobum Gregorium *data* 25. Novemb. 1669, *ubi hæc ſunt verba.*

* *Barrovius* Provinciam ſuam Publice præ‑ ‘legendi remiſit cuidam nomine *Newtono* ‘*Cantabrigienſi*, quem tanquam Virum acutiſſi‑ ‘mi ingenii, in Præfatione Prælectionum Op‑ ‘ticarum memorat, & qui, antequam edere‑ ‘tur *Mercatoris* Logarithmotechnia, Metho‑ ‘dum invenerat eandem, eamque ad omnes ‘Curvas generaliter, & ad Circulum diver‑ ‘ſimode applicârat.’

Quinetiam D. Collinſius *in Epiſtola ad D.* Strode, *data* 26. Julii 1672, *ſic ſcribit.*

† Menſe *Septembri* 1668, *Mercator* Loga‑ ‘rithmotechniam edidit ſuam, quæ ſpeci‑ ‘men hujus Methodi (i. e. Serierum Infini‑ ‘tarum) in unica tantum Figura, nempe, ‘Quadratura Hyperbolæ continet. Haud ‘multo poſtquam in publicum prodiret liber, ‘exemplar unum Cl. *Walliſio Oxoniam* miſi, ‘qui ſuum de eo judicium in *Actis Philo‑* ‘*ſophicis* ſtatim fecit: alterum *Barrovio Can‑* ‘*tabrigiam*, qui quaſdam *Newtoni* chartas, qui ‘jam *Barrovium* in Mathematicis Prælecti‑ ‘onibus

* Mr. *Barrow* hath reſign'd his Lecture's Place *to one* Mr. *Newton* of *Cambridge*, whom he mentions in his Optic Preface as a very ingenious Perſon; one who hath, before Mr. *Mercator*'s Logarithmotechnia was extant, invented the ſame Method, and applied it generally to all Curves, and divers ways to the Circle.

'onibus publicis excipit, extemplo remisit: 'Ex quibus & aliis, quæ olim ab Auctore 'cum *Barrovio* communicata fuerant, patet 'illam Methodum a dicto *Newtono* aliquot 'annis ante excogitatam, & modo generali 'applicatam fuisse: ita ut ejus ope in quavis 'Figura Curvilinea proposita, quæ una vel 'pluribus Proprietatibus definitur, Quadra- 'tura vel Area dictæ Figuræ, accurata si 'possibile sit, sin minus infinite propinqua; 'Evolutio vel Longitudo lineæ Curvæ; Cen- 'trum gravitatis Figuræ; Solida ejus rota- 'tione genita, & eorum Superficies; sine 'ulla Radicum Extractione obtineri queant.

Ubi obiter notemus in hac Collinsii *historiola, methodum argumentandi usurpatam a D.* Newtono *in Tractatu suo* De Quadratura Curvarum, *quasi digito monstrari, dum dicit hanc Methodum exhibere Quadraturam Figurarum accuratam, si modo fieri possit, sin minus in infinitum approximantem,*

† In *September* 1668, Mr. *Mercator* publish'd his Logarithmotechnia, containing a Specimen of this Method, (that is, of Infinite Series) in one only Figure, to wit in the Quadrature of the Hyperbola. Not long after the Book came out, I sent one of them to Dr. *Wallis* at *Oxford*, who forthwith gave his sense of it in the Philos. Transactions: another of them I sent to Dr. *Barrow* at *Cambridge*, who forthwith sent me up some Papers of Mr. *Newton*, who is since become the Doctor's Successor in the Mathematical Lectures there. By which, and former communications made thereof from the Author to the Doctor, it appears that the said Method was invented some Years before by the said Mr. *Newton*, and generally applied: so that thereby in any Curvilinear Figure propos'd, that is determin'd by one or more common Properties, by the same Method may be obtain'd the Quadrature or Area of the said Figure, accurately when it can be done, but always infinitely near; the Evolution, or Length of the said Curve Line; the Centre of Gravity of the Figure; its round Solids made by Rotation, and their Surfaces; and all perform'd without any Extraction of Roots.

tem, atque ista omnia fieri sine ulla Extractione Radicum; Hæc enim ipsa est argumentatio in dicto Tractatu observata: & propterea hanc Methodum saltem Anno 1672 *coætaneam extitisse non est dubitandum.*

Inveni etiam in exemplari Epistolæ, a D. Collinsio *ad* D. Davidem Gregorium *prædicti* Jacobi *fratrem, datæ* 11. Aug. 1676, *præter eadem fere quæ* D. Strode *scripserat, etiam verba sequentia:*

* Supradicta Serierum Infinitarum do-'ctrina, a *Newtono* biennium ante excogi-'tata fuit, quam ederetur *Mercatoris* Lo-'garithmotechnia, & generaliter omnibus 'Figuris applicata.

Simulque affirmat se olim cum quibusdam Academicis Parisiensibus *hæc eadem scripto communicasse.*

Quapropter, cum D. Mercator *Librum suum Anno* 1668 *in lucem ediderit, sequitur eandem Doctrinam Infinitarum Serierum* Figuris *omnibus generaliter applicatam fuisse Anno* 1666.

Denique in Epistola, circa idem tempus ad D. Oldenburgum *scripta, asserit* Collinsius, *Jacobum Gregorium* non nisi conspecta aliqua e Seriebus

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* The said Doctrine of Infinite Series was invented by Mr. *Newton*, about two Years before the Publication of Mr. *Mercator*'s Logarithmotechnia, and generally applied to all Curves.

bus *Newtoni*, quam illi impertierat, in eandem Serierum Methodum incidisse.

Ex Newtoni *autem Schedis quibusdam a me visis intellexi, quod is Quadraturam Circuli, Hyperbolæ, & aliarum quarundam Curvarum per Series Infinitas ex* Wallisii *nostri* Arithmetica Infinitorum, *per Interpolationem Serierum ejus, primo deduxit, idque Anno* 1665; *deinde Methodum excogitavit easdem Series per Divisiones & Extractiones Radicum inveniendi, quam Anno sequente generalem reddidit.*

Et cum scriptus fuerit hic Tractatus, quo tempore hæc recens inventa essent, ideo Cl. Auctor dignatus est multa in eo dilucidare, ad Resolutionem Æquationum per Infinitas Series spectantia, quæ in aliis Libris frustra quæras.

His subjunximus diversa Epistolarum Auctoris fragmenta, quæ ad easdem Doctrinas pertinent, quæq; olim inter Cl. Wallisii *Opera in lucem prodiere. Epistolas haud dedi integras, ut evitarem repetitionem non necessariam multarum rerum, quæ supra in Tractatu* De Analysi per Æquationes Infinitas *traditæ sunt: Quinimo Exempla quædam in iis Epistolis prætermisi ipsius Cl. Auctoris monitu, Regulas suas per se satis claras esse credentis, neque ullam dilucidationem desiderare.*

PRÆFATIO.

Inter D. Collinsii *schedas reperi autographum Epistolæ datæ* 8. Novemb. 1676, *cujus fragmentum sub finem adjeci, & dignum luce putavi; quoniam in eo memoratur solutio Problematis admodum generalis in Comparatione Curvarum usûs, quæ perficitur in Cor.* 2. *Prop.* 10. *Tract.* De Quadratura Curvarum: *Unde Lector intelligat solutionem illam Auctori jam tum innotuisse.*

Hisce Fragmentis annexus est ille ipse Tractatus De Quadratura Curvarum; *una cum altero* De Enumeratione Linearum Tertii Ordinis, *quorum uterque primum typis mandatus est Anno* 1704, *ad finem* Optices *eximiæ ejusdem Cl Auctoris.*

Coronidis loco subjicitur Tractatulus, cui titulo est, Methodus Differentialis, *quem Cl. Auctoris permissu ex ejus autographo descripsi; Complectitur autem Doctrinam describendi Curvas ex datis Differentiis differentiarum Ordinatarum. Hæc* Methodus Differentialis *innititur Problemati ducendi Curvam Parabolici generis per data quotcunque puncta; de quo Cl. Auctor olim mentionem fecerat in Epistola sua ad D.*Oldenburgum 1676 *missa; & cujus solutionem dedit in Lem.* 5. *Lib.* 3. Princip. Philos. *non tamen prorsus eandem quoad Constructionem cum ea quam impræsentiarum tradimus.*

Hujus

PRÆFATIO.

Hujus Geometriæ Newtonianæ *non minimam esse laudem duco, quod dum per limites* Rationum Primarum *&* Ultimarum *argumentatur, æque demonstrationibus Apodicticis ac illa Veterum munitur; utpote quæ haud innititur duriusculæ illi Hypothesi quantitatum Infinite parvarum vel Indivisibilium, quarum Evanescentia obstat quominus eas tanquam quantitates speculemur. Neque parum præcellere videtur, quod tam paucis innixa* Propositionibus *tam late sese extendat hæc Mathesis, intra se omnia* Problemata *difficiliora vulgatas Methodos eludentia complectens; siquidem quicquid proponitur poterit Geometrice per alicujus Curvæ Aream effingi; vel per Methodum* Universalem *Extrahendi* Radices *ex* Æquatione *quavis erui; vel ad summum, ducendo Curvam per terminos quantitatum datarum solvi.*

W. Jones.

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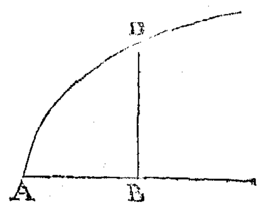
DE ANALYSI

Per Æquationes Numero Terminorum INFINITAS.

Methodum generalem, quam de Curvarum quantitate per Infinitam terminorum Seriem mensuranda, olim excogitaveram, in sequentibus breviter explicatam potius quam accuratè demonstratam habes.

BASI AB Curvæ alicujus AD, sit Applicata BD perpendicularis: Et vocetur $AB = x$, $BD = y$, & sint a, b, c, &c. Quantitates datæ, & m, n, Numeri Integri. Deinde,



Curvarum Simplicium Quadratura.

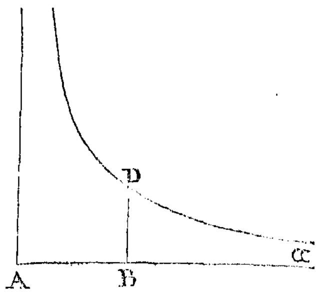
REGULA I.

Si $ax^{\frac{m}{n}} = y$; *Erit* $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = *Areæ ABD.*

Res Exemplo patebit.

1. Si x^2 ($= 1x^{\frac{2}{1}}$) $= y$, hoc est, $a = 1 = n$, & $m = 2$; Erit $\frac{1}{3}x^3 = $ ABD.

2. Si $4\sqrt{x}\ (= 4x^{\frac{1}{2}}) = y$; Erit $\frac{8}{3}x^{\frac{3}{2}}\ (= \frac{8}{3}\sqrt[2]{x^3}) =$ ABD.
3. Si $\sqrt[3]{x^5}\ (= x^{\frac{5}{3}}) = y$; Erit $\frac{3}{8}x^{\frac{8}{3}}\ (= \frac{3}{8}\sqrt[3]{x^8}) =$ ABD.
4. Si $\frac{1}{x^2}\ (= x^{-2}) = y$, id est, si $a = 1 = n$, & $m = -2$;



Erit $(\frac{1}{-1}x^{\frac{-1}{1}} =) - x^{-1}\ (= \frac{-1}{x}) = \alpha$BD, infinite versus α protensæ, quam Calculus ponit negativam, propterea quod jacet ex altera parte Lineæ BD.

5. Si $\frac{1}{\sqrt{x^3}}\ (= x^{-\frac{3}{2}}) = y$; Erit $(\frac{2}{-1}x^{-\frac{1}{2}} =) \frac{2}{-\sqrt{x}} =$ BDα.

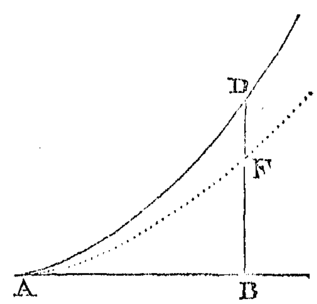
6. Si $\frac{1}{x}\ (= x^{-1}) = y$; Erit $\frac{1}{0}x^{\frac{0}{1}} = \frac{1}{0}x^0 = \frac{1}{0} \times 1 = \frac{1}{0} =$ Infinitæ, qualis est Area Hyperbolæ ex utraque parte Lineæ BD.

Compositarum Curvarum Quadratura ex Simplicibus.

REGULA II.

Si valor ipsius y *ex pluribus istiusmodi Terminis componitur, Area etiam componetur ex Areis quæ a singulis Terminis emanant.*

Exempla Prima.



Si $x^2 + x^{\frac{3}{2}} = y$; Erit $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ ABD.

Etenim si semper sit $x^2 =$ BF, et $x^{\frac{3}{2}} =$ FD, erit, ex præcedente Regula, $\frac{1}{3}x^3 =$ superficiei AFB descriptæ per Lineam BF, et $\frac{2}{5}x^{\frac{5}{2}} =$ AFD descriptæ per DF; Quare $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ toti ABD.

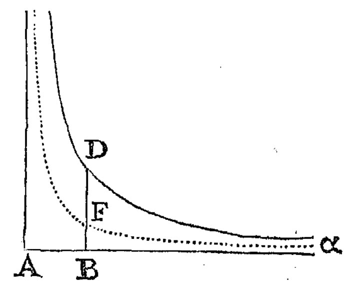
Sic si $x^2 - x^{\frac{3}{2}} = y$; Erit $\frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}} =$ ABD.

Et si $3x - 2x^2 + x^3 - 5x^4 = y$; Erit $\frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5 =$ ABD.

Ex-

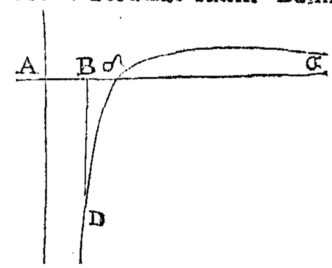
Exempla Secunda.

Si $x^{-2}+x^{-\frac{3}{2}}=y$; Erit $-x^{-1}-2x^{-\frac{1}{2}}=\alpha BD$. Vel ſi $x^{-2}-x^{-\frac{3}{2}}=y$; Erit $-x^{-1}+2x^{-\frac{1}{2}}=\alpha BD$.



Quarum ſigna ſi mutaveris habebis *Affirmativum* valorem ($x^{-1}+2x^{-\frac{1}{2}}$ vel $x^{-1}-2x^{-\frac{1}{2}}$) ſuperficiei αBD, modo tota cadat ſupra baſim $AB\alpha$.

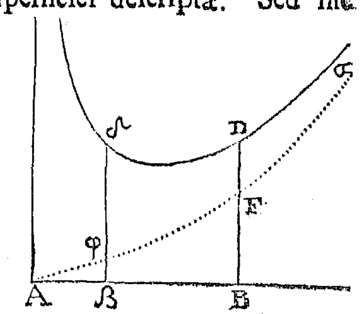
Sin aliqua pars cadat infra (quod fit cum Curva decuſſat ſuam Baſin inter B et α, ut hic vides in δ,) iſta parte a parte ſuperiori ſubducta, habebis valorem Differentiæ: Earum vero Summam ſi cupis, quære utramque Superficiem ſeorſim, et adde. Quod idem in reliquis hujus Regulæ exemplis notandum volo.



Exempla Tertia.

Si $x^2+x^{-2}=y$; Erit $\frac{1}{3}x^3-x^{-1}=$ Superficiei deſcriptæ. Sed hic notandum eſt, quod dictæ Superficei partes ſic inventæ jacent ex diverſo latere Lineæ BD.

Nempe, poſito $x^2=BF$, & $x^{-2}=FD$; Erit $\frac{1}{3}x^3=ABF$ Superficiei per BF deſcriptæ, & $-x^{-1}=DF\alpha$ Superficiei deſcriptæ per DF.



Et

Et hoc ſemper accidit cum Indices $(\frac{m+n}{n})$ rationum Baſis x in valore Superficiei quæſitæ, ſint variis Signis affecti. In hujuſmodi Caſibus, pars aliqua $BD\delta\beta$ Superficiei media (quæ ſola dari poterit, cum Superficies ſit utrinque infinita) ſic invenitur.

Subtrahe Superficiem ad minorem Baſin $A\beta$ pertinentem, a Superficie ad majorem Baſin AB pertinente, & habebis $\beta BD\delta$ Superficiem differentiæ Baſium inſiſtentem. Sic in hoc Exemplo. (*Vide Fig. Præcedentem.*)

Si $AB = 2$, & $A\beta = 1$; Erit $\beta BD\delta = \frac{17}{6}$:

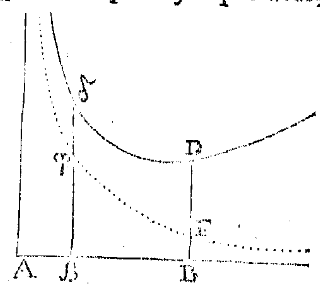
Etenim Superficies ad AB pertinens (viz. $ABF - DF\alpha$) erit $\frac{8}{3} - \frac{1}{2}$ ſive $\frac{13}{6}$; et ſuperficies ad $A\beta$ pertinens (viz. $A\phi\beta - \delta\phi\alpha$) erit $\frac{1}{3} - 1$, ſive $-\frac{2}{3}$: et earum differentia (viz. $ABF - DF\alpha - A\phi\beta + \delta\phi\alpha = \beta BD\delta$) erit $\frac{13}{6} + \frac{2}{3}$ ſive $\frac{17}{6}$.

Eodem modo, ſi $A\beta = 1$, $AB = x$; Erit $\beta BD\delta = \frac{2}{3} + \frac{1}{3}x^3 - x^{-1}$.

Sic ſi $2x^3 - 3x^5 - \frac{2}{3}x^{-4} + x^{-\frac{3}{5}} = y$, & $A\beta = 1$;

Erit $\beta BD\delta = \frac{1}{2}x^4 - \frac{1}{2}x^6 + \frac{2}{9}x^{-3} + \frac{5}{2}x^{\frac{2}{5}} - \frac{49}{18}$.

Denique notari poterit quod ſi quantitas x^{-1} in valore ipſius y reperiatur, iſte Terminus (cum Hyperbolicam ſuperficiem generat) ſeorſim a reliquius conſiderandus eſt.



Ut ſi $x^2 + x^{-3} + x^{-1} = y$: Sit $x^{-1} = BF$, & $x^2 + x^{-3} = FD$, ac $A\beta = 1$; Et erit $\delta\phi FD = \frac{1}{6} + \frac{1}{3}x^3 - \frac{1}{2}x^{-2}$, utpote quæ ex Terminis $x^2 + x^{-3}$ generatur.

Quare, ſi reliqua Superficies $\beta\phi FB$, quæ Hyperbolica eſt, ex Calculo aliquo ſit data, dabitur tota $\beta BD\delta$.

Aliarum

Aliarum Omnium Quadratura.

REGULA III.

Sin valor ipſius y, *vel aliquis ejus Terminus ſit præcedentibus magis compoſitus, in Terminos Simpliciores reducendus eſt; operando in Literis ad eundem Modum quo Arithmetici in Numeris Decimalibus dividunt, Radices extrahunt, vel affectas Æquationes ſolvunt; & ex iſtis Terminis quæſitam Curvæ Superficiem, per præcedentes Regulas deinceps elicies.*

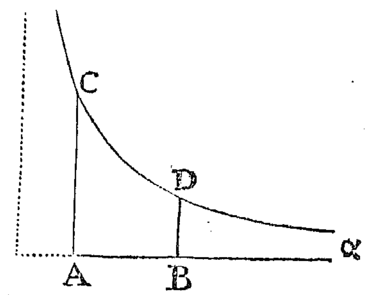
Exempla Dividendo.

Sit $\frac{aa}{b+x} = y$; Curva nempe exiſtente Hyperbola.

Jam ut Æquatio iſta a Denominatore ſuo liberetur, Diviſionem ſic inſtituo.

$$b+x)\ aa + 0\ \Big(\frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4}\ \&c.$$

$$\begin{array}{l}
aa + \frac{aax}{b} \\
\hline
0 - \frac{aax}{b} + 0 \\
\quad - \frac{aax}{b} - \frac{aax^2}{b^2} \\
\hline
\qquad 0 + \frac{aax^2}{b^2} + 0 \\
\qquad + \frac{aax^2}{b^2} + \frac{aax^3}{b^3} \\
\hline
\qquad\quad 0 - \frac{aax^3}{b^3} + 0 \\
\qquad\quad - \frac{aax^3}{b^3} - \frac{aax^4}{b^4} \\
\hline
\qquad\qquad 0 + \frac{aax^4}{b^4} \\
\qquad\qquad \&c.
\end{array}$$



Et sic vice hujus $y=\frac{aa}{b+x}$, nova prodit $y=\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$, &c. serie istac infinite continuata; Adeoque (per Regulam Secundam) Area quæsita ABDC æqualis erit ipsi $\frac{a^2x}{b}-\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$ &c. infinitæ etiam seriei, cujus tamen Termini pauci initiales sunt in usum quemvis satis exacti, si modo x sit aliquoties minor quam b.

Eodem modo, si sit $\frac{1}{1+xx}=y$, Dividendo prodibit

$y=1-x^2+x^4-x^6+x^8$ &c. Unde (per Regulam Secundam)

erit ABDC $=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9$ &c.

Vel si Terminus xx ponatur in divisore primus, hoc modo $xx+1$), prodibit $x^{-2}-x^{-4}+x^{-6}-x^{-8}$ &c. pro valore ipsius y; Unde (per Regulam Secundam)

erit BDα $=-x^{-1}+\frac{1}{3}x^{-3}-\frac{1}{5}x^{-5}+\frac{1}{7}x^{-7}$ &c. Priori modo procede cum x est satis parva, posteriori cum satis magna supponitur.

Denique si $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}=y$; Dividendo prodit

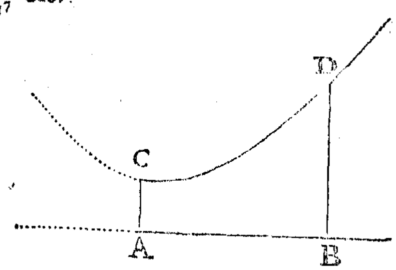
$2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$ &c. unde erit

ABDC $=\frac{4}{3}x^{\frac{3}{2}}-x^2+\frac{14}{5}x^{\frac{5}{2}}-\frac{13}{3}x^3$ &c.

Exempla Radicem Extrahendo.

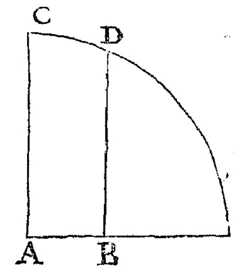
Si sit $\sqrt{aa+xx}=y$, Radicem sic extraho,

$$
\begin{array}{l}
aa+xx\ \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\right.\ \&c. \\
aa \\
\hline
o+xx \\
\quad xx+\frac{x^4}{4a^2} \\
\hline
\quad o-\frac{x^4}{4a^2} \\
\quad -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
\qquad o+\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
\qquad +\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\
\hline
\qquad\quad o-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}} \\
\qquad\qquad \&c.
\end{array}
$$



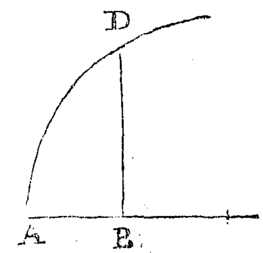
Unde,

Unde, pro Æquatione $\sqrt{aa+xx}=y$, nova producitur, viz. $y = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Et (per Reg. 2.) Area quæsita ABDC erit $= ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. Et hæc est Quadratura Hyperbolæ.



Eodem modo, si sit $\sqrt{aa-xx}=y$, ejus Radix erit $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. Adeoque Area quæsita ABDC erit æqualis $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. Et hæc est Quadratura Circuli.

Vel si ponas $\sqrt{x-xx}=y$; erit Radix æqualis infinitæ seriei $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}}$ &c. Et Area quæsita ABD æqualis erit $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$ &c. sive $x^{\frac{1}{2}}$ in $\frac{2}{3}x - \frac{1}{5}x^2 - \frac{1}{28}x^3 - \frac{1}{72}x^4 - \frac{5}{704}x^5$ &c. Et hæc est Areæ Circuli Quadratura.



Si $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$, (Cujus Quadratura dat Longitudinem curvæ Ellipticæ;) Extrahendo radicem utramq; prodit

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8} \text{ \&c.}$$

Et Dividendo, sicut fit in Fractionibus Decimalibus, habes

$$\begin{array}{lllll} 1 + \frac{1}{2}bx^2 & + \frac{3}{8}b^2x^4 & + \frac{5}{16}b^3x^6 & + \frac{35}{128}b^4x^8 & \text{\&c.} \\ \quad + \frac{1}{2}a & + \frac{1}{4}ab & + \frac{3}{16}ab^2 & + \frac{5}{32}ab^3 & \\ & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & - \frac{3}{64}a^2b^2 & \\ & & + \frac{1}{16}a^3 & + \frac{1}{32}a^3b & \\ & & & - \frac{5}{128}a^4 & \end{array}$$

Adeoque Aream quæsitam

$$\begin{array}{llll} x & + \frac{1}{6}bx^3 & + \frac{3}{40}b^2x^5 & \text{\&c.} \\ & + \frac{1}{6}a & + \frac{1}{20}ab & \\ & & - \frac{3}{40}a^2 & \end{array}$$

Sed.

Sed observandum est, quod Operatio non raro abbreviatur per debitam Æquationis præparationem, ut in allato Exemplo $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}} = y$. Si utramque partem fractionis per $\sqrt{1-bxx}$ multiplices prodibit

$\frac{\sqrt{1 {+a \atop -b} x^2 - abx^4}}{1+bx^2} = y$, & reliquum opus perficitur extrahendo Radicem Numeratoris tantum, & dividendo per Denominatorem.

Ex hisce, credo, satis patebit modus reducendi quemlibet valorem ipsius y (quibuscunque Radicibus vel Denominatoribus sit perplexus, ut hic videre est;

$$x^3 + \frac{\sqrt{x - \sqrt{1-xx}}}{\sqrt[3]{ax^2+x^3}} - \frac{\sqrt[5]{x^3+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+x^2} - \sqrt[2]{2x-x^{\frac{2}{3}}}} = y)$$ in series Infinitas

simplicium Terminorum, ex quibus, per Regulam Secundam, quæsita Superficies cognoscetur.

Exempla per Resolutionem Æquationum.

NUMERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

Quia tota difficultas in Resolutione latet, modum quo ego utor in Æquatione Numerali primum illustrabo.

Sit $y^3 - 2y - 5 = 0$, resolvenda : Et sit 2, numerus qui minus quam decima sui parte differt a Radice quæsita. Tum pono $2 + p = y$, & substituo hunc ipsi valorem in Æquationem, & inde nova prodit $p^3 + 6p^2 + 10p - 1 = 0$, cujus Radix p exquirenda est, ut quotienti addatur : Nempe (neglectis $p^3 + 6p^2$ ob parvitatem) $10p - 1 = 0$, sive $p = 0{,}1$ prope veritatem est ; itaque scribo 0,1 in quotiente, & suppono $0{,}1 + q = p$, & hunc ejus valorem, ut prius substituo, unde prodit $q^3 + 6{,}3q^2 + 11{,}23q + 0{,}061 = 0$.

Et cum $11{,}23q + 0{,}061$ veritati prope accedit, sive fere sit q æqualis $-0{,}0054$ (dividendo nempe donec tot eliciantur Figuræ, quot locis primæ Figuræ hujus & principalis quotientis exclusive distant) scribo $-0{,}0054$ in inferiori parte quotientis, cum negativa sit.

Et

Et supponens $-0{,}0054 + r = q$, hunc ut prius substituo, & operationem sic produco quo usq; placuerit. Verum si ad bis tot figuras tantum quot in Quotiente jam reperiuntur una dempta, operam continuare cupiam, pro q substituo $-0{,}0054 + r$ in hanc $6{,}3q^2 + 11{,}23q + 0{,}061$, scilicet primo ejus termino (q^3) propter exilitatem suam

$y^3 - 2y - 5 = 0$		$+2{,}10000000$ $-0{,}00544853$ $+2{,}09455147 = y$
$2 + p = y$	$+y^3$	$+8 + 12p + 6p^2 + p^3$
	$+2y$	$-4 - 2p$
	-5	-5
	Summa	$-1 + 10p + 6p^2 + p^3$
$0{,}1 + q = p$	$+p^3$	$+0{,}001 + 0{,}03q + 0{,}3q^2 + q^3$
	$+6p^2$	$+0{,}06 \quad + 1{,}2 \quad + 6{,}0$
	$+10p$	$+1, \quad + 10,$
	-1	$-1,$
	Summa	$+0{,}061 + 11{,}23q + 6{,}3q^2 + q^3$
$-0{,}0054 + r = q$	$+6{,}3q^2$	$+0{,}000183708 - 0{,}06804r + 6{,}3r^2$
	$+11{,}23q$	$-0{,}060642 \quad + 11{,}23$
	$+0{,}061$	$+0{,}061$
	Summa	$+0{,}000541708 + 11{,}16196r + 6{,}3r^2$
$-0{,}00004854 + s = r$		

neglecto, & prodit $6{,}3r^2 + 11{,}16196r + 0{,}000541708 = 0$ fere, sive (rejecto $6{,}3r^2$) $r = \frac{-0{,}000541708}{11{,}16196} = -0{,}00004853$ fere, quam scribo in negativa parte Quotientis. Denique negativam partem Quotientis ab Affirmativa subducens habeo $2{,}09455147$ Quotientem quæsitam.

Æquationes plurium dimensionum nihilo secius resolvuntur, & operam sub fine, ut hic factum fuit levabis, si primos ejus terminos gradatim omiseris.

Præterea notandum est quod in hoc exemplo, si dubitarem an $0{,}1 = p$ veritati satis accederet, pro $10p - 1 = 0$, finxissem $6p^2 + 10p - 1 = 0$, & ejus radicis primam figuram in Quotiente scripsissem; & secundam vel tertiam Quotientis figuram sic explorare convenit, ubi in Æquatione ista ultimo resultante quadratum coefficientis penultimi termini, non sit decies majus quam factus ex ultimo termino ducto in coefficientem termini antepenultimi.

Imo laborem plerumque minues præsertim in Æquationibus plurimarum dimensionum, si figuras omnes Quotienti addendas dicto modo (hoc est extrahendo minorem radicum, ex tribus ultimis terminis Æquationis novissime resultantis) exquiras: Isto enim modo figuras duplo plures qualibet vice Quotienti lucraberis.

Hæc Methodus resolvendi Æquationes pervulgata an sit nescio, certe mihi videtur præ reliquis simplex, & usui accommodata. Demonstratio ejus ex ipso modo operandi patet, unde cum opus sit, in memoriam facile revocatur.

Æquationes in quibus vel aliqui vel nulli Termini desint, eadem fere facilitate tractantur; & Æquatio semper relinquitur, cujus Radix una cum acquisita Quotiente adæquat Radicem Æquationis primo propositæ. Unde Examinatio Operis hic æque poterit institui ac in reliqua Arithmetica, auferendo nempe Quotientem a Radice primæ Æquationis (sicut Analystis notum est) ut Æquatio ultima vel Termini ejus duo tresve ultimi producantur inde. Quicquid laboris hic est, istud in Operatione substituendi quantitates unas pro aliis reperietur: Id quod varie perficias, at sequentem modum maxime expeditum puto, præsertim ubi Numeri Coefficientes constant ex pluribus Figuris.

Sit $p+3$ substituenda pro y in hanc $y^4-4y^3+5y^2-12y+17=0$. Et cum ista possit resolvi in hanc formam

$\overline{\overline{\overline{y-4}\times y+5}\times y-12}\times y+17=0$. Æquatio nova sic generabitur $\overline{p-1}\times\overline{p+3}=p^2+2p-3$. et p^2+2p+2 in $p+3=p^3+5p^2+8p+6$. et p^3+5p^2+8p-6 in $p+3=p^4+8p^3+23p^2+18p-18$. et $p^4+8p^3+23p^2+18p-1=0$, quæ quærebatur.

LITERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

His in numeris sic ostensis: Sit Æquatio literalis
$y^3+a^2y-2a^3+axy-x^3=0$, resolvenda.

Primum inquiro valorem ipsius y cum x sit nulla, hoc est, elicio Radicem hujus Æquationis $y^3+a^2y-2a^3=0$, & invenio esse $+a$. Itaque scribo $+a$ in Quotiente, & supponens $+a+p=y$, substituo pro y valorem ejus, & Terminos inde resultantes ($p^3+3ap^2+4a^2p$, &c.) margini appono; Ex quibus assumo $+4a^2p+a^2x$ terminos utique ubi p & x seorsim sunt minimarum dimensionum, & eos nihilo fere æquales esse suppono, sive $p=-\frac{1}{4}x$ fere, vel $p=-\frac{1}{4}x+q$. Et scribens

$-\frac{1}{4}x$

$-\frac{1}{4}x$ in Quotiente, ſubſtituo $-\frac{1}{4}x+q$ pro p; Et terminos inde reſultantes iterum in margine ſcribo, ut vides in annexo ſchemate, & inde aſſumo Quantitates $+4a^2q-\frac{1}{16}ax^2$, in quibus utiq; q & x ſeorſim ſunt minimarum dimenſionum, & fingo $q=\frac{xx}{64a}$ fere, ſive $q=+\frac{xx}{64a}+r$; & adnectens $+\frac{xx}{64a}$ Quotienti, ſubſtituo $\frac{xx}{64a}+r$ pro q; & ſic procedo quo uſque placuerit.

$$y^3+a^2y-2a^3+axy-x^3=0$$

$$y=a-\frac{x}{4}+\frac{x^2}{64a}-\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}\ \&c.$$

$+a+p=y$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3$
	$+a^2y$	$+a^3+a^2p$
	$+axy$	$+a^2x+axp$
	$-2a^3$	$-2a^3$
	$-x^3$	$-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$
	$+4a^2p$	$-a^2x+4a^2q$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$
	$+4a^2q$	$+\frac{1}{16}ax^2+4a^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+\frac{3}{16}x^2q$	$-\frac{3x^4}{1024a}+\frac{3}{16}x^2r$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$

$$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\ (+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$$

Sin duplo tantum plures Quotienti terminos, uno dempto, jungendos adhuc vellem: Primo termino (q^3) Æquationis noviſſime reſultantis miſſo, & iſta etiam parte ($-\frac{3}{4}xq^2$) ſecundi, ubi x eſt tot dimenſionum quot in penultimo termino Quotientis; In reliquos terminos ($3aq^2+4a^2q$, &c.)

margini

margini adſcriptos ut vides, ſubſtituo $\frac{x^2}{64a} + r$ pro q; & ex ultimis duobus terminis ($\frac{15x^4}{4096a} - \frac{131}{128}x^3 + \frac{9}{32}x^2r - \frac{1}{2}axr + 4a^2r$) Æquationis inde reſultantis, facta diviſione $4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2$) $+ \frac{131}{128}x^3 - \frac{15x^4}{4096a}$ (elicio $+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ Quotienti adnectendos.

Denique Quotiens iſta ($a - \frac{x}{4} + \frac{xx}{64a}$, &c.) per Regulam ſecundam, dabit $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3}$, &c. pro Area quæſita, quæ ad veritatem tanto magis accedit, quanto x ſit minor.

Alius modus eaſdem Reſolvendi.

Sin valor Areæ tanto magis ad veritatem accedere debet quanto x ſit major; Exemplum eſto $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Itaque hanc reſoluturus excerpo terminos $y^3 + x^2y - 2x^3$ in quibus x & y vel ſeorſim, vel ſimul multiplicatæ, ſunt & plurimarum, & æqualium ubique dimenſionum; & ex iis quaſi nihilo æqualibus Radicem elicio. Hanc invenio eſſe x, & in Quotiente ſcribo. Vel quod eodem recidit, ex $y^3 + y - 2$ (unitate pro x ſubſtituta) Radicem extraho quæ hic prodit 1, & eam per x multiplico, & factum (x) in Quotiente ſcribo. Denique pono $x + p = y$, & ſic procedo ut in priori Exemplo, donec habeam Quotientem $x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$, &c. adeoque Aream $\frac{x^2}{2} - \frac{ax}{4}$ $+ \boxed{\frac{aa}{64x}} - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$, de qua vide exempla tertia Regulæ ſecundæ. Lucis gratia dedi hoc exemplum in omnibus idem cum priori, modo x & a ſibi invicem ibi ſubſtituantur, ut non opus eſſet aliud Reſolutionis exemplum hic adjungere.

Area autem ($\frac{xx}{2} - \frac{ax}{4} + \boxed{\frac{aa}{64x}}$ &c. terminatur ad Curvam quæ juxta Aſymptoton aliquam in infinitum ſerpit; & Termini initiales ($x - \frac{1}{4}a$) valoris extracti de y, in Aſymptoton iſtam ſemper terminantur; unde portionem Aſymptoti facile invenies. Idem ſemper notandum eſt cum Area deſignatur terminis plus pluſque diviſis per x continue, præterquam quod vice Aſymptoti rectæ quandoque habeatur Parabola Conica, vel alia magis compoſita.

Sed

Sed hunc modum missum faciens, utpote particularem, quia non applicabilem Curvis in orbem ad instar Ellipsium flexis; de altero modo per exemplum $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, supra ostenso (scilicet quo demensiones ipsius x in numeratoribus quotientis perpetuo augeantur) annotabo sequentia.

1. Si quando accidit quod valor ipsius y, cum x nullum esse fingitur, sit quantitas surda vel penitus ignota, licebit illam litera aliqua designare. Ut in exemplo, $y^3+a^2y+axy - 2a^3 - x^3=0$, si radix hujus $y^3+a^2y - 2a^3$ fuisset surda vel ignota, finxissem quamlibet (b) pro ea ponendam; et resolutionem ut sequitur perfecissem. Scribens b in Quotiente, suppono $b + p = y$, & istum pro y substituo, ut vides; unde nova $p^3 + 3bp^2$, &c. resultat, rejectis terminis $b^3+a^2b - 2a^3$, qui nihilo sunt æquales, propterea quod b supponitur Radix hujus $y^3 + a^2y - 2a^3 = 0$. Deinde termini $3b^2p + a^2p + abx$ dant $-\frac{abx}{3b^2+a^2}$ quotienti apponendum, et $-\frac{abx}{3b^2+a^2} + q$ substituendum pro p, &c.

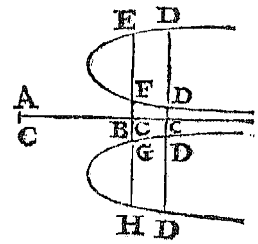
$y^3 + aay + axy - 2a^3 - x^3 = 0.$ Sit $cc = 3b^2 + a^2$.		
$y = b - \frac{abx}{c^2} + \frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} - \frac{a^5bx^3}{c^8} + \frac{a^5b^3x^3}{c^{10}}$ &c.		
$b + p = y$	y^3	$+ b^3 + 3b^2p + 3bp^2 + p^3$
	$+ axy$	$+ abx + axp$
	$+ aay$	$+ aab + aap$
	$- x^3$	$- x^3$
	$- 2a^3$	$- 2a^3$
$\frac{-abx}{cc} + q = p$	p^3	$- \frac{a^3b^3x^3}{c^6}$ &c.
	$+ 3bp^2$	$+ \frac{3a^2b^3x^2}{c^4} - \frac{6ab^2x}{c^2}q$ &c.
	$+ axp$	$- \frac{a^2bx^2}{c^2} + axq$
	$+ ccp$	$- abx + ccq$
	$- x^3$	$- x^3$
	$+ abx$	$+ abx$
$c^2 + ax - \frac{6ab^2x}{cc}$) $\frac{a^4bx^2}{c^4} + x^3 + \frac{a^3b^3x^3}{c^6}$ ($\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8}$ &c.		

Completo opere, sumo numerum aliquem pro a, & hanc $y^3 + a^2y - 2a^3 = 0$, sicut de numerali æquatione ostensum supra resolvo; & radicem ejus pro b substituo.

2. Si dictus valor sit nihil, hoc est si in æquatione resolvenda nullus sit terminus nisi qui per x vel y sit multiplicatus, ut in hac $y^3 - axy + x^3 = 0$; tum terminos $(- axy + x^3)$ seligo in quibus x seorsim & y etiam seorsim si fieri potest, alias per x multiplicata, sit minimarum dimensionum. Et

illi dant $+\frac{xx}{a}$ pro primo termino quotientis, & $\frac{xx}{a}+p$ pro y substituendum. In hac $y^3-a^2y+axy-x^3=0$, licebit primum terminum quotientis vel ex $-a^2y-x^3$, vel ex y^3-a^2y elicere.

3. Si valor iste sit imaginarius, ut in hac $y^4+y^2-2y+6-x^2y^2-2x+x^2+x^4=0$, augeo vel imminuo quantitatem x donec dictus valor evadat realis.



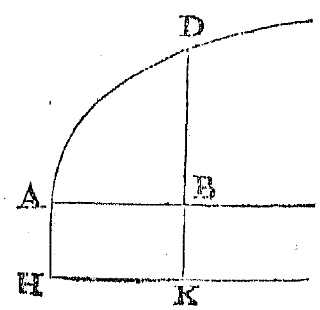
Sic in annexo schemate, cum AC (x) nulla est, tum CD (y) est imaginaria.

Sin minuatur AC per datam AB, ut BC fiat x; tum posito quod BC (x) sit nulla, CD (y) erit valore quadruplici (CE, CF, CG, vel CH) realis; quarum radicum (CE, CF, CG, vel CH) quælibet potest esse primus terminus quotientis, prout superficies BEDC, BFDC, BGDC, vel BHDC desideratur. In aliis etiam casibus, si quando hæsitas, te hoc modo extricabis.

Deniq; si index potestatis ipsius x vel y sit fractio, reduco ipsum ad integrum: ut in hoc exemplo $y^3-xy^{\frac{1}{2}}+x^{\frac{4}{3}}=0$. Posito $y^{\frac{1}{2}}=v$, & $x^{\frac{1}{3}}=z$, resultabit $v^6-z^3v+z^4=0$, cujus radix est $v=z+z^3$, &c. sive (restituendo valores) $y^{\frac{1}{2}}=x^{\frac{1}{3}}+x$, &c. & quadrando $y=x^{\frac{2}{3}}+2x^{\frac{4}{3}}$, &c.

Et hæc de areis curvarum investigandis dicta sufficiant. Imo cum Problemata omnia de curvarum Longitudine, de quantitate & superficie solidorum, deque Centro Gravitatis, possunt eo tandem reduci ut quæratur quantitas Superficiei planæ linea curva terminatæ, non opus est quicquam de iis adjungere. In istis autem quo ego operor modo dicam breviss1me.

Applicatio prædictorum ad reliqua istiusmodi Problemata.



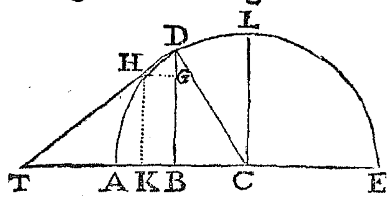
Sit ABD curva quævis, & AHKB rectangulum cujus latus AH vel BK est unitas. Et cogita rectam DBK uniformiter ab AH motam, areas ABD & AK describere; & quod BK (1) sit momentum quo AK (x) & BD (y) momentum quo ABD gradatim augetur; & quod ex momento BD perpetim dato, possis, per prædictas regulas, aream ABD ipso descriptam investigare, sive cum AK (x) momento 1 descripta conferre.

Jam

Jam qua ratione Superficies ABD ex momento ſuo perpetim dato, per præcedentes regulas elicitur, eadem quælibet alia quantitas ex momento ſuo ſic dato elicietur. Exemplo res fiet clarior.

Longitudines Curvarum invenire.

Sit ADLE circulus cujus arcus AD longitudo eſt indaganda. Ducto tangente DHT, & completo indefinite parvo rectangulo HGBK, & poſito AE $= 1 = 2$AC. Erit ut BK ſive GH, momentum Baſis AB (x), ad HD momentum Arcus AD :: BT : DT :: BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) :: 1 (BK) :



$\frac{1}{2\sqrt{x-xx}}$ (DH). Adeoque $\frac{1}{2\sqrt{x-xx}}$ ſive $\frac{\sqrt{x-xx}}{2x-2xx}$ eſt momentum Arcus AD.

Quod reductum fit $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ &c.

Quare, per regulam ſecundam, longitudo Arcus AD eſt

$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}$ &c.

ſive $x^{\frac{1}{2}}$ in $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5$, &c.

Non ſecus ponendo CB eſſe x, & radium CA eſſe 1, invenies Arcum LD eſſe $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

Sed notandum eſt quod unitas iſta quæ pro momento ponitur eſt Superficies cum de Solidis, & linea cum de ſuperficiebus, & punctum cum de lineis (ut in hoc exemplo) agitur.

Nec vereor loqui de unitate in punctis, ſive lineis infinite parvis, ſiquidem proportiones ibi jam contemplantur Geometræ, dum utuntur methodis Indiviſibilium.

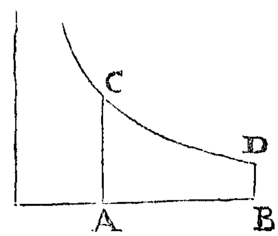
Ex his fiat conjectura de ſuperficiebus & quantitatibus ſolidorum, ac de Centris Gravitatum.

Invenire prædictorum converſum.

Verum ſi e contra ex area vel longitudine &c. Curvæ alicujus data longitudo Baſis AB deſideratur, ex æquationibus per præcedentes regulas inventis extrahatur radix de x.

Inventio Basis ex Area data.

Ut si ex area ABDC Hyperbolæ ($\frac{1}{1+x} = y$) data, cupiam basim AB investigare, area ista z nominata, extraho radicem hujus z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglectis illis terminis in quibus x est plurium dimensionum quam z in quotiente desideratur.



Ut si vellem quod z ad quinque tantum dimensiones in quotiente ascendat, negligo omnes $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. et radicem hujus tantum $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$ extraho.

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ &c.		
$z + p = x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{4}z^4 - z^3p$, &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ x$	$+z + p$
	$- z$	$-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{2}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$-z\ p$	$-\frac{1}{2}z^3 - zq$
	$+\ p$	$+\frac{1}{2}z^2 + q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 - \frac{1}{20}z^5\ (\ \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

Analysin ut vides exhibui propter adnotanda duo sequentia.

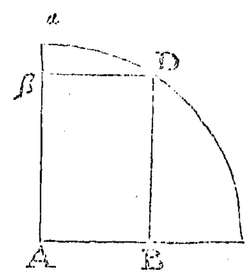
1. Quod inter substituendum, istos terminos semper omitto quos nulli deinceps usui fore prævideam. Cujus rei regula esto, quod post primum terminum ex qualibet quantitate sibi collaterali resultantem non addo plures terminos dextrorsum quam istius primi termini index dimensionis

ab

ab indice dimensionis maximæ unitatibus distat. Ut in hoc exemplo, ubi maxima dimensio est 5, omisi omnes terminos post z^5, post z^4 posui unicum, & duos tantum post z^3. Cum radix extrahenda (x) sit parium ubique, vel imparium dimensionum, hæc esto regula; Quod post primum terminum ex qualibet quantitate sibi collaterali resultantem non addo plures terminos dextrorsum, quam istius primi termini index dimensionis ab indice dimensionis maximæ binis unitatibus distat; vel ternis unitatibus, si indices dimensionum ipsius x unitatibus ubique ternis a se invicem distant, & sic de reliquis.

2. Cum video p, q, vel r, &c. in æquatione novissime resultante esse unius tantum dimensionis, ejus valorem, hoc est, reliquos terminos quotienti addendos, per divisionem quæro. Ut hic vides factum.

Inventio Basis ex data Longitudine Curvæ.



Si ex dato arcu αD Sinus AB desideratur; æquationis $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. supra inventæ, (posito nempe $AB = x$, $\alpha D = z$, & $A\alpha = 1$,) radix extracta erit $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

Et præterea si Cosinum $A\beta$ ex isto arcu dato cupis, fac $A\beta\ (= \sqrt{1-xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

De Serie progressionum continuanda.

Hic obiter notetur, quod 5 vel 6 terminis istarum radicum cognitis, eas plerumque ex analogia observata poteris ad arbitrium producere.

Sic hanc $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. produces dividendo ultimum terminum per hos ordine numeros 2, 3, 4, 5, 6, 7, &c.

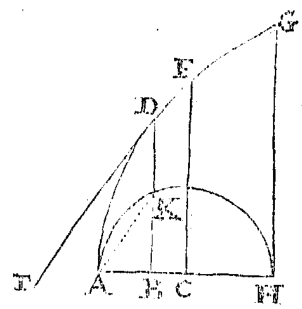
Et hanc $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. per hos 2×3, 4×5, 6×7, 8×9, 10×11, &c.

Et hanc $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. per hos 1×2, 3×4, 5×6, 7×8, 9×10, &c.

Et hanc $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. multiplicando per hos $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, &c. Et sic in reliquis.

Applicatio prædictorum ad Curvas Mechanicas.

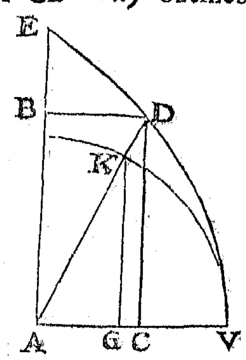
Et hæc de curvis Geometricis dicta sufficiant. Quinetiam curva etiamsi Mechanica sit, methodum tamen nostram nequaquam respuit.



Exemplo sit Trochoides, ADFG, cujus vertex A, & axis AH, & AKH rota qua describitur. Et quæratur Superficies ABD. Jam posito AB $= x$, BD $= y$, ut supra, & AH $= 1$; primo quæro Longitudinem ipsius BD. Nempe ex natura Trochoidis est KD = arcui AK. Quare tota BD = BK + arc. AK. Sed est BK $(= \sqrt{x - xx}) = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. & (ex prædictis) arcus AK $= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$, &c. Ergo tota BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. Et (per Reg. 2.) area ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}}$, &c.

Vel brevius sic: Cum recta AK tangenti TD parallela sit, erit AB ad BK sicut momentum lineæ AB ad momentum lineæ BD, hoc est $x : \sqrt{x - xx} :: 1 : \frac{1}{x}\sqrt{x - xx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$, &c. Quare (per Reg. 2.) BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$, &c. Et superficies ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}}$, &c.

Non dissimili modo (posito C centro circuli, & CB $= x$) obtinebis aream CBDF, &c.



Sit area ABDV Quadratricis VDE (cujus vertex est V, & A centrum circuli interioris VK cui aptatur) invenienda. Ducta qualibet AKD, demitto perpendiculares DB, DC, KG. Eritque KG : AG :: AB (x) : BD (y), sive $\frac{x \times AG}{KG} = y$. Verum ex natura Quadratricis est BA (= DC) = arcui VK, sive VK $= x$. Quare posito AV $= 1$, erit GK $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. ex supra ostensis, & GA $= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c.

Adeoque

Adeoque $y \left(= \frac{x \times AG}{KG}\right) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6}$ &c. ſive, diviſione facta, $y = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6$, &c. et (per Reg. 2.) area $AVDB = x - \frac{1}{9}x^3 - \frac{1}{225}x^5 - \frac{2}{6615}x^7$, &c.

Sic longitudo Quadratricis VD, licet calculo difficiliori, determinabilis eſt.

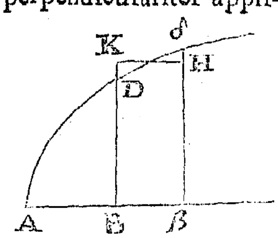
Nec quicquam hujuſmodi ſcio ad quod hæc methodus idque variis modis, ſeſe non extendit. Imo tangentes ad curvas Mechanicas (ſi quando id non alias fiat) hujus ope ducantur. Et quicquid vulgaris Analyſis per æquationes ex finito terminorum numero conſtantes (quando id ſit poſſibile) perficit, hæc per æquationes infinitas ſemper perficiat: Ut nil dubitaverim nomen Analyſis etiam huic tribuere. Ratiocinia nempe in hac non minus certa ſunt quam in illa, nec æquationes minus exactæ; licet omnes earum terminos, nos homines & rationis finitæ nec deſignare neque ita concipere poſſimus, ut quantitates inde deſideratas exacte cognoſcamus: Sicut radices ſurdæ finitarum æquationum nec numeris nec quavis arte Analytica ita poſſunt exhiberi ut alicujus quantitas a reliquis diſtincta exacte cognoſcatur.

Denique ad Analyticam merito pertinere cenſeatur cujus beneficio curvarum areæ & longitudines &c. (id modo fiat) exacte & Geometrice determinentur. Sed iſta narrandi non eſt locus, reſpicienti duo præ reliquis demonſtranda occurrunt.

1. *Demonſtratio quadraturæ curvarum ſimplicium in Regula prima.*

Præparatio pro Regula prima demonſtranda.

Sit itaque curvæ alicujus $AD\delta$ Baſis $AB = x$, perpendiculariter applicata $BD = y$, & area $ABD = z$, ut prius. Item ſit $B\beta = o$, $BK = v$, & rectangulum $B\beta HK$ (ov) æquale ſpatio $B\beta\delta D$.



Eſt ergo $A\beta = x + o$, & $A\delta\beta = z + ov$. His præmiſſis, ex relatione inter x & z ad arbitrium aſſumpta quæro y iſto, quem ſequentem vides, modo.

Pro lubitu ſumatur $\frac{2}{3}x^{\frac{3}{2}} = z$, ſive $\frac{4}{9}x^3 = zz$. Tum $x + o$ $(A\beta)$ pro x, & $z + ov$ $(A\delta\beta)$ pro z ſubſtitutis, prodibit $\frac{4}{9}$ in $x^3 + 3x^2o + 3xo^2 + o^3 =$ (ex natura curvæ) $z^2 + 2zov + o^2v^2$.

Et

Et ſublatis ($\frac{4}{9}x^3$ & zz) æqualibus, reliquiſque per o diviſis, reſtat $\frac{4}{9}$ in $3x^2 + 3xo + o^2 = 2zv + ov^2$. Si jam ſupponamus $B\beta$ in infinitum diminui & evaneſcere, ſive o eſſe nihil, erunt v & y æquales, & termini per o multiplicati evaneſcent, quare reſtabit $\frac{4}{9} \times 3xx = 2zv$, ſive $\frac{2}{3}xx$ ($= zy$) $= \frac{2}{3}x^{\frac{3}{2}}y$, ſive $x^{\frac{1}{2}}$ ($= \frac{x^2}{x^{\frac{3}{2}}}$) $= y$. Quare e contra ſi $x^{\frac{1}{2}} = y$, erit $\frac{2}{3}x^{\frac{3}{2}} = z$.

Demonſtratio.

Vel generaliter, ſi $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; ſive, ponendo $\frac{na}{m+n} = c$, & $m+n = p$, ſi $cx^{\frac{p}{n}} = z$, ſive $c^n x^p = z^n$: tum $x+o$ pro x, & $z+ov$ (ſive, quod perinde eſt, $z+oy$) pro z, ſubſtitutis, prodit c^n in $x^p + pox^{p-1}$, &c. $= z^n + noyz^{n-1}$, &c. reliquis nempe terminis, qui tandem evaneſcerent, omiſſis. Jam ſublatis $c^n x^p$ & z^n æqualibus, reliquiſque per o diviſis, reſtat $c^n px^{p-1} = nyz^{n-1}$ ($= \frac{nyz^n}{z} = \frac{nyc^n x^p}{cx^{\frac{p}{n}}}$ ſive, dividendo per $c^n x^p$, erit $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$ ſive $pcx^{\frac{p-n}{n}} = ny$; vel reſtituendo $\frac{na}{m+n}$ pro c, & $m+n$ pro p, hoc eſt, m pro $p-n$, & na pro pc, fiet $ax^{\frac{m}{n}} = y$. Quare e contra, ſi $ax^{\frac{m}{n}} = y$, erit $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$. Q. E. D.

Inventio curvarum quæ poſſunt quadrari.

Hinc in tranſitu notetur modus quo curvæ tot quot placuerit, quarum areæ ſunt cognitæ, poſſunt inveniri; ſumendo nempe quamlibet æquationem pro relatione inter aream z & baſin x ut inde quæratur applicata y. Ut ſi ſupponas $\sqrt{aa+xx} = z$, ex calculo invenies $\frac{x}{\sqrt{aa+xx}} = y$. Et ſic de reliquis.

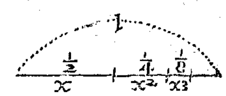
2. *Demonſtratio reſolutionis æquationum affectarum.*

Alterum demonſtrandum eſt literalis æquationum affectarum reſolutio. Nempe quod Quotiens, cum x ſit ſatis parva, quo magis producitur eo magis ad veritatem accedit, ut defectus (p, q, vel r, &c.) quo diſtat ab exacto valore

valore ipſius y, tandem evadat minor quavis data quantitate; & in infinitum producta ſit ipſi y æqualis. Quod ſic patebit.

1. Quoniam ex ultimo termino æquationum quarum p, q, r, &c. ſunt radices, quantitas illa in qua x eſt minimæ dimenſionis (hoc eſt, pluſquam dimidium iſtius ultimi termini, ſi ſupponis x ſatis parvam eſſe) in qualibet operatione perpetuo tollitur: iſte ultimus terminus (per 1. 10. *Elem.*) tandem evadet minor quavis data quantitate; & prorſus evaneſcet ſi opus infinite continuatur.

Nempe ſi $x = \frac{1}{2}$, erit x dimidium omnium $x + x^2 + x^3 + x^4$, &c. et x^2 dimidium omnium $x^2 + x^3 + x^4 + x^5$, &c. Itaque ſi $x \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x + x^2 + x^3$, &c. et x^2 pluſquam dimidium omnium $x^2 + x^3 + x^4$, &c. Sic ſi $\frac{x}{b} \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x + \frac{x^2}{b} + \frac{x^3}{bb}$, &c. Et ſic de reliquis. Et numeros coefficientes quod attinet, illi plerumque decreſcunt perpetuo, vel ſi quando increſcant, tantum opus eſt ut x aliquoties adhuc minor ſupponatur.



2. Si ultimus terminus alicujus æquationis continuo diminuatur donec tandem evaneſcat, una ex ejus radicibus etiam diminuetur donec cum ultimo termino ſimul evaneſcat.

3. Quare quantitatum p, q, r, &c. unus valor continuo decreſcit donec tandem, cum opus in infinitum producitur, penitus evaneſcat.

4. Sed valores iſtarum p, q, vel r, &c. una cum quotiente eatenus extracta adæquant radices æquationis propoſitæ (Sic in reſolutione æquationis $y^3 + aay + axy - 2a^3 - x^3 = 0$, ſupra oſtenſa, percipies $y = a + p = a - \frac{1}{4}x + q = a - \frac{1}{4}x + \frac{xx}{64a} + r$, &c.) Unde ſatis liquet propoſitum quod quotiens infinite producta eſt una ex valoribus de y.

Idem patebit ſubſtituendo quotientem pro y in æquationem propoſitam. Videbis enim terminos illos ſeſe perpetuo deſtruere in quibus x eſt minimarum dimenſionum.

CUM in Epistolis D. Newtoni, *vel in lucem jamdudum editis, vel quæ in manus nostras inciderunt, reperiantur aliqua quæ ad hanc Doctrinam pertinent, ea excerpere & huic Tractatui adjungere visum est.*

EXCERPTA
Ex Epiſtolis D. *NEWTONI* Ad Methodum FLUXIONUM, ET SERIERUM INFINITARUM Spectantibus.

Fragmentum **Epiſtolæ ad D.* Oldenburgium 13 Junii 1676 *miſſæ*.

FRactiones in Infinitas Series reducuntur per diviſionem; & quantitates radicales per extractionem radicum, perinde inſtituendo operationes iſtas in ſpeciebus ac inſtitui ſolent in decimalibus numeris. Hæc ſunt fundamenta harum reductionum; ſed extractiones radicum, multum abbreviantur per hoc *Theorema*.

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Ubi $P+PQ$ ſignificat quantitatem cujus Radix, vel etiam dimenſio quævis, vel radix dimenſionis, inveſtiganda eſt. P, primum terminum quantitatis ejus; Q, reliquos terminos diviſos per primum. Et $\frac{m}{n}$, numeralem indicem dimenſionis ipſius $P+PQ$: Sive dimenſio illa integra ſit; ſive (ut ita loquar) fracta; ſive affirmativa, ſive negativa. Nam, ſicut Analyſtæ, pro aa, aaa, &c. ſcribere ſolent a^2, a^3, &c. ſic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.a^5}$, &c. ſcribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, ſcribo a^{-1}, a^{-2}, a^{-3}.

Et

* *Extat Epiſtola in Tom.* 3. *Operum* Walliſii.

Et sic pro $\frac{aa}{\sqrt{c:\,a^3+b^2x}}$ scribo $aa\times\overline{a^3+b^2x}|^{-\frac{1}{3}}$; & pro $\frac{a^2b}{\sqrt{c:\,\overline{a^3+b^2x}\times\overline{a^3+b^2x}}}$, scribo $a^2b\times\overline{a^3+b^2x}|^{-\frac{2}{3}}$: In quo ultimo casu, si $\overline{a^3+b^2x}|^{-\frac{2}{3}}$ concipiatur esse $\overline{P+PQ}|^{\frac{m}{n}}$ in Regula; erit $P=a^3$, $Q=\frac{b^2x}{a^3}$, $m=-2$, & $n=3$. Denique, pro terminis inter operandum inventis in quoto, usurpo A, B, C, D, &c. nempe A pro primo termino $P^{\frac{m}{n}}$, B pro secundo $\frac{m}{n}AQ$, & sic deinceps. Cæterum usus Regulæ patebit exemplis.

Exempl. 1. Est $\sqrt{c^2+x^2}$ (seu $\overline{c^2+x^2}|^{\frac{1}{2}}$) $=c+\frac{x^2}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}+\frac{7x^{10}}{256c^9}$ &c. Nam, in hoc casu, est $P=c^2$, $Q=\frac{x^2}{c^2}$, $m=1$, $n=2$, A $(=P^{\frac{m}{n}}=\overline{cc}|^{\frac{1}{2}})=c$, B $(=\frac{m}{n}AQ)=\frac{x^2}{2c}$, C $(=\frac{m-n}{2n}BQ)=-\frac{x^4}{8c^3}$, & sic deinceps.

Exempl. 2. Est $\sqrt{5:\,c^5+c^4x-x^5}$ (i. e. $\overline{c^5+c^4x-x^5}|^{\frac{1}{5}}$) $=c+\frac{c^4x-x^5}{5c^4}-\frac{2c^8x^2+4c^4x^6-2x^{10}}{25c^9}+$ &c. Ut patebit substituendo in allatam Regulam, 1 pro m, 5 pro n, c^5 pro P, & $\frac{c^4x-x^5}{c^5}$ pro Q. Potest etiam $-x^5$ substitui pro P, & $\frac{c^4x+c^5}{-x^5}$ pro Q, et tunc evadet $\sqrt{5:\,c^5+c^4x-x^5}=-x+\frac{c^4x+c^5}{5x^4}+\frac{2c^8x^2+4c^9x+c^{10}}{25x^9}+$ &c. Prior modus eligendus est, si x valde parvum sit; posterior, si valde magnum.

Exempl. 3. Est $\frac{N}{\sqrt{3:\,y^3-a^2y}}$ (hoc est, $N\times\overline{y^3-a^2y}|^{-\frac{1}{3}}$) æqualis $N\times\overline{\frac{1}{y}+\frac{a^2}{3y^3}+\frac{2a^4}{9y^5}+\frac{14a^6}{81y^7}+\&c.}$ Nam $P=y^3$, $Q=\frac{-aa}{yy}$, $m=-1$, $n=3$. A $(=P^{\frac{m}{n}}=y^{3\times-\frac{1}{3}})=y^{-1}$, hoc est $\frac{1}{y}$. B $(=\frac{m}{n}AQ=-\frac{1}{3}\times\frac{1}{y}\times\frac{-aa}{yy})=\frac{aa}{3y^3}$, &c.

Exempl. 4. Radix cubica ex quadrato-quadrato ipsius $d+e$, (hoc est, $\overline{d+e}|^{\frac{4}{3}}$) est $d^{\frac{4}{3}}+\frac{4ed^{\frac{1}{3}}}{3}+\frac{2e^2}{9d^{\frac{2}{3}}}-\frac{4e^3}{81d^{\frac{5}{3}}}+$ &c.

Nam $P=d$, $Q=\frac{e}{d}$, $m=4$, $n=3$, A $(=P^{\frac{m}{n}})=d^{\frac{4}{3}}$, &c.

Exempl. 5. Eodem modo simplices etiam potestates eliciuntur. Ut si quadrato-cubus ipsius $d+e$, (hoc est, $\overline{d+e}|^5$, seu $\overline{d+e}|^{\frac{5}{1}}$) desideretur: erit, juxta Regulam, $P=d$, $Q=\frac{e}{d}$, $m=5$, & $n=1$; adeoque A $(=P^{\frac{m}{n}})=d^5$, B $(=\frac{m}{n}AQ)=5d^4e$, & sic $C=10d^3e^2$, $D=10d^2e^3$, $E=5de^4$, $F=e^5$, & G $(=\frac{m-5n}{6n}FQ)=0$. Hoc est, $\overline{d+e}|^5=d^5+5d^4e+10d^3e^2+10d^2e^3+5de^4+e^5$.

Exem.

Exempl. 6. Quinetiam Divisio, sive simplex sit, sive repetita, per eandem Regulam perficitur. Ut si $\frac{1}{d+e}$ (hoc est $\overline{d+e}|^{-1}$ sive $\overline{d+e}|^{-\frac{1}{1}}$) in seriem simplicium terminorum resolvendum sit: Erit juxta Regulam $P=d$, $Q=\frac{e}{d}$, $m=-1$, $n=1$, & $A\,(=P^{\frac{m}{n}}=d^{-\frac{1}{1}})=d^{-1}$ seu $\frac{1}{d}$, $B\,(=\frac{m}{n}AQ=-1\times\frac{1}{d}\times\frac{e}{d}=-\frac{e}{d^2}$, & sic $C=\frac{ee}{d^3}$, $D=-\frac{e^3}{d^4}$, &c. Hoc est $\frac{1}{d+e}=\frac{1}{d}-\frac{e}{d^2}+\frac{e^2}{d^3}-\frac{e^3}{d^4}+$ &c.

Exempl. 7. Sic et $\overline{d+e}|^{-3}$ (hoc est unitas ter divisa per $d+e$, vel semel per cubum ejus,) evadit $\frac{1}{d^3}-\frac{3e}{d^4}+\frac{6e^2}{d^5}-\frac{10e^3}{d^6}+$ &c.

Exempl. 8. Et $N\times\overline{d+e}|^{-\frac{1}{3}}$, (hoc est N divisum per radicem cubicam ipsius $d+e$,) evadit $N\times\overline{\frac{1}{d^{\frac{1}{3}}}-\frac{e}{3d^{\frac{4}{3}}}+\frac{2e^2}{9d^{\frac{7}{3}}}-\frac{14e^3}{81d^{\frac{10}{3}}}+\&c.}$

Exempl. 9. Et $N\times\overline{d+e}|^{-\frac{3}{5}}$ (hoc est N divisum per radicem quadrato-cubicam ex cubo ipsius $d+e$, sive $\frac{N}{\sqrt{5:\,d^3+3d^2e+3de^2+e^3}}$) evadit $N\times\frac{1}{d^{\frac{3}{5}}}-\frac{3e}{5d^{\frac{8}{5}}}+\frac{12e^2}{25d^{\frac{13}{5}}}-\frac{52e^3}{125d^{\frac{18}{5}}}+$ &c.

Per eandem Regulam Genesis Potestatum, Divisiones per Potestates aut per quantitates radicales, & Extractiones radicum altiorum in numeris etiam commode instituuntur.

Extractiones Radicum affectarum in speciebus imitantur earum extractiones in numeris, sed methodus *Vietæ* & *Oughtredi* nostri huic negotio minus idonea est: Quapropter aliam excogitare adactus sum. [*Hujus specimen exhibetur in Tractatu præcedente Pag.* 8.]

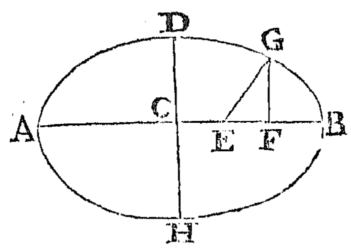
Quomodo ex æquationibus, sic ad infinitas series reductis, Areæ & Longitudines curvarum, Contenta & Superficies solidorum, vel quorumlibet segmentorum figurarum quarumvis, eorumque Centra gravitatis determinantur; & quomodo etiam Curvæ omnes Mechanicæ ad ejusmodi æquationes infinitarum serierum reduci possint, indeque Problemata circa illas resolvi perinde ac si Geometricæ essent; nimis longum foret describere, sufficiat specimina quædam talium Problematum recensuisse: Inque iis, brevitatis gratia, literas A, B, C, D, &c. pro terminis seriei, sicut sub initio, nonnunquam usurpabo.

1. *Si ex dato sinu recto, vel sinu verso, Arcus desideretur*: Sit radius r, & sinus rectus x: Eritque Arcus $=x+\frac{x^3}{6r^2}+\frac{3x^5}{40r^4}+\frac{5x^7}{112r^6}$ &c. hoc est, $=x+\frac{1\times1\times xx}{2\times3\times rr}A+\frac{3\times3xx}{4\times5rr}B+\frac{5\times5xx}{6\times7rr}C+\frac{7\times7xx}{8\times9rr}D+$ &c.

Vel, sit d diameter, ac x sinus versus ; & erit Arcus æqualis $d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c. hoc est, $= \sqrt{dx}$ in $1 + \frac{x}{6d} + \frac{3x^2}{40d^2} + \frac{5x^3}{112d^3} +$ &c.

2. Si vicissim, *ex dato Arcu desideretur sinus* : Sit radius r, & arcus z : Eritque sinus rectus $= z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{362880r^8} -$ &c. hoc est, $= z - \frac{zz}{2 \times 3rr}A - \frac{zz}{4 \times 5rr}B - \frac{zz}{6 \times 7rr}C -$ &c. Et sinus versus $= \frac{z^2}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{40320r^7} +$ &c. hoc est, $= \frac{zz}{1 \times 2r} - \frac{zz}{3 \times 4rr}A - \frac{zz}{5 \times 6rr}B - \frac{zz}{7 \times 8rr}C -$ &c.

3. *Si Arcus capiendus sit in ratione data ad alium Arcum* : Esto diameter $= d$, chorda arcus dati $= x$, & arcus quæsitus ad arcum illum datum ut n ad 1 ; Eritque arcus quæsiti Chorda $= nx + \frac{1-nn}{2 \times 3dd}xxA + \frac{9-nn}{4 \times 5dd}xxB + \frac{25-nn}{6 \times 7dd}xxC + \frac{36-nn}{8 \times 9dd}xxD + \frac{49-nn}{10 \times 11dd}xxE +$ &c. Ubi nota, quod cum n est numerus impar, series desinet esse infinita, & evadet eadem quæ prodit per Vulgarem Algebram ad multiplicandum datum angulum per istum numerum n.



4. Si in Axe alterutro AB, Ellipseos ADB (cujus centrum C, & axis alter DH) detur punctum aliquod E, circa quod recta EG occurrens Ellipsi in G, motu angulari feratur ; & ex data area sectoris Elliptici BEG, quæratur recta GF, quæ a puncto G ad axem AB normaliter demittitur : Esto BC $= q$, DC $= r$, EB $= t$, ac duplum areæ BEG $= z$; et erit GF $= \frac{1}{t}z - \frac{q}{6r^2t^4}z^3 + \frac{10q^2 - 9qt}{120r^4t^7}z^5 - \frac{280q^3 + 504q^2t - 225qt^2}{5040r^6t^{10}}z^7 +$ &c. Sic itaque Astronomicum illud *Kepleri* Problema resolvi potest.

5. In eadem Ellipsi, si statuatur CD $= r$, $\frac{CB^2}{CD} = c$, & CF $= x$: Erit arcus

$$\text{Ellipticus DG} = x + \begin{array}{l} \frac{1}{6c^2} \\ \\ \\ \\ \\ \end{array} x^3 + \begin{array}{l} \frac{1}{10rc^3} \\ -\frac{1}{40c^4} \\ \\ \\ \\ \end{array} x^5 + \begin{array}{l} \frac{1}{14r^2c^4} \\ -\frac{1}{28rc^5} \\ +\frac{1}{112c^6} \\ \\ \\ \end{array} x^7 + \begin{array}{l} \frac{1}{18r^3c^5} \\ -\frac{1}{24r^2c^6} \\ +\frac{1}{48rc^7} \\ -\frac{5}{1152c^8} \\ \\ \end{array} x^9 + \begin{array}{l} \frac{1}{22r^4c^6} \\ -\frac{1}{22r^3c^7} \\ +\frac{3}{88r^2c^8} \\ -\frac{5}{352rc^9} \\ +\frac{7}{2816c^{10}} \end{array} x^{11} + \text{\&c.}$$

Hic numerales Coefficientes supremorum terminorum ($\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{14}$, &c.) sunt in Musica progressione : Et numerales Coefficientes omnium inferiorum in unaquaque columna prodeunt multiplicando continuo numeralem Coefficientem supremi termini per terminos hujus progressionis

$\frac{\frac{1}{2}n - 1}{2}$

$\frac{\frac{1}{2}n-1}{2}$, $\frac{\frac{3}{3}n-3}{4}$, $\frac{\frac{5}{4}n-5}{6}$, $\frac{\frac{7}{5}n-7}{8}$, $\frac{\frac{9}{6}n-9}{10}$, &c. Ubi n ſignificat numerum dimenſionum ipſius c in denominatore iſtius ſupremi termini. E. g. ut terminorum infra $\frac{1}{22r^4c^6}$, numerales coefficientes inveniantur, pono $n=6$, ducoque $\frac{1}{22}$ (numeralem coefficientem ipſius $\frac{1}{22r^4c^6}$) in $\frac{\frac{1}{2}n-1}{2}$, hoc eſt, in 1; & prodit $\frac{1}{22}$, numeralis coefficiens termini proxime inferioris: dein duco hunc $\frac{1}{22}$ in $\frac{\frac{3}{3}n-3}{4}$, ſive in $\frac{n-3}{4}$, hoc eſt, in $\frac{3}{4}$; & prodit $\frac{3}{88}$ numeralis coefficiens tertii termini in iſta columna. Atque iſta $\frac{3}{88}\times\frac{\frac{5}{4}n-5}{6}$ facit $\frac{5}{352}$ numeralem coefficientem quarti termini; & $\frac{5}{352}\times\frac{\frac{7}{5}n-7}{8}$ facit $\frac{7}{2816}$ numeralem coefficientem infimi termini. Idem in aliis ad infinitum uſque columnis præſtari poteſt: Adeoque valor ipſius DG per hanc Regulam pro lubitu produci.

Adhæc, ſi BF dicatur x, ſitque r latus rectum Ellipſeos, & $e=\frac{r}{AB}$; Erit Arcus Ellipticus

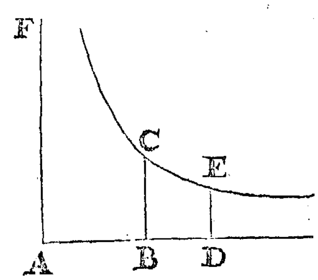
$$BG=\sqrt{rx\text{ in }1+\left.\frac{2-\frac{3}{2}e}{3r}\right\}x\;\left.\frac{-2+3e-\frac{5}{8}e^2}{5r^2}\right\}x^2\;\left.\frac{+4-9e+\frac{23}{4}e^2-\frac{7}{16}e^3}{7r^3}\right\}x^3\;\left.\frac{-10+30e-\frac{123}{4}e^2+\frac{91}{8}e^3-\frac{45}{128}e^4}{9r^4}\right\}x^4,+\&c.}$$

Quare, ſi ambitus totius Ellipſeos deſideretur; Biſeca CB in F, & quære Arcum DG, per prius Theorema, & Arcum BG per poſterius.

6. Si, vice verſa, *ex dato arcu Elliptico* DG, *quæratur Sinus ejus* CF; tum dicto CD $=r$, $\frac{CB^2}{CD}=c$, & arcu illo DG $=z$; Erit

$$CF=z-\frac{1}{6c^2}z^3-\left.\begin{matrix}\frac{1}{10rc^3}\\+\frac{13}{120c^4}\end{matrix}\right.z^5-\left.\begin{matrix}\frac{1}{14r^2c^4}\\+\frac{71}{420rc^5}\\-\frac{493}{5040c^6}\end{matrix}\right.z^7-\&c.$$

Quæ autem de Ellipſi dicta ſunt, omnia facile accommodantur ad Hyperbolam; mutatis tantum ſignis ipſorum c & e ubi ſunt imparium dimenſionum.

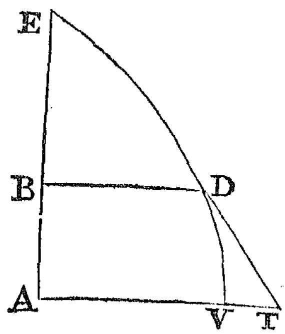


7. Præterea, ſi ſit CE Hyperbola, cujus Aſymptoti AD, AF rectum angulum FAD conſtituant; & ad AD erigantur utcunque perpendicula BC, DE occurrentia Hyperbolæ in C & E: & AB dicatur a, BC b, & area BCED z;

Erit

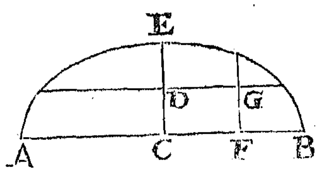
Erit BD $= \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +$ &c. Ubi coefficientes denominatorum prodeunt multiplicando terminos hujus Arithmeticæ progressionis, 1, 2, 3, 4, 5, &c. in se continuo. Et hinc ex Logarithmo dato potest numerus ei competens invenire.

8. Esto VDE *Quadratrix*, cujus vertex est V, existente A centro & AE semi-diametro Circuli ad quem aptatur, & angulo VAE recto: Demissoque ad AE perpendiculo quovis DB, & acta Quadratricis Tangente DT occurrente axi ejus AV in T: Dic AV $= a$, & AB $= x$; Eritq; DB $= a - \frac{x^2}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5} -$ &c. Et VT $= \frac{x^2}{3a} + \frac{x^4}{15a^3} + \frac{2x^6}{189a^5} +$ &c. Et Area AVDB $= ax - \frac{x^3}{9a} - \frac{x^5}{225a^3} - \frac{2x^7}{6615a^5} -$ &c. Et Arcus VD $= x + \frac{2x^3}{27a^2} + \frac{14x^5}{2025a^4} + \frac{604x^7}{893025a^6} +$ &c. Unde vicissim, ex dato BD, vel VT, aut area AVDB, arcuve VD, per resolutionem affectarum æquationum erui potest x seu AB.



9. Esto denique AEB *Sphæroides*, revolutione Ellipseos AEB circa axem AB genita, & secta planis quatuor, AB per axem transeunte, DG parallelo AB, CDE perpendiculariter bisecante axem, & FG parallelo CE: sitque recta CB $= a$, CE $= c$, CF $= x$, & FG $= y$. Et Sphæroideos segmentum CDGF dictis quatuor planis comprehensum, erit

$$+ 2cx\,y - \frac{x}{3c}y^3 - \frac{x}{20c^3}y^5 - \frac{x}{56c^5}y^7 - \frac{5x}{576c^7}y^9 - \text{\&c.}$$
$$- \frac{cx^3}{3a^2} - \frac{x^3}{18ca^2} - \frac{x^3}{40c^3a^2} - \frac{5x^3}{336c^5a^2} - \text{\&c.}$$
$$- \frac{cx^5}{20a^4} - \frac{x^5}{40ca^4} - \frac{3x^5}{160c^3a^4} - \text{\&c.}$$
$$- \frac{cx^7}{56a^6} - \frac{5x^7}{336ca^6} - \text{\&c.}$$
$$- \frac{5cx^9}{576a^8} - \text{\&c.}$$
$$- \text{\&c.}$$



Ubi numerales coefficientes supremorum terminorum ($2, -\frac{1}{3}, -\frac{1}{20}, -\frac{1}{56}, -\frac{5}{576}$, &c.) in infinitum producuntur multiplicando primum coeffitem 2 continuo per terminos hujus progressionis

$-\frac{1\times1}{2\times3}, \frac{1\times3}{4\times5}, \frac{3\times5}{6\times7}, \frac{5\times7}{8\times9}, \frac{7\times9}{10\times11}$, &c. Et numerales coefficientes terminorum in unaquaque columna descendentium in infinitum producuntur multiplicando continuo coefficientem supremi termini in prima columna per eandem progressionem, in secunda autem per terminos hujus

$\frac{1\times1}{2\times3}, \frac{3\times3}{4\times5}, \frac{5\times5}{6\times7}, \frac{7\times7}{8\times9}$, &c. in tertia per terminos hujus $\frac{3\times1}{2\times3}, \frac{5\times3}{4\times5}, \frac{7\times5}{6\times7}, \frac{9\times7}{8\times9}$, &c. in quarta per terminos hujus $\frac{5\times1}{2\times3}, \frac{7\times3}{4\times5}, \frac{9\times5}{6\times7}$, &c. in quinta per terminos hujus $\frac{7\times1}{2\times3}, \frac{9\times3}{4\times5}, \frac{11\times5}{6\times7}$; &c. Et sic in infinitum.

Et

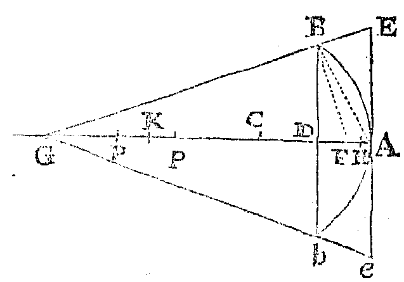
Et eodem modo ſegmenta aliorum ſolidorum deſignari, & valores eorum aliquando commode per ſeries quaſdem numerales in infinitum produci poſſunt.

Ex his videre eſt, quantum fines Analyſeos per hujuſmodi infinitas æquationes amplientur: Quippe quæ, earum beneficio, ad omnia pæne dixerim problemata (ſi numeralia *Diophanti* & ſimilia excipias) ſeſe extendit.

Non tamen omnino univerſalis evadit, niſi per ulteriores quaſdam methodos eliciendi ſeries infinitas. Sunt enim quædam Problemata in quibus non liceat ad ſeries infinitas per diviſionem vel extractionem radicum ſimplicium affectarumve, pervenire. Sed quomodo in iſtis caſibus procedendum ſit, jam non vacat dicere; ut neque alia quædam tradere quæ circa reductionem infinitarum ſerierum in finitas, ubi rei natura tulerit, excogitavi. Nam parcius ſcribo, quod hæ ſpeculationes diu mihi faſtidio eſſe cœperint, adeo ut ab iiſdem jam per quinque fere annos abſtinuerim.

Unum tamen addam: quod poſtquam Problema aliquod ad infinitam æquationem deducitur, poſſint inde variæ approximationes in uſum Mechanicæ, nullo fere negotio formari; quæ, per alias methodos quæſitæ, multo labore temporiſque diſpendio conſtare ſolent.

Cujus rei exemplo eſſe poſſunt Tractatus *Hugenii* aliorumque de Quadratura circuli. Nam, ut ex data arcus chorda A, & dimidii arcus chorda B, arcum illum proxime aſſequaris; finge arcum illum eſſe z, & circuli radium r; juxtaque ſuperiora erit A (nempe duplum ſinus dimidii z) $= z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} -$ &c. Et $B = \frac{1}{2}z - \frac{z^3}{2 \times 16 \times 6r^2} + \frac{z^5}{2 \times 16 \times 16 \times 120r^4} -$ &c. Duc jam B in numerum fictitium n, & a producto aufer A, & reſidui ſecundum terminum (nempe $-\frac{nz^3}{2 \times 16 \times 6r^2} + \frac{z^3}{4 \times 6r^2}$,) eo ut evaneſcat, pone $= 0$; indeque emerget $n = 8$, et erit $8B - A = 3z * - \frac{3z^5}{64 \times 120r^4} +$ &c. hoc eſt $\frac{8B - A}{3} = z$; errore tantum exiſtente $\frac{z^5}{7680r^4} -$ &c. in exceſſu. Quod eſt Theorema *Hugenianum*.



Inſuper, ſi in Arcus Bb, ſagitta AD indefinite producta, quæratur punctum G, a quo actæ rectæ GB, Gb abſcindant Tangentem Ee quam proxime æqualem Arcui iſti: Eſto circuli centrum C, diameter $AK = d$, & ſagitta $AD = x$: Et erit DB $(= \sqrt{dx - x^2})$

$$= d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}} - \text{\&c.}$$

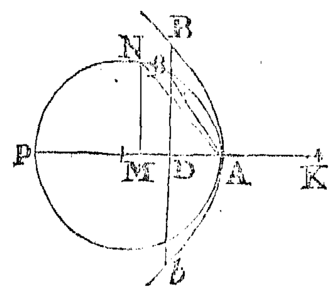
Et AE $(= AB) = d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c.

Et AE — DB : AD : : AE : AG ; Quare AG $= \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$ — vel + &c. Finge ergo AG $= \frac{3}{2}d - \frac{1}{5}x$; et viciſſim erit DG $(\frac{3}{2}d - \frac{6}{5}x)$: DB : : DA : AE — DB. Quare AE — DB $= \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}}$ + &c. Adde DB ; & prodit

AE $= d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}}$ + &c. Hoc aufer de valore ipſius AE ſupra habito, & reſtabit error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}}$ + vel — &c. Quare in AG, cape AH quintam partem DA, et KG = HC, et actæ GBE, G*be* abſcindent Tangentem E*e* quam proxime æqualem arcui BA*b* ; errore tantum exiſtente $\frac{16x^3}{525d^3}\sqrt{dx}$ + vel — &c. multo minore ſcilicet quam in Theoremate *Hugenii.* Quod ſi fiat 7AK : 3AH : : DH : n; et capiatur KG = CH — n, erit error adhuc multo minor.

Atque ita, ſi Circuli ſegmentum aliquod BA*b* per Mechanicam deſignandum eſſet : Primo reducerem Aream iſtam in Infinitam ſeriem, puta hanc B*b*A $= \frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}}$ — &c. Dein quærerem conſtructiones Mechanicas quibus hanc ſeriem proxime aſſequerer ; cujuſmodi ſunt hæ : Age rectam AB, et erit ſegmentum B*b*A $= \overline{\tfrac{2}{3}\mathrm{AB} + \mathrm{BD}} \times \frac{4}{5}\mathrm{AD}$ proxime ; exiſtente ſcilicet errore tantum $\frac{x^3}{70d^2}\sqrt{dx}$ + &c. in defectu : Vel proximius, erit ſegmentum illud (biſecto AD in F, & acta recta BF) $= \frac{4\mathrm{BF} + \mathrm{AB}}{15} \times 4\mathrm{AD}$; exiſtente errore ſolummodo $\frac{x^3}{560d^2}\sqrt{dx}$ + &c. qui ſemper minor erit quam $\frac{1}{1500}$ totius ſegmenti, etiamſi ſegmentum illud ad uſque ſemicirculum augeatur.

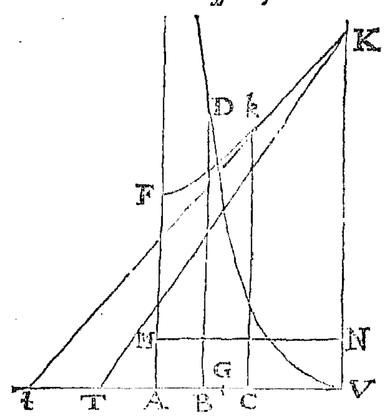
Sic et in Ellipſi BA*b*, [*Vid. Fig. Præcedent.*] cujus vertex A, axis alteruter AK, & latus rectum AP ; cape PG $= \frac{1}{2}\mathrm{AP} + \frac{19\mathrm{AK} - 21\mathrm{AP}}{10\mathrm{AK}} \times \mathrm{AP}$. In Hyperbola vero, cape PG $= \frac{1}{2}\mathrm{AP} + \frac{19\mathrm{AK} + 21\mathrm{AP}}{10\mathrm{AK}} \times \mathrm{AP}$. Et acta recta GBE abſcindet tangentem AE quam proxime æqualem arcui Elliptico vel Hyperbolico AB, dummodo Arcus ille non ſit nimis magnus.



Et pro Area ſegmenti Hyperbolici B*b*A ; in DP cape MD $= \frac{3\mathrm{AD}^2}{4\mathrm{AK}}$, & ad D & M erige perpendicula Dβ, MN occurrentia ſemicirculo ſuper Diametro AP deſcripto : Eritque $\frac{4\mathrm{AN} + \mathrm{A}\beta}{15} \times 4\mathrm{AD} = \mathrm{B}b\mathrm{A}$ proxime : Vel proximius, erit $\frac{21\mathrm{AN} + 4\mathrm{A}\beta}{75} \times 4\mathrm{AD} = \mathrm{B}b\mathrm{A}$; ſi modo capiatur DM $= \frac{5\mathrm{AD}^2}{7\mathrm{AK}}$.

Frag.

Fragmentum Epistolæ D. Newtoni, *ad D.* Oldenburgium 24 Octob. 1676 *missæ.*



LOngitudo *Cissoidis* sic construitur. Sit VD *Cissois*, AV Diameter Circuli ad quem aptatur, V vertex, AF Asymptota ejus, ac DB perpendiculare quodvis ad AV demissum. Cum semi-axe AF = AV, & semi-parametro AG = ⅓AV, describatur Hyperbola F*k*K; & inter AB & AV sumpta AC media proportionali, erigantur ad C & V perpendicula C*k*, VK Hyperbolæ occurrentia in *k* & K; Et agantur rectæ KT, *kt* tangentes Hyperbolam in eisdem K & *k*, et occurrentes AV in T & *t*; Et ad AV constituatur rectangulum AVNM æquale spatio TK*kt*. Et *Cissoidis* VD longitudo erit Sextupla altitudinis VN. Demonstratio perbrevis est.

[*Quæ sequuntur scripta sunt in explicationem Epistolæ præcedentis.*]

Quod vero attinet ad Inventionem terminorum *p*, *q*, *r*, (*vide* pag. 25 & 8 :) in extractione Radicis affectæ, primum *p* sic eruo.

B Fig. 1.

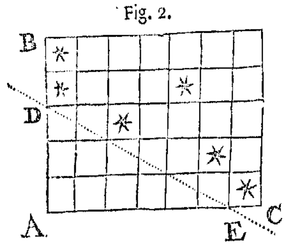
x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
o	y	y^2	y^3	y^4

A C

Descripto Angulo recto BAC, latera ejus BA, CA divido in partes æquales; & inde normales erigo distribuentes angulare spatium in æqualia parallelograma vel quadrata, quæ concipio denominata esse a dimensionibus duarum indefinitarum specierum, puta x & y, regulariter ascendentium a termino A; prout vides in Fig. 1. inscriptas. Ubi y denotat Radicem extrahendam; et x alteram indefinitam quantitatem, ex cujus potestatibus series conficienda;

enda eſt. Deinde, cum Æquatio aliqua proponitur, parallelogramma ſingulis ejus terminis correſpondentia inſignio nota aliqua: Et Regula ad duo vel forte plura ex inſignitis parallelogrammis applicata; quorum unum ſit humillimum in columna ſiniſtra juxta AB, & alia ad Regulam dextrorſum ſita, cæteraque omnia non contingentia Regulam ſupra eam jaceant; Seligo terminos Æquationis per parallelogramma contingentia Regulam deſignatos, & inde quæro quantitatem Quotienti addendam.

Fig. 2.



Sic ad extrahendam Radicem y, ex $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$; parallelogramma hujus terminis reſpondentia ſigno nota aliqua $*$; ut vides in Fig. 2. Dein applico Regulam DE ad inferiorem e locis ſignatis in ſiniſtra columna; eamque ab inferioribus ad ſuperiora dextrorſum gyrare facio, donec alium ſimiliter vel forte plura e reliquis ſignatis locis cœperit attingere. Videoque loca ſic attracta eſſe x^3, x^2y^2, & y^6. E terminis itaque, $y^6 - 7a^2x^2y^2 + 6a^3x^6$ tanquam nihilo æqualibus (& inſuper ſi placet reductis ad $v^6 - 7v^2 + 6 = 0$, ponendo $y = v\sqrt{ax}$,) quæro valorem y, & invenio quadruplicem, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, & $-\sqrt{2ax}$, quorum quemlibet pro primo termino Quotientis accipere licet, prout e radicibus quampiam extrahere decretum eſt.

Sic Æquatio $y^3 + axy + aay - x^3 - 2a^3 = 0$, quam reſolvebam in priori Epiſtola, dat $-2a^3 + aay + y^3 = 0$, & inde $y = a$ proxime: Cum itaque a ſit primus terminus valoris y, pono p pro cæteris omnibus in infinitum, & ſubſtituo $a + p = y$. (Obvenient hic aliquando difficultates nonnullæ; ſed ex iis, credo, lector ſe proprio marte extricabit.) Subſequentes vero termini q, r, s, &c. eodem modo ex æquationibus ſecundis, tertiis, cæteriſque eruuntur, quo primus p e prima, ſed cura leviori; quia cæteri valores y ſolent prodire dividendo terminum involventem infimam poteſtatem indefinitæ quantitatis x per Coefficientem radicis p, q, r, aut s.

Intellexti credo ex ſuperioribus, regreſſionem ab Areis curvarum ad Lineas rectas, fieri per hanc extractionem Radicis affectæ. Sed duo alii ſunt modi quibus idem perficio.

Eorum unus affinis eſt Computationibus quibus colligebam approximationes ſub finem alterius Epiſtolæ, & intelligi poteſt per hoc exemplum. Proponatur Æquatio ad Aream Hyperbolæ $z = x + \frac{1}{2}xx + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$, &c. Et partibus ejus multiplicatis in ſe, emerget $z^2 = x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5$, &c. $z^3 = x^3 + \frac{3}{2}x^4 + \frac{7}{4}x^5$, &c. $z^4 = x^4 + 2x^5$, &c. $z^5 = x^5$, &c. Jam de z aufero $\frac{1}{2}zz$, & reſtat $z - \frac{1}{2}zz = x - \frac{1}{6}x^3 - \frac{5}{24}x^4 - \frac{13}{60}x^5$, &c. Huic addo $\frac{1}{6}z^3$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 = x + \frac{1}{24}x^4 + \frac{3}{40}x^5$, &c. Aufero $\frac{1}{24}z^4$, & reſtat $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 = x - \frac{1}{120}x^5$, &c. Addo $\frac{1}{120}z^5$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5 = x$ quamproxime; ſive $x = z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c.

Eodem

Eodem modo, Series de una indefinita quantitate, in aliam transferri possunt. Quemadmodum si posito r radio circuli, x sinu recto arcus z, & $x + \frac{x^3}{6rr} + \frac{3x^5}{40r^4} +$ &c. longitudine arcus istius; atque hanc Seriem a Sinu recto ad Tangentem vellem transferre: Quæro longitudinem Tangentis $\frac{rx}{\sqrt{rr-xx}}$, & reduco in infinitam Seriem $x + \frac{x^3}{2rr} + \frac{3x^5}{8r^4} +$ &c. Vocetur hæc quantitas, t. Colligo potestates ejus $t^3 = x^3 + \frac{3x^5}{2rr}$ &c. $t^5 = x^5 +$ &c. Aufero autem t de z, & restat $z - t = -\frac{1}{3}x^3 - \frac{3}{10}x^5 -$ &c. Addo $\frac{1}{3}t^3$, & fit $z - t + \frac{1}{3}t^3 = \frac{1}{5}x^5 +$ &c. Aufero $\frac{1}{5}t^5$, & restat $z - t + \frac{1}{3}t^3 - \frac{1}{5}t^5 = 0$ quamproxime. Quare est $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 -$ &c. Sed siquis in usus Trigonometricos me jussisset exhibere expressionem Arcus per Tangentem; eam non hoc circuitu, sed directa methodo quæsivissem.

Per hoc genus Computi colliguntur etiam Series ex duabus vel pluribus indefinitis quantitatibus constantes; & Radices affectarum Æquationum magna ex parte extrahuntur. Sed ad hunc posteriorem usum adhibeo potius methodum in altera Epistola descriptam tanquam generaliorem, & (Regulis pro Elisione superfluorum terminorum habitis) paulo magis expeditam.

Pro Regressione vero ab Areis ad Lineas rectas, & similibus, possunt hujusmodi *Theoremata* adhiberi.

THEOREMA I. Sit $z = ay + by^2 + cy^3 + dy^4 + ey^5$, &c.

Et vicissim erit
$$y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{3a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^9}z^5 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Aream Hyperbolæ, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c. Et substitutis in Regula 1 pro a, $-\frac{1}{2}$ pro b, $\frac{1}{3}$ pro c, $-\frac{1}{4}$ pro d, & $\frac{1}{5}$ pro e; vicissim exurgit, $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 +$ &c.

THEOREMA II. Sit $z = ay + by^3 + cy^5 + dy^7 + ey^9 +$ &c.

Et vicissim erit
$$y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3b^2 - ac}{a^7}z^5 + \frac{8abc - a^2d - 12b^3}{a^{10}}z^7 + \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}}z^9 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Arcum circuli, $z = y + \frac{y^3}{6rr} + \frac{3y^5}{40r^4} + \frac{5y^7}{112r^6} + \&c.$ Et ſubſtitutis in Regula 1 pro a, $\frac{1}{6r^2}$ pro b, $\frac{3}{40r^4}$ pro c, $\frac{5}{112r^6}$ pro d &c; orietur $y = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \&c.$

Fragmentum **Epiſtolæ* D. Newtoni, *ad* D. Wallisium *Anno* 1692, *miſſæ.*

SUB finem Epiſtolæ anni 1676 [*Hæc ſunt verba* Walliſii] ſcribit [D. *Newtonus*] etiam Problema determinandi Curvas per conditiones Tangentium in ſua poteſtate eſſe, una cum aliis difficilioribus; ad quæ ſolvenda ſe uſum eſſe dicit duplici methodo, una concinniore, altera generaliore; & utramque literis tranſpoſitis celat: quæ in ordinem redactæ hanc ſententiam exhibent. *Una methodus conſiſtit in Extractione fluentis quantitatis ex æquatione ſimul involvente fluxionem ejus. Altera tantum in aſſumptione ſeriei pro quantitate qualibet incognita ex qua cætera commode derivari poſſunt; & in collatione terminorum homologorum æquationis reſultantis ad eruendos terminos aſſumptæ Seriei.* Harum methodorum Secunda ex verbis jam recitatis abſque ulteriore explicatione intelligi poteſt; priorem ab Authore jam accepi ut ſequitur.

Hæc methodus, ait, ejuſdem eſt generis cum ea pro extrahendo radices ex æquationibus affectis ſuperius deſcripta. Pone quod Problema reſolvendum reducatur ad æquationem fluentes quantitates y & z una cum earum fluxionibus $\dot{y}$ & $\dot{z}$ involventem, & quod fluxio ipſius z uniformis ſit. Ut hæc fluxio ex æquatione evaneſcat, pro ea ponatur unitas, & manebit æquatio ſolas y, z & $\dot{y}$ involvens, *quam Reſolvendam* vocat. Proponitur, inventio ipſius y in Serie infinita convergente, quæ ſolam z involvet. Hoc in aliquibus æquationibus impoſſibile eſt, in aliis præparationem æquationum requirit, ubi vero directe confici poſſit reſolutio eſt hujuſmodi.

PROBLEMA

* *Extat Epiſtola in Tom.* 2. *Operum* Walliſii.

PROBLEMA.

Ex æquatione fluxionem radicis involvente radicem extrahere.

RESOLUTIO.

Termini omnes, ex eodem æquationis latere consistentes, æquentur nihilo, & ipsarum y & $\dot{y}$ dignitates (si opus sit) exaltentur vel deprimantur, sic ut earum indices nec alicubi negativi sint, nec tamen altiores quam ad hunc effectum requiritur; & sit kz^λ terminus infimæ dignitatis eorum qui neque per y neque per ejus fluxionem $\dot{y}$ neque per earum dignitatem quamvis multiplicantur. Sit $lz^\mu y^\alpha \dot{y}^\beta$ terminus alius quilibet, & omnes ordine terminos percurrendo collige ex singulis seorsim numerum $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ sic, ut tot habeas ejusmodi numeros quot sunt termini. Horum numerorum maximus vocetur ν, & z^ν erit dignitas primi termini Seriei. Pro ejus coefficiente ponatur a, & in æquatione quæ *resolvenda* dicitur scribe az^ν pro y, & $\nu az^{\nu-1}$ pro $\dot{y}$; ac termini omnes resultantes in quibus z ejusdem est dignitatis ac in termino kz^λ, sub propriis signis collecti, ponantur æquales nihilo. Nam hæc æquatio debite reducta dabit coefficientem a. Sic habes az^ν terminum primum Seriei.

OPERATIO SECUNDA.

Pro reliquis omnibus hujus Seriei terminis nondum inventis pone p, & habebis Æquationem $y = az^\nu + p$, & inde etiam Æquationem $\dot{y} = \nu az^{\nu-1} + \dot{p}$. In resolvenda, pro y & $\dot{y}$ scribe hos eorum valores & habebis Resolvendam novam, ubi p officium præstat ipsius y: & ex hac Resolvenda primum extrahes terminum Seriei p eodem modo atque terminum primum Seriei totius $y = az^\nu + p$ ex Resolvenda prima extraxisti.

OPERATIO TERTIA ET SEQUENTES.

Dein tertiam Resolvendam eadem ratione invenias atque secundam invenisti, & ex ea terminum tertium Seriei totius extrahes. Et similiter Resolvendam quartam invenies, & ex ea quartum Seriei terminum, & sic in infinitum. Series autem sic inventa erit radix Æquationis quam extrahere oportuit.

EXEMPLUM.

EXEMPLUM.

Ex Æquatione $y^2\dot{z}^2 - z^2\dot{z}\dot{y} - d^2\dot{z}\dot{z} + dz\dot{z}^2 = 0$, extrahenda sit radix y. Pone $\dot{z} = 1$, & Æquatio evadet $y^2 - z^2\dot{y} - dd + dz = 0$, quæ est Resolvenda. Jam vero terminus infimus in quo nec y neque $\dot{y}$ reperitur, est dd, qui ipsi kz^λ æquatus dat $\lambda = 0$. Terminis reliquis y^2, $-z^2\dot{y}$ pone $lz^\mu y^\alpha \dot{y}^\beta$ æqualem successive, & inde in primo casu habebis $\mu = 0$, $\alpha = 2$, $\beta = 0$, in secundo $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ fit in primo casu o, in secundo -1. Unde ν est o, & az^ν & $\nu az^{\nu-1}$ sunt a & o; quarum ultimæ duæ a & o in Resolvenda pro y & $\dot{y}$ scriptæ, producunt $aa + oz^2 - dd + dz$; & termini aa & $-dd$, in quibus index dignitatis z est λ seu o, positi æquales nihilo dant $a = d$. Unde primus Seriei terminus az^ν evadit d.

OPERATIO SECUNDA.

Pro terminis reliquis pone p, & habebis æquationem $y = d + p$, & inde $\dot{y} = \dot{p}$; qui valores in Resolvenda pro y & $\dot{y}$ substituti dant Resolvendam novam $2dp + pp - zz\dot{p} + dz = 0$, ubi p & $\dot{p}$ vices subeunt ipsarum y & $\dot{y}$. Terminus unicus in quo nec p neque $\dot{p}$ reperitur est dz, qui cum termino kz^λ collatus dat $\lambda = 1$. Terminis reliquis $2dp$, pp & $-zz\dot{p}$ pone $lz^\mu p^\alpha \dot{p}^\beta$ æqualem successive; & inde in primo casu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in secundo $\mu = 0$, $\alpha = 2$, & $\beta = 0$; & in tertio $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit primo casu 1, in secundo $\frac{1}{2}$, in tertio o. Unde ν est 1, & az^ν & $\nu az^{\nu-1}$ sunt az & a. Termini duo ultimi az & a in Resolvenda pro p & $\dot{p}$ respective scripti, producunt $2daz + a^2z^2 - az^2 + dz$. Et termini $2daz$ & dz in quibus index dignitatis z est λ seu 1, positi æquales nihilo, dant $a = -\frac{1}{2}$. Unde az, terminus primus Seriei p fit $-\frac{1}{2}z$.

OPERATIO TERTIA.

Pro terminis reliquis nondum inventis pone q & habebis æquationem $p = -\frac{1}{2}z + q$, & inde $\dot{p} = -\frac{1}{2} + \dot{q}$: Qui valores pro p & $\dot{p}$ in Resolvenda novissima substituti producunt Resolvendam novam $2dq - zq + qq + \frac{3}{4}zz - zz\dot{q} = 0$. Ubi q & $\dot{q}$ vices supplent ipsorum y & $\dot{y}$. Terminus unicus in quo neque q nec $\dot{q}$ reperitur est $\frac{3}{4}zz$, qui cum kz^λ collatus dat $\lambda = 2$. Terminis reliquis $2dq$, $-zq$, $+qq$, $-zz\dot{q}$ pone $lz^\mu q^\alpha \dot{q}^\beta$ æqualem successive; & inde in primo casu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in secundo, $\mu = 1$,

$\mu = 1$, $\alpha = 1$, $\beta = 0$; in tertio, $\mu = 0$, $\alpha = 2$, $\beta = 0$: in quarto $\mu = 2$, $\alpha = 0$, $\beta = 1$: & inde $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit in primo casu 2, in secundo, tertio, & quarto 1. Et hinc v est 2, vel az^v & vaz^{v-1} sunt az^2 & $2az$: qui valores in Resolvenda pro q & $\dot{q}$ substituti dant $2daz^2 - az^3 + aaz^4 + \frac{3}{4}zz - 2az^3$; & termini $2dazz + \frac{3}{4}zz$ in quibus index dignitatis z est λ seu 2, positi æquales nihilo, dant $a = -\frac{3}{8d}$. Unde az^v terminus primus Seriei q evadit $-\frac{3zz}{8d}$.

OPERATIO QUARTA.

Pro reliquis Seriei terminis nondum inventis pone r, & habebis æquationes $q = -\frac{3zz}{8d} + r$, & $\dot{q} = -\frac{3z}{4d} + \dot{r}$; & inde resolvendam novam $2dr + \frac{9z^3}{8d} - zr + \frac{9z^4}{64dd} - \frac{3zzr}{4d} + rr - zz\dot{r} = 0$; & ex ea per Methodum superiorem habebis $-\frac{9z^3}{16dd}$ terminum primum Seriei r. Et sic pergitur in infinitum.

Est igitur radix extrahenda $y = d + p = d - \frac{1}{2}z + q = d - \frac{1}{2}z - \frac{3zz}{8d} + r = d - \frac{1}{2}z - \frac{3zz}{8d} - \frac{9z^3}{16dd} -$ &c. Et operationem continuando producere licet radicem ad terminos plures.

Et eadem methodo, dicit *Newtonus*, radices æquationum, fluxiones secundas, tertias, quartas, ($\ddot{y}$, $\dddot{y}$, $\ddddot{y}$,) aliasque involventium, extrahi posse.

His utitur radicum extractionibus ubi aliæ Methodi nil prosunt. Nam in Epistola prædicta anni 1676 docet, quod in Solutione problematum de Tangentibus inversorum, casus aliqui dantur in quibus hæc Methodus generalis non requiritur: & particulariter, si in triangulo rectangulo quod ab ordinata, tangente, & interjacente parte abscissæ constituitur, relatio duorum quorumlibet e lateribus tribus per æquationem quamvis definiatur; Problema absque Methodo hacce generali solvi poterit.

Methodi autem hæ omnes tam particulares quam generales collectim sumptæ, solutionem exhibent secundæ partis problematis, quod *Newtonus* sub initio istius Epistolæ his verbis proposuit. *Data æquatione quotcunque fluentes quantitates involvente fluxiones invenire, & vice versa.* Nam tota fluxionum Methodus in hujus directa & inversa solutione consistit.

Part of a Letter from Sir *Is. Newton*, to Mr. *J. Collins*, *Novemb.* 8. 1676.

THere is no Curve-line expreſs'd by any Equation of three terms, tho' the unknown quantities affect one another in it, or the Indices of their Dignities be ſurd quantities (ſuppoſe $ax^{\lambda} + bx^{\mu}y^{\sigma} + cy^{\tau} = 0$, *where* x *ſignifies the Baſe,* y *the Ordinate,* λ, μ, σ, τ *the Indices of the dignities of* x *and* y, *and* a, b, c *known quantities with their ſigns + or —,) I ſay, there is no ſuch Curve-line, but I can, in leſs than half a quarter of an hour, tell whether it may be Squar'd, or what are the ſimpleſt Figures it may be compar'd with, be thoſe Figures Conic Sections, or others: And then by a direct and ſhort way, (I dare ſay the ſhorteſt the nature of the thing admits of, for a general one,) I can compare them. And ſo if any two Figures expreſs'd by ſuch Equations be propounded, I can, by the ſame Rule compare them if they may be compar'd. This may ſeem a bold aſſertion, becauſe it's hard to ſay a Figure may, or may not, be Squar'd, or Compar'd with another; but it's plain to me by the fountain I draw it from, tho' I will not undertake to prove it to others. The ſame Method extends to Equations of four Terms, and others alſo, but not ſo generally.*

Fragmentum Epiſtolæ D. Newtoni *ad D.* Collinſium, Novemb. 8. 1676, *Latine redditum.*

NUlla extat Curva cujus Æquatio ex tribus conſtat terminis, in qua, licet quantitates incognitæ ſe mutuo afficiant, vel Indices dignitatum ſint ſurdæ quantitates (v. g. $ax^{\lambda} + bx^{\mu}y^{\sigma} + cy^{\tau} = 0$, ubi x deſignat Baſin, y Ordinatam, λ, μ, σ, τ Indices dignitatum ipſius x & y, & a, b, c quantitates cognitas una cum ſignis ſuis + vel —) nulla inquam hujuſmodi eſt Curva, de qua, an Quadrari poſſit, necne, vel quænam ſint Figuræ ſimpliciſſimæ quibuſcum comparari poſſit, ſive ſint Conicæ ſectiones ſive aliæ magis complicatæ, intra horæ octantem reſpondere non poſſim. Deinde * methodo directa & brevi, imo methodorum omnium generalium breviſſima eas comparare queo. Quinetiam ſi duæ quævis Figuræ per hujuſmodi Æquationes expreſſæ proponantur, per eandem Regulam, eas, modo comparari poſſint, comparo.

Affirmatio quidem videri poteſt temeraria, propterea quod perdifficile ſit dictu an Figura Quadrari vel cum alia comparari poſſit, necne; mihi autem manifeſtum eſt, ex eo unde deduxi fonte, quanquam id aliis demonſtrare in me ſuſcipere nollem. Eadem methodus Æquationes quatuor terminorum aliaſque complectitur, haud tamen adeo generaliter.

* *Methodum habes in Coroll.* 2. *Prop.* 10. *Tract. ſequentis.*

TRACTATUS
DE
Quadratura Curvarum.

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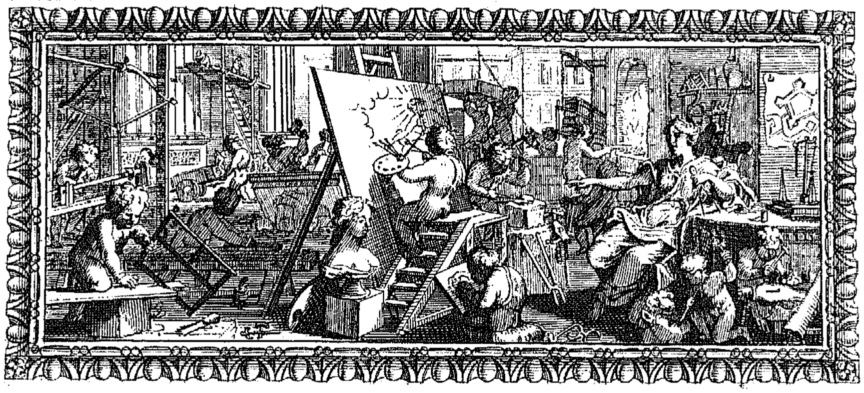


INTRODUCTIO

AD

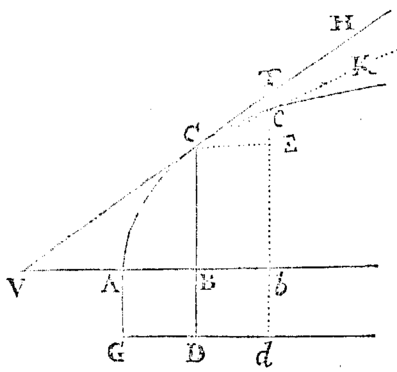
Quadraturam Curvarum.

Quantitates Mathematicas non ut ex partibus quam minimis conſtantes, ſed ut motu continuo deſcriptas hic conſidero. Lineæ deſcribuntur ac deſcribendo generantur non per appoſitionem partium ſed per motum continuum punctorum, ſuperficies per motum linearum, ſolida per motum ſuperficierum, anguli per rotationem laterum, tempora per fluxum continuum, & ſic in cæteris. Hæ Geneſes in rerum natura locum vere habent & in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectarum immobilium geneſin docuerunt rectangulorum.

Conſiderando igitur quod quantitates æqualibus temporibus creſcentes & creſcendo genitæ, pro velocitate majori vel minori qua creſcunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; & has motuum vel incrementorum velocitates nominando *Fluxiones* & quantitates genitas nominando *Fluentes*, incidi paulatim *Annis* 1665 & 1666 in Methodum Fluxionum qua hic uſus ſum in Quadratura Curvarum.

Fluxiones ſunt quam proxime ut Fluentium augmenta æqualibus temporis particulis quam minimis genita, &, ut accurate loquar, ſunt in prima ratione augmentorum naſcentium; exponi autem poſſunt per lineas quaſcunque quæ ſunt ipſis proportionales.

Ut ſi areæ ABC, ABDG Ordinatis BC, BD ſuper baſi AB uniformi cum motu progredientibus deſcribantur, harum arearum fluxiones erunt inter ſe ut Ordinatæ deſcribentes BC & BD, & per Ordinatas illas exponi poſſunt, propterea quod Ordinatæ illæ ſunt ut arearum augmenta naſcentia.



Progrediatur ordinata BC de loco ſuo BC in locum quemvis novum bc. Compleatur parallelogrammum BCEb, ac ducatur recta VTH quæ Curvam tangat in C ipſiſque bc & BA productis occurrat in T & V: & Abſciſſæ AB, Ordinatæ BC, & Lineæ Curvæ ACc augmenta modo genita erunt Bb, Ec & Cc; & in horum augmentorum naſcentium ratione prima ſunt latera trianguli CET, ideoque fluxiones ipſarum AB, BC & AC ſunt ut trianguli illius CET latera CE, ET & CT & per eadem latera exponi poſſunt, vel quod perinde eſt per latera trianguli conſimilis VBC.

Eodem recidit ſi ſumantur fluxiones in ultima ratione partium evaneſcentium. Agatur recta Cc & producatur eadem ad K. Redeat Ordinata bc in locum ſuum priorem BC, & coeuntibus punctis C & c, recta CK coincidet cum tangente CH, & triangulum evaneſcens CEc in ultima ſua forma evadet ſimile triangulo CET, & ejus latera evaneſcentia CE, Ec & Cc erunt ultimo inter ſe ut ſunt trianguli alterius CET latera CE, ET & CT, & propterea in hac ratione ſunt fluxiones linearum AB, BC & AC. Si puncta C & c parvo quovis intervallo ab invicem diſtant recta CK parvo intervallo a tangente CH diſtabit. Ut recta CK cum tangente CH coincidat & rationes ultimæ linearum CE, Ec & Cc inveniantur, debent puncta C & c coire & omnino coincidere. Errores quam minimi in rebus Mathematicis non ſunt contemnendi.

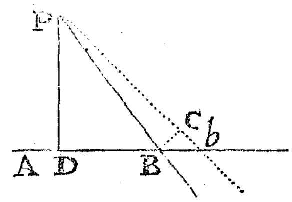
Simili argumento ſi circulus centro B radio BC deſcriptus in longitudinem Abſciſſæ AB ad angulos rectos uniformi cum motu ducatur, fluxio ſolidi geniti ABC erit ut circulus ille generans, & fluxio ſuperficiei ejus erit ut perimeter Circuli illius & fluxio lineæ curvæ AC conjunctim.

Nam

Nam quo tempore solidum ABC generatur ducendo circulum illum in longitudinem abscissæ AB, eodem superficies ejus generatur ducendo perimetrum circuli illius in longitudinem Curvæ AC. Hujus methodi accipe etiam exempla quæ sequuntur.

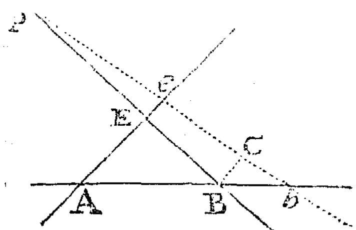
Recta PB circa polum datum P revolvens secet aliam positione datam rectam AB: quæritur proportio fluxionum rectarum illarum AB & PB.

Progrediatur recta PB de loco suo PB in locum novum Pb. In Pb capiatur PC ipsi PB æqualis, & ad AB ducatur PD sic, ut angulus bPD æqualis sit angulo bBC; & ob similitudinem triangulorum bBC, bPD erit augmentum Bb ad augmentum Cb ut Pb ad Db. Redeat jam Pb in locum suum priorem PB ut augmenta illa evanescant, & evanescentium ratio ultima, id est ratio ultima Pb ad Db, ea erit quæ est PB ad DB, existente angulo PDB recto, & propterea in hac ratione est fluxio ipsius AB ad fluxionem ipsius PB.



Recta PB circa datum Polum P revolvens secet alias duas positione datas rectas AB & AE in B & E: quæritur proportio fluxionum rectarum illarum AB & AE.

Progrediatur recta revolvens PB de loco suo PB in locum novum Pb rectas AB, AE in punctis b & e secantem, & rectæ AE parallela BC ducatur ipsi Pb occurrens in C, & erit Bb ad BC ut Ab ad Ae, & BC ad Ee ut PB ad PE, & conjunctis rationibus Bb ad Ee ut Ab×PB ad Ae×PE. Redeat jam linea Pb in locum suum priorem PB, & augmentum evanescens Bb erit ad augmentum evanescens Ee ut AB×PB ad AE×PE, ideoque in hac ratione est fluxio rectæ AB ad fluxionem rectæ AE.



Hinc si recta revolvens PB lineas quasvis Curvas positione datas secet in punctis B & E, & rectæ jam mobiles AB, AE Curvas illas tangant in Sectionum punctis B & E: erit fluxio Curvæ quam recta AB tangit ad fluxionem Curvæ quam recta AE tangit ut AB×PB ad AE×PE. Id quod etiam eveniet si recta PB Curvam aliquam positione datam perpetuo tangat in puncto mobili P.

Fluat quantitas x *uniformiter & invenienda sit fluxio quantitatis* x^n. Quo tempore quantitas x fluendo evadit $x+o$, quantitas x^n evadet $\overline{x+o}|^n$, id est per methodum serierum infinitarum, $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \&c.$

Et

Et augmenta o & $nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. ſunt ad invicem ut 1 & $nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +$ &c. Evaneſcant jam augmenta illa, & eorum ratio ultima erit 1 ad nx^{n-1} : ideoque fluxio quantitatis x eſt ad fluxionem quantitatis x^n ut 1 ad nx^{n-1}.

Similibus argumentis per methodum rationum primarum & ultimarum colligi poſſunt fluxiones linearum ſeu rectarum ſeu curvarum in caſibus quibuſcunque, ut & fluxiones ſuperficierum, angulorum & aliarum quantitatum. In finitis autem quantitatibus Analyſin ſic inſtituere, & finitarum naſcentium vel evaneſcentium rationes primas vel ultimas inveſtigare, conſonum eſt Geometriæ Veterum : & volui oſtendere quod in Methodo Fluxionum non opus ſit figuras infinite parvas in Geometriam introducere. *Peragi tamen poteſt Analyſis in figuris quibuſcunque ſeu finitis ſeu infinite parvis quæ figuris evaneſcentibus finguntur ſimiles, ut & in figuris quæ per Methodos Indiviſibilium pro infinite parvis haberi ſolent, modo caute procedas.*

Ex Fluxionibus invenire Fluentes Problema difficilius eſt, & ſolutionis primus gradus æquipollet Quadraturæ Curvarum ; de qua ſequentia olim ſcripſi.

D E

DE
Quadratura Curvarum.

Quantitates indeterminatas ut motu perpetuo creſcentes vel decreſcentes, id eſt ut fluentes vel defluentes in ſequentibus conſidero, deſignoque literis z, y, x, v, & earum fluxiones ſeu celeritates creſcendi noto iiſdem literis punctatis $\dot z$, $\dot y$, $\dot x$, $\dot v$, Sunt & harum fluxionum fluxiones ſeu mutationes magis aut minus celeres quas ipſarum z, y, x, v, fluxiones ſecundas nominare licet & ſic deſignare $\ddot z$, $\ddot y$, $\ddot x$, $\ddot v$, & harum fluxiones primas ſeu ipſarum z, y, x, v fluxiones tertias ſic $\dddot z$, $\dddot y$, $\dddot x$, $\dddot v$, & quartas ſic $\ddddot z$, $\ddddot y$, $\ddddot x$, $\ddddot v$. Et quemadmodum $\dddot z$, $\dddot y$, $\dddot x$, $\dddot v$ ſunt fluxiones quantitatum $\ddot z$, $\ddot y$, $\ddot x$, $\ddot v$, & hæ ſunt fluxiones quantitatum $\dot z$, $\dot y$, $\dot x$, $\dot v$, & hæ ſunt fluxiones quantitatum primarum z, y, x, v: ſic hæ quantitates conſiderari poſſunt ut fluxiones aliarum quas ſic deſignabo, $\overset{\prime}{z}$, $\overset{\prime}{y}$, $\overset{\prime}{x}$, $\overset{\prime}{v}$, & hæ ut fluxiones aliarum $\overset{\prime\prime}{z}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{x}$, $\overset{\prime\prime}{v}$, & hæ ut fluxiones aliarum $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{v}$. Deſignant igitur $\overset{\prime\prime}{z}$, $\overset{\prime}{z}$, z, $\dot z$, $\ddot z$, $\dddot z$, $\ddddot z$, $\overset{\cdot\cdot\cdot\cdot\cdot}{z}$, &c. ſeriem quantitatum quarum quælibet poſterior eſt fluxio præcedentis & quælibet prior eſt fluens quantitas fluxionem habens ſubſequentem. Similis eſt ſeries $\sqrt{\overset{\prime\prime}{az-zz}}$, $\sqrt{\overset{\prime}{az-zz}}$, $\sqrt{az-zz}$, $\sqrt{\overset{\cdot}{az-zz}}$, $\sqrt{\overset{\cdot\cdot}{az-zz}}$, $\sqrt{\overset{\cdot\cdot\cdot}{az-zz}}$, ut & ſeries $\overset{\prime\prime}{\frac{az+zz}{a-z}}$, $\overset{\prime}{\frac{az+zz}{a-z}}$, $\frac{az+zz}{a-z}$, $\overset{\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot\cdot}{\frac{az+zz}{a-z}}$.

Et notandum eſt quod quantitas quælibet prior in his ſeriebus eſt ut area figuræ curvilineæ cujus ordinatim applicata rectangula eſt quantitas poſterior & abſciſſa eſt z: uti $\sqrt{\overset{\prime}{az-zz}}$ area curvæ cujus ordinata eſt $\sqrt{az-zz}$ & abſciſſa z. Quo autem ſpectant hæc omnia patebit in Propoſitionibus quæ ſequuntur.

PROP. I. PROB. I.

Data æquatione quotcunque fluentes quantitates involvente, invenire fluxiones.

Solutio.

Multiplicetur omnis æquationis terminus per indicem dignitatis quantitatis cujusque fluentis quam involvit, & in singulis multiplicationibus mutetur dignitatis latus in fluxionem suam, & aggregatum factorum omnium sub propriis signis erit æquatio nova.

Explicatio.

Sunto a, b, c, d &c. quantitates determinatæ & immutabiles, & proponatur æquatio quævis quantitates fluentes z, y, x &c. involvens, uti $x^3 - xy^2 + a^2z - b^3 = 0$. Multiplicentur termini primo per indices dignitatum x, & in singulis multiplicationibus pro dignitatis latere, seu x unius dimensionis, scribatur $\dot{x}$, & summa factorum erit $3\dot{x}x^2 - \dot{x}y^2$. Idem fiat in y & prodibit $-2x\dot{y}y$. Idem fiat in z & prodibit $aa\dot{z}$. Ponatur summa factorum æqualis nihilo, & habebitur æquatio $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Dico quod hac æquatione definitur relatio fluxionum.

Demonstratio.

Nam sit o quantitas admodum parva & sunto $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, quantitatum z, y, x momenta id est incrementa momentanea synchrona. Et si quantitates fluentes jam sunt z, y & x hæ post momentum temporis incrementis suis $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ auctæ, evadent $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$, quæ in æquatione prima pro z, y, & x scriptæ dant æquationem $x^3 + 3x^2o\dot{x} + 3xo^2\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2z + a^2o\dot{z} - b^3 = 0$.

Subducatur æquatio prior, & residuum divisum per o erit $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$. Minuatur quantitas o in infinitum, & neglectis terminis evanescentibus restabit $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Q. E. D.

Explicatio

Explicatio plenior.

Ad eundem modum si æquatio esset $x^3 - xy^2 + a^2\sqrt{ax-yy} - b^3 = 0$, produceretur $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax-yy}}} = 0$. Ubi si fluxionem $\dot{\overline{\sqrt{ax-yy}}}$ tollere velis, pone $\sqrt{ax-yy} = z$, & erit $ax - y^2 = z^2$, & (per hanc Propositionem) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$ seu $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, hoc est $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-yy}} = \dot{\overline{\sqrt{ax-yy}}}$. Et inde $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x}-2a^2\dot{y}y}{2\sqrt{ax-yy}} = 0$.

Et per operationem repetitam pergitur ad fluxiones secundas, tertias & sequentes. Sit æquatio $zy^3 - z^4 + a^4 = 0$, & fiet per operationem primam $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, per secundam $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = 0$, per tertiam $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$.

Ubi vero sic pergitur ad fluxiones secundas, tertias & sequentes, convenit quantitatem aliquam ut uniformiter fluentem considerare, & pro ejus fluxione prima unitatem scribere, pro secunda vero & sequentibus nihil. Sit æquatio $zy^3 - z^4 + a^4 = 0$, ut supra; & fluat z uniformiter, sitque ejus fluxio unitas, & fiet per operationem primam $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$, per secundam $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = 0$, per tertiam $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = 0$.

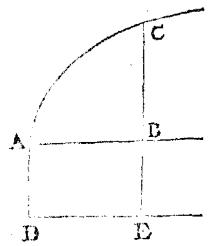
In hujus autem generis æquationibus concipiendum est quod fluxiones in singulis terminis sint ejusdem ordinis, id est vel omnes primi ordinis $\dot{y}$, $\dot{z}$, vel omnes secundi $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$, vel omnes tertii $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. Et ubi res aliter se habet complendus est ordo per subintellectas fluxiones quantitatis uniformiter fluentis. Sic æquatio novissima complendo ordinem tertium fit $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z\dot{z}^3 = 0$.

PROP.

PROP. II. PROB. II.

Invenire Curvas quæ quadrari possunt.

Sit ABC figura invenienda, BC Ordinatim applicata rectangula, & AB abscissa. Producatur CB ad E ut sit BE = 1, & compleatur parallelogrammum ABED: & arearum ABC, ABED fluxiones erunt ut BC & BE. Assumatur igitur æquatio quævis qua relatio arearum definiatur, & inde dabitur relatio ordinatarum BC & BE per Prop. 1. Q. E. I.



Hujus rei exempla habentur in Propositionibus duabus sequentibus.

PROP. III. THEOR. I.

Si pro abscissa AB & area AE seu AB×1 promiscue scribatur z, & si pro $e+fz^n+gz^{2n}+hz^{3n}+$ &c. scribatur R: sit autem area Curvæ $z^\theta R^\lambda$, erit ordinatim applicata BC æqualis

$$\overline{\theta e + \begin{smallmatrix}\theta\\ +\lambda n\end{smallmatrix} \times fz^n + \begin{smallmatrix}\theta\\ +2\lambda n\end{smallmatrix} \times gz^{2n} + \begin{smallmatrix}\theta\\ +3\lambda n\end{smallmatrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1}.$$

Demonstratio.

Nam si sit $z^\theta R^\lambda = v$, erit (per Prop. 1) $\theta\dot{z}z^{\theta-1}R^\lambda + \lambda z^\theta\dot{R}R^{\lambda-1} = \dot{v}$. Pro R^λ in primo æquationis termino & z^θ in secundo scribe $RR^{\lambda-1}$ & $zz^{\theta-1}$, & fiet $\theta\dot{z}R + \lambda z\dot{R}$ in $z^{\theta-1}R^{\lambda-1} = \dot{v}$. Erat autem $R = e+fz^n+gz^{2n}$ $+hz^{3n}+$&c. et inde (per Prop. 1) fit $\dot{R} = nf\dot{z}z^{n-1} + 2ng\dot{z}z^{2n-1} + 3nh\dot{z}z^{3n-1} +$ &c. quibus substitutis & scripta BE seu 1 pro $\dot{z}$, fiet

$$\overline{\theta e + \begin{smallmatrix}\theta\\ +\lambda n\end{smallmatrix} \times fz^n + \begin{smallmatrix}\theta\\ +2\lambda n\end{smallmatrix} \times gz^{2n} + \begin{smallmatrix}\theta\\ +3\lambda n\end{smallmatrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1} = \dot{v} = \mathrm{BC}.$$

Q. E. D.

PROP.

PROP. IV. THEOR. II.

Si Curvæ abſciſſa AB ſit z, & ſi pro $e+fz^{n}+gz^{2n}+$ &c. ſcribatur R, & pro $k+lz^{n}+mz^{2n}+$ &c. ſcribatur S; ſit autem area Curvæ $z^{\theta}R^{\lambda}S^{\mu}$: Erit ordinatim applicata BC æqualis

$$\left.\begin{array}{l} \theta ek + \substack{\theta\\+\lambda n}\times fkz^{n} + \substack{\theta\\+2\lambda n}\times gkz^{2n} \ldots * \ldots\ldots * \ldots \\ \quad + \substack{\theta\\+\mu n}\times elz^{n} + \substack{\theta\\+\lambda n\\+\mu n}\times flz^{2n} + \substack{\theta\\+2\lambda n\\+\mu n}\times glz^{3n} \ldots * \ldots \\ \qquad\qquad + \substack{\theta\\+2\mu n}\times emz^{2n} + \substack{\theta\\+\lambda n\\+2\mu n}\times fmz^{3n} + \substack{\theta\\+2\lambda n\\+2\mu n}\times gmz^{4n} \end{array}\right\}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$$

Demonſtratur ad modum Propoſitionis ſuperioris.

PROP. V. THEOR. III.

Si Curvæ abſciſſa AB ſit z, & pro $e+fz^{n}+gz^{2n}+hz^{3n}+$ &c. ſcribatur R: ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^{n}+cz^{2n}+dz^{3n}+\&c.}$ & ponatur $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=v$, &c.

$$\begin{array}{ll} \text{Erit Area} = z^{\theta}R^{\lambda}\ \text{in} & +\dfrac{\frac{1}{n}a}{re} \\ & +\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}z^{n} \\ & +\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}z^{2n} \\ & +\dfrac{\frac{1}{n}d-\overline{s+2}\times fC-\overline{t+1}\times gB-vhA}{\overline{r+3}\times e}z^{3n} \\ & +\dfrac{-\overline{s+3}\times fD-\overline{t+2}\times gC-\overline{v+1}\times hB}{\overline{r+4}\times e}z^{4n} \\ & +\ \&c. \end{array}$$

Ubi A, B, C, D, &c. denotant totas coefficientes datas terminorum ſingulorum in ſerie cum ſignis ſuis $+$ & $-$, nempe

A primi termini coefficientem $\dfrac{\frac{1}{n}a}{re}$

B ſecundi termini coefficientem $\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}$

C tertii termini coefficientem $\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}$

Et ſic deinceps.

Demonstratio.

Sunto juxta Propositionem tertiam.

$$\begin{array}{ll} & \text{CURVARUM ORDINATÆ} \\ 1\,. & \theta eA + \overline{\theta+\lambda n}\times fAz^{n} + \overline{\theta+2\lambda n}\times gAz^{2n} + \overline{\theta+3\lambda n}\times hAz^{3n},\ \&c. \\ 2\ldots & + \overline{\theta+n}\times eBz^{n} + \overline{\theta+n+\lambda n}\times fBz^{2n} + \overline{\theta+n+2\lambda n}\times gBz^{3n}\ \&c. \\ 3\ldots\ldots\ldots & + \overline{\theta+2n}\times eCz^{2n} + \overline{\theta+2n+\lambda n}\times fCz^{3n}\ \&c. \\ 4\ldots\ldots\ldots\ldots\ldots & + \overline{\theta+3n}\times eDz^{3n}\ \&c. \end{array}\left.\vphantom{\begin{array}{l}1\\2\\3\\4\\5\end{array}}\right\}\times z^{\theta-1}R^{\lambda-1}.\left\{\begin{array}{l}\text{AREÆ} \\ Az^{\theta}R^{\lambda} \\ Bz^{\theta+n}R^{\lambda} \\ Cz^{\theta+2n}R^{\lambda} \\ Dz^{\theta+3n}R^{\lambda}\end{array}\right.$$

Et si summa Ordinatarum ponatur æqualis Ordinatæ $a + bz^{n} + cz^{2n} + dz^{3n} +$ &c. in $z^{\theta-1}R^{\lambda-1}$, summa arearum $z^{\theta}R^{\lambda}$ in $A + Bz^{n} + Cz^{2n} + Dz^{3n} +$ &c. æqualis erit areæ Curvæ cujus ista est Ordinata. Æquentur igitur Ordinatarum termini correspondentes,

$$\begin{aligned} \text{et fiet } a &= \theta eA, \\ b &= \overline{\theta+\lambda n}\times fA + \overline{\theta+n}\times eB, \\ c &= \overline{\theta+2\lambda n}\times gA + \overline{\theta+n+\lambda n}\times fB + \overline{\theta+2n}\times eC, \\ &\&c. \end{aligned}$$

$$\begin{aligned} \text{et inde } A &= \frac{a}{\theta e} \\ B &= \frac{b - \overline{\theta+\lambda n}\times fA}{\overline{\theta+n}\times e} \\ C &= \frac{c - \overline{\theta+2\lambda n}\times gA - \overline{\theta+n+\lambda n}\times fB}{\overline{\theta+2n}\times e} \end{aligned}$$

Et sic deinceps in infinitum.

Pone jam $\frac{\theta}{n} = r$, $r + \lambda = s$, $s + \lambda = t$, &c. et in area $z^{\theta}R^{\lambda}$ in $A + Bz^{n} + Cz^{2n} + Dz^{3n} +$ &c. scribe ipsorum A, B, C, &c. valores inventos, & prodibit series proposita. Q. E. D.

Et notandum est quod Ordinata omnis duobus modis in seriem resolvitur. Nam index n vel affirmativus esse potest vel negativus.

Proponatur Ordinata $\frac{3k - lzz}{zz\sqrt{kz - lz^{3} + mz^{4}}}$: Hæc vel sic scribi potest

$z^{-\frac{5}{2}} \times \overline{3k - lz^{2}} \times \overline{k - lz^{2} + mz^{3}}|^{-\frac{1}{2}}$,

vel sic $z^{-2} \times \overline{-l + 3kz^{-2}} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}}$.

In

In casu priore est $a = 3k$, $b = 0$, $c = -l$, $e = k$, $f = 0$, $g = -l$, $h = m$, $\lambda = \frac{1}{2}$, $n = 1$, $\theta - 1 = -\frac{3}{2}$, $\theta = -\frac{1}{2} = r$, $s = -1$, $t = -\frac{1}{2}$, $v = 0$.

In posteriore est $a = -l$, $b = 0$, $c = 3k$, $e = m$, $f = -l$, $g = 0$, $h = k$, $\lambda = \frac{1}{2}$, $n = -1$, $\theta - 1 = -2$, $\theta = -1$, $r = 1$, $s = 1\frac{1}{2}$, $t = 2$, $v = 2\frac{1}{2}$. Tentandus est casus uterque. Et si ferierum alterutra ob terminos tandem deficientes abrumpitur ac terminatur habebitur area Curvæ in terminis finitis. Sic in exempli hujus priore casu scribendo in ferie valores ipsorum a, b, c, e, f, g, h, λ, θ, r, s, t, v, termini omnes post primum evanescunt in infinitum & area Curvæ prodit $-2\sqrt{\frac{k - lz^2 + mz^3}{z^3}}$. Et hæc area ob signum negativum adjacet abscissæ ultra ordinatam productæ. Nam area omnis affirmativa adjacet tam abscissæ quam ordinatæ, negativa vero cadit ad contrarias partes ordinatæ & adjacet abscissæ productæ, manente scilicet signo Ordinatæ. Hoc modo series alterutra & nonnunquam utraque semper terminatur & finita evadit si Curva geometrice quadrari potest. At si Curva talem quadraturam non admittit, series utraque continuabitur in infinitum, & earum altera converget & aream dabit approximando, præterquam ubi r (propter aream infinitam) vel nihil est vel numerus integer & negativus, vel ubi $\frac{z^n}{e}$ æqualis est unitati. Si $\frac{z^n}{e}$ minor est unitate, converget series in qua index n affirmativus est: sin $\frac{z^n}{e}$ unitate major est, converget series altera. Si in uno casu area adjacet abscissæ ad usque ordinatam ductæ, in altero adjacet abscissæ ultra ordinatam productæ.

Nota insuper quod si Ordinata contentum est sub factore rationali Q & factore surdo irreducibili R^π & factoris surdi latus R non dividit factorem rationalem Q; erit $\lambda - 1 = \pi$ & $R^{\lambda - 1} = R^\pi$. Sin factoris surdi latus R dividit factorem rationalem semel, erit $\lambda - 1 = \pi + 1$ & $R^{\lambda - 1} = R^{\pi + 1}$: si dividit bis, erit $\lambda - 1 = \pi + 2$ & $R^{\lambda - 1} = R^{\pi + 2}$: si ter, erit $\lambda - 1 = \pi + 3$, & $R^{\lambda - 1} = R^{\pi + 3}$: & sic deinceps.

Si Ordinata est fractio rationalis irreducibilis cum denominatore ex duobus vel pluribus terminis composito: resolvendus est denominator in divisores suos omnes primos. Et si divisor sit aliquis cui nullus alius est æqualis, Curva quadrari nequit: Sin duo vel plures sint divisores æquales, rejiciendus est eorum unus, & si adhuc alii duo vel plures sint sibi mutuo æquales & prioribus inæquales, rejiciendus est etiam eorum unus, & sic in aliis omnibus æqualibus si adhuc plures sint: deinde divisor qui relinquitur vel contentum sub divisoribus omnibus qui relinquuntur, si plures sunt, ponendum est pro R, & ejus quadrati reciprocum R^{-2} pro $R^{\lambda - 1}$, præterquam ubi contentum illud est quadratum vel cubus vel quadrato quadratum, &c. quo casu ejus latus ponendum est pro R & potestatis index 2 vel 3 vel 4 negative sumptus pro λ, et Ordinata ad denominatorem R^2 vel R^3 vel R^4 vel R^5 &c. reducenda. Ut

Ut ſi Ordinata ſit $\frac{z^5+z^4-8z^3}{z^5+z^4-5z^3-z^2+8z-4}$; quoniam hæc fractio irreducibilis eſt & denominatoris diviſores ſunt pares, nempe $z-1$, $z-1$, $z-1$ & $z+2$, $z+2$, rejicio magnitudinis utriuſque diviſorem unum, & reliquorum $z-1$, $z-1$, $z+2$ contentum z^3-3z+2 pono pro R & ejus quadrati reciprocum $\frac{1}{R^2}$ ſeu R^{-2} pro $R^{\lambda-1}$. Dein Ordinatam ad denominatorem R^2 ſeu $R^{1-\lambda}$ reduco, & fit $\frac{z^6-9z^4+8z^3}{\overline{z^3-3z+2}|^2}$, i.e. $z^3\times\overline{8-9z+z^3}\times\overline{2-3z+z^3}|^{-2}$. Et inde eſt $a=8$, $b=-9$, $c=0$, $d=1$, &c. $e=2$, $f=-3$, $g=0$, $h=1$, $\lambda-1=-2$, $\lambda=-1$, $\eta=1$, $\theta-1=3$, $\theta=4=r$, $s=3$, $t=2$, $v=1$. Et his in ſerie ſcriptis prodit area $\frac{z^4}{z^3-3z+2}$, terminis omnibus in tota ſerie poſt primum evaneſcentibus.

Si denique Ordinata eſt fractio irreducibilis & ejus denominator contentum eſt ſub factore rationali $\mathcal{Q}$ & factore ſurdo irreducibili R^π, inveniendi ſunt lateris R diviſores omnes primi, & rejiciendus eſt diviſor unus magnitudinis cujuſque & per diviſores qui reſtant, ſiqui ſint, multiplicandus eſt factor rationalis $\mathcal{Q}$: & ſi factum æquale eſt lateri R vel lateris illius poteſtati alicui cujus index eſt numerus integer, eſto index ille m, & erit $\lambda-1=-\pi-m$, & $R^{\lambda-1}=R^{-\pi-m}$.

Ut ſi Ordinata ſit $\frac{3q^5-q^4x+9q^2x^2-q^2x^3-6x^3}{\overline{q^2-x^2}\times\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, quoniam factoris ſurdi latus R ſeu $q^3+q^2x-qx^2-x^3$ diviſores habet $q+x$, $q+x$, $q-x$ qui duarum ſunt magnitudinum, rejicio diviſorem unum magnitudinis utriuſque & per diviſorem $q+x$ qui relinquitur, multiplico factorem rationalem q^2-x^2. Et quoniam factum $q^3+q^2x-qx^2-x^3$ æquale eſt lateri R, pono $m=1$, & inde, cum π ſit $\frac{1}{3}$, fit $\lambda-1=-\frac{4}{3}$. Ordinatam igitur reduco ad denominatorem $R^{-\frac{4}{3}}$, & fit

$$x^0\times\overline{3q^6+2q^5x+8q^4x^2+8q^3x^3-7q^2x^4-6qx^5}\times\overline{q^3+q^2x-qx^2-x^3}|^{-\frac{4}{3}}$$

Unde eſt $a=3q^6$, $b=2q^5$ &c. $e=q^3$, $f=q^2$ &c. $\theta-1=0$, $\theta=1=\eta$, $\lambda=-\frac{1}{3}$, $r=1$, $s=\frac{2}{3}$, $t=\frac{1}{3}$, $v=0$. Et his in ſerie ſcriptis prodit area $\frac{3q^2x+3x^3}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, terminis omnibus in ſerie tota poſt tertium evaneſcentibus.

PROP. VI. THEOR. IV.

Si Curvæ abſciſſa AB ſit z, & ſcribantur R pro $e+fz^\eta+gz^{2\eta}+hz^{3\eta}+$&c. & S pro $k+lz^\eta+mz^{2\eta}+nz^{3\eta}+$&c. ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ in $a+bz^\eta+cz^{2\eta}+dz^{3\eta}+$&c. et ſi terminorum, e, f, g, h, &c. et k, l, m, n, &c. rectangula ſint

$$\begin{matrix} ek & fk & gk & hk & \text{\&c.} \\ el & fl & gl & hl & \text{\&c.} \\ em & fm & gm & hm & \text{\&c.} \\ en & fn & gn & hn & \text{\&c.} \end{matrix}$$

Et

Et si rectangulorum illorum coefficientes numerales sint respective

$$\frac{\theta}{n}=r. \quad r+\lambda=s. \quad s+\lambda=t. \quad t+\lambda=v. \quad \&c.$$

$$r+\mu=s'. \quad s+\mu=t'. \quad t+\mu=v'. \quad v+\mu=w'. \quad \&c.$$

$$s'+\mu=t''. \quad t'+\mu=v''. \quad v'+\mu=w''. \quad w'+\mu=x''. \quad \&c.$$

$$t''+\mu=v'''. \quad v''+\mu=w'''. \quad w''+\mu=x'''. \quad x''+\mu=y'''. \quad \&c.$$

Area Curvæ erit hæc

$$z^{\theta} R^{\lambda} S^{\mu} \text{ in } + \frac{\frac{1}{n}a}{rek}$$

$$+ \frac{\frac{1}{n}b \begin{smallmatrix} -sfk\,A \\ -s'el \end{smallmatrix}}{\overline{r+1}\times ek} z^{n}$$

$$+ \frac{\frac{1}{n}c \begin{smallmatrix} & -tgk\,A \\ -\overline{s+1}\times fk\,B & -t'fl \\ -\overline{s'+1}\times el & -t''em \end{smallmatrix}}{\overline{r+2}\times ek} z^{2n}$$

$$+ \frac{\frac{1}{n}d \begin{smallmatrix} & & -vhk\,A \\ & -\overline{t+1}\times gk\,B & -v'gl \\ -\overline{s+2}\times fk\,C & -\overline{t'+1}\times fl & -v''fm \\ -\overline{s'+2}\times el & -\overline{t''+1}\times em & -v'''en \end{smallmatrix}}{\overline{r+3}\times ek} z^{3n}$$

$$+ \&c.$$

Ubi A denotat termini primi coefficientem datam $\frac{\frac{1}{n}a}{rek}$ cum signo suo $+$ vel $-$, B coefficientem datam secundi, C coefficientem datam tertii, & sic deinceps. Terminorum vero, $a, b, c,$ &c. $e, f, g,$ &c. $k, l, m,$ &c. unus vel plures deesse possunt.

Demonstratur Propositio ad modum præcedentis, & quæ ibi notantur hic obtinent. Pergit autem series talium Propositionum in infinitum, & progressio seriei manifesta est.

PROP. VII. THEOR. V.

Si pro $e+fz^{n}+gz^{2n}+$ &c. scribatur R ut supra, & in Curvæ alicujus Ordinata $z^{\theta \pm n\sigma} R^{\lambda \pm \tau}$ maneant quantitates datæ $\theta, n, \lambda, e, f, g,$ &c. et pro σ ac τ scribantur successive numeri quicunque integri: & si detur Area unius ex Curvis quæ per Ordinatas innumeras sic prodeuntes designantur si Ordinatæ sunt duorum nominum in vinculo radicis, vel si dentur Areæ duarum ex Curvis si Ordinatæ sunt trium nominum in vinculo radicis, vel Areæ trium ex Curvis si Ordinatæ sunt quatuor nominum in vinculo radicis, & sic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Pro nominibus hic habeo terminos omnes in vinculo radicis tam deficientes quam plenos quorum indices dignitatum ſunt in progreſſione arithmetica. Sic Ordinata $\sqrt{a^4 - ax^3 + x^4}$ ob terminos duos inter a^4 & $-ax^3$ deficientes pro quinquinomio haberi debet. At $\sqrt{a^4 + x^4}$ binomium eſt, & $\sqrt{a^4 + x^4 - \frac{x^8}{a^4}}$ trinomium, cum progreſſio jam per majores differentias procedat. Propoſitio vero ſic demonſtratur.

CAS. I.

Sunto Curvarum duarum Ordinatæ $pz^{\theta-1}R^{\lambda-1}$ & $qz^{\theta+n-1}R^{\lambda-1}$, & Areæ pA & qB, exiſtente R quantitate trium nominum $e+fz^n+gz^{2n}$. Et cum per Prop. 3. ſit $z^\theta R^\lambda$ area Curvæ cujus Ordinata eſt $\overline{\theta e + \overline{\theta+\lambda n} \times fz^n + \overline{\theta+2\lambda n} \times gz^{2n}}$ in $z^{\theta-1}R^{\lambda-1}$, ſubduc Ordinatas & Areas priores de Area & Ordinata poſteriori, & manebit $\overline{\theta e - p + \overline{\theta \times f - q}_{+\lambda n} \times z^n + \theta_{+2\lambda n} \times gz^{2n}} \times z^{\theta-1}R^{\lambda-1}$ Ordinata nova Curvæ, et $\overline{z^\theta R^\lambda - pA - qB}$ ejuſdem Area. Pone $\theta e = p$, & $\theta f + \lambda nf = q$, & Ordinata evadet $\overline{\theta+2\lambda n} \times gz^{2n} \times z^{\theta-1}R^{\lambda-1}$, & Area $z^\theta R^\lambda - \theta eA - \theta fB - \lambda nfB$. Divide utramque per $\theta g + 2\lambda ng$, & Aream prodeuntem dic C, & aſſumpta utcunque r, erit rC Area Curvæ cujus Ordinata eſt $rz^{\theta+2n-1}R^{\lambda-1}$. Et qua ratione ex Areis pA & qB Aream rC Ordinatæ $rz^{\theta+2n-1}R^{\lambda-1}$ congruentem invenimus, licebit ex Areis qB & rC Aream quartam puta sD, Ordinatæ $sz^{\theta+3n-1}R^{\lambda-1}$ congruentem invenire, & ſic deinceps in infinitum. Et par eſt ratio progreſſionis ab Areis B & A in partem contrariam pergentis. Si terminorum θ, $\theta+\lambda n$, & $\theta+2\lambda n$ aliquis deficit & ſeriem abrumpit, aſſumatur Area pA in principio progreſſionis unius & Area qB in principio alterius, & ex his duabus Areis dabuntur Areæ omnes in progreſſione utraque. Et contra, ex aliis duabus Areis aſſumptis fit regreſſus per Analyſin ad Areas A & B, adeo ut ex duabus datis cæteræ omnes dentur. Q. E. O.

Hic eſt caſus Curvarum ubi ipſius z index θ augetur vel diminuitur perpetua additione vel ſubductione quantitatis n. Caſus alter eſt Curvarum ubi index λ augetur vel diminuitur unitatibus.

CAS. II.

Ordinatæ $pz^{\theta-1}R^\lambda$ et $qz^{\theta+n-1}R^\lambda$, quibus Areæ pA et qB jam reſpondeant, ſi in R ſeu $\overline{e+fz^n+gz^{2n}}$ ducantur ac deinde ad R viciſſim applicentur, evadunt $\overline{pe+pfz^n+pgz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $\overline{qez^n+qfz^{2n}+qgz^{3n}} \times z^{\theta-1}R^{\lambda-1}$. Et (per Prop. 3.) eſt $az^\theta R^\lambda$ area Curvæ cujus Ordinata eſt $\overline{\theta ae + \overline{\theta+\lambda n} \times afz^n + \overline{\theta+2\lambda n} \times agz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $bz^{\theta+n}R^\lambda$ area Curvæ

cujus

cujus Ordinata est $\overline{\overline{\theta+n}\times bez^n+\overline{\theta+n+\lambda n}\times bfz^{2n}+\overline{\theta+n+2\lambda n}\times bgz^{3n}}\times z^{\theta-1}R^{\lambda-1}$.

Et harum quatuor Arearum summa est $pA+qB+az^\theta R^\lambda+bz^{\theta+n}R^\lambda$, & summa respondentium Ordinatarum

$$\overline{\begin{array}{llll} \theta ae+\overline{\theta+\lambda n}\times afz^n+ & \overline{\theta+2\lambda n}\times agz^{2n}+ & \overline{\theta+n+2\lambda n}\times bgz^{3n} & \times z^{\theta-1}R^{\lambda-1}. \\ +pe+\overline{\theta+n}\times be & +\overline{\theta+n+\lambda n}\times bf & +1\times qg & \\ \quad +1\times pf & +1\times pg & & \\ \quad +1\times qe & +1\times qf & & \end{array}}$$

Si terminus primus tertius & quartus ponantur seorsim æquales nihilo, per primum fiet $\theta ae+pe=0$, seu $-\theta a=p$, per quartum $-\theta b-nb-2\lambda nb=q$, & per tertium (eliminando p & q) $\frac{2ag}{f}=b$. Unde secundus fit $\frac{\lambda naff-4\lambda nage}{f}$, adeoq; summa quatuor Ordinatarum est $\frac{2\lambda naff-4\lambda nage}{f}z^{\theta+n-1}R^{\lambda-1}$, & summa totidem respondentium Arearum est $az^\theta R^\lambda+\frac{2ag}{f}z^{\theta+n}R^\lambda-\theta aA+\frac{2\theta+2n+4\lambda n}{f}agB$. Dividantur hæ summæ per $\frac{2\lambda naff-4\lambda nage}{f}$, & si Quotum posterius dicatur D, erit D Area curvæ cujus Ordinata est Quotum prius $z^{\theta+n-1}R^{\lambda-1}$. Et eadem ratione ponendo omnes Ordinatæ terminos præter primum æquales nihilo potest Area Curvæ inveniri cujus Ordinata est $z^{\theta-1}R^{\lambda-1}$. Dicatur Area ista C, & qua ratione ex Areis A & B inventæ sunt Areæ C ac D, ex his Areis C ac D inveniri possunt aliæ duæ E & F Ordinatis $z^{\theta-1}R^{\lambda-2}$ et $z^{\theta+n-1}R^{\lambda-2}$ congruentes, & sic deinceps in infinitum. Et per Analysin contrariam regredi licet ab Areis E & F ad Areas C ac D, & inde ad Areas A & B, aliasque quæ in progressione sequuntur. Igitur si index λ perpetua unitatum additione vel subductione augeatur vel minuatur, & ex Areis quæ Ordinatis sic prodeuntibus respondent duæ simplicissimæ habentur; dantur aliæ omnes in infinitum. Q. E. O.

CAS. III.

Et per casus hosce duos conjunctos, si tam index θ perpetua additione vel subductione ipsius n, quam index λ perpetua additione vel subductione unitatis, utcunque augeatur vel minuatur, dabuntur Areæ singulis prodeuntibus Ordinatis respondentes. Q. E. O.

CAS. IV.

Et simili argumento si Ordinata constat ex quatuor nominibus in vinculo radicali & dantur tres Arearum, vel si constat ex quinque nominibus & dantur quatuor Arearum, & sic deinceps: dabuntur Areæ omnes quæ ad-

dendo

dendo vel ſubducendo numerum n indici θ vel unitatem indici λ generari poſſunt. Et par eſt ratio Curvarum ubi Ordinatæ ex binomiis conſtantur, & Area una earum quæ non ſunt Geometrice quadrabiles datur. Q. E. O.

PROP. VIII. THEOR. VI.

Si pro $e + fz^n + gz^{2n} +$ &c. et $k + lz^n + mz^{2n} +$ &c. ſcribantur R & S ut ſupra, & in Curvæ alicujus Ordinata $z^{\theta \pm n\sigma} R^{\lambda \pm \tau} S^{\mu \pm \upsilon}$ maneant quantitates datæ θ, n, λ, μ, e, f, g, k, l, m, &c. et pro σ, τ, & υ, ſcribantur ſucceſſive numeri quicunque integri: & ſi dentur Areæ duarum ex curvis quæ per Ordinatas ſic prodeuntes deſignantur ſi quantitates R & S ſunt binomia, vel ſi dentur Areæ trium ex curvis ſi R & S conjunctim ex quinque nominibus conſtant, vel Areæ quatuor ex curvis ſi R & S conjunctim ex ſex nominibus conſtant, & ſic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Demonſtratur ad modum propoſitionis ſuperioris.

PROP. IX. THEOR. VII.

Æquantur Curvarum Areæ inter ſe quarum Ordinatæ ſunt reciproce ut fluxiones Abſciſſarum.

Nam contenta ſub Ordinatis & fluxionibus Abſciſſarum erunt æqualia, & fluxiones Arearum ſunt ut hæc contenta.

COROL. I.

Si aſſumatur relatio quævis inter Abſciſſas duarum Curvarum, & inde per Prop. I. quæratur relatio fluxionum Abſciſſarum, & ponantur Ordinatæ reciproce proportionales fluxionibus, inveniri poſſunt innumeræ Curvæ quarum Areæ ſibi mutuo æquales erunt.

COROL. II.

Si enim Curva omnis cujus hæc eſt Ordinata $z^{\theta-1} \times \overline{e + fz^n + gz^{2n} + \&c.}|^{\lambda}$, aſſumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu} = s$, & $z^s = x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $\frac{\nu}{n} x^{\frac{\nu\theta - n}{n}} \times \overline{e + fx^{\nu} + gx^{2\nu} + \&c.}|^{\lambda}$.

COROL.

COROL. III.

Et Curva omnis cujus Ordinata est $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}$, assumendo quantitatem quamvis pro ν & ponendo $\frac{n}{\nu}=s$, & $z^{s}=x$, migrat in aliam sibi æqualem cujus Ordinata est $\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$.

COROL. IV.

Et Curva omnis cujus Ordinata est $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}\times\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu}$, assumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu}=s$, & $z^{s}=x$, migrat in aliam sibi æqualem cujus Ordinata est

$$\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}\times\overline{k+lx^{\nu}+mx^{2\nu}+\&c.}|^{\mu}$$

COROL. V.

Et Curva omnis cujus Ordinata est $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}$, ponendo $\frac{1}{z}=x$, migrat in aliam sibi æqualem cujus Ordinata est $\frac{1}{x^{\theta+1}}\times\overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}$ id est $\frac{1}{x^{\theta+1+n\lambda}}\times\overline{f+ex^{n}}|^{\lambda}$ si duo sunt nomina in vinculo radicis, vel $\frac{1}{x^{\theta+1+2n\lambda}}\times\overline{g+fx^{n}+ex^{2n}}|^{\lambda}$ si tria sunt nomina; & sic deinceps.

COROL. VI.

Et Curva omnis cujus Ordinata est $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}\times\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu}$, ponendo $\frac{1}{z}=x$, migrat in aliam sibi æqualem cujus Ordinata est

$$\frac{1}{x^{\theta+1}}\times\overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}\times\overline{k+lx^{-n}+mx^{-2n}+\&c.}|^{\mu}$$

id est $\frac{1}{x^{\theta+1+n\lambda+n\mu}}\times\overline{f+ex^{n}}|^{\lambda}\times\overline{l+kx^{n}}|^{\mu}$ si bina sunt nomina in vinculis radicum, vel $\frac{1}{x^{\theta+1+2n\lambda+n\mu}}\times\overline{g+fx^{n}+ex^{2n}}|^{\lambda}\times\overline{l+kx^{n}}|^{\mu}$ si tria sunt nomina in vinculo radicis prioris ac duo in vinculo posterioris: & sic in aliis.

Et nota quod Areæ duæ æquales in noviſſimis hiſce duobus Corollariis jacent ad contrarias partes Ordinatarum. Si Area in alterutra Curva adjacet Abſciſſæ, Area huic æqualis in altera Curva adjacet Abſciſſæ productæ.

COROL. VII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definiatur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha}\times\overline{e+fy^{n}z^{\delta}+gy^{2n}z^{2\delta}+hy^{3n}z^{3\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{n}z^{\delta}+my^{2n}z^{2\delta}+\&c.}$$

hæc Figura aſſumendo $s=\frac{n-\delta}{n}$, $x=\frac{1}{s}z^{s}$, & $\lambda=\frac{n-\delta}{\alpha\delta-\beta n}$, migrat in aliam ſibi æqualem cujus Abſciſſa x, ex data Ordinata v, determinatur per æquationem non affectam $\frac{1}{s}v^{\alpha\lambda}\times\overline{e+fv^{n}+gv^{2n}+\&c.}|^{\lambda}\times\overline{k+lv^{n}+mv^{2n}+\&c.}|^{-\lambda}=x$.

COROL. VIII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definitur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha}\times\overline{e+fy^{n}z^{\delta}+gy^{2n}z^{2\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{n}z^{\delta}+my^{2n}z^{2\delta}+\&c.}$$
$$+z^{\gamma}\times\overline{p+qy^{n}z^{\delta}+ry^{2n}z^{2\delta}+\&c.}$$

hæc Figura aſſumendo $s=\frac{n-\delta}{n}$, $x=\frac{1}{s}z^{s}$, $\mu=\frac{\alpha\delta+\beta n}{n-\delta}$, & $\nu=\frac{\alpha\delta+\gamma n}{n-\delta}$, migrat in aliam ſibi æqualem cujus Abſciſſa x ex data Ordinata v determinatur per æquationem minus affectam $v^{\alpha}\times\overline{e+fv^{n}+gv^{2n}+\&c.}=s^{\mu}x^{\mu}\times\overline{k+lv^{n}+mv^{2n}+\&c.}$

$$+s^{\nu}x^{\nu}\times\overline{p+qv^{n}+rv^{2n}+\&c.}$$

COROL. IX.

Curva omnis cujus Ordinata eſt

$$\frac{\pi z^{\theta-1}\times\overline{\nu e+\overline{\nu+n}fz^{n}+\overline{\nu+2n}gz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}}{}\ \text{in}$$
$$\overline{a+b\times\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\tau}}|^{\omega},$$

ſi ſit $\theta=\lambda\nu$, & aſſumantur $x=\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}|^{\pi}$, $\sigma=\frac{\tau}{\pi}$, & $\vartheta=\frac{\lambda-\pi}{\pi}$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Et nota quod Ordinata prior in hoc Corollario evadit ſimplicior ponendo $\lambda=1$, vel ponendo $\tau=1$, & efficiendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel etiam ponendo $\omega=-1$, & $\lambda=1=\tau=\sigma=\pi$, ut alios caſus prætercam.

COROL.

COROL. X.

Pro $\overline{ez^{\nu}+fz^{\nu+n}+gz^{\nu+2n}+\&c.}$ $\overline{\nu ez^{\nu-1}+\overline{\nu+n}fz^{\nu+n-1}+\overline{\nu+2n}gz^{\nu+2n-1}+\&c.}$ $\overline{k+lz^{n}+mz^{2n}+\&c.}$ et $\overline{nlz^{n-1}+2nmz^{2n-1}+\&c.}$ ſcribantur R, $\dot{r}$, S & $\dot{s}$ reſpective, & Curva omnis cujus Ordinata eſt $\overline{\pi S\dot{r}+\phi R\dot{s}}\times R^{\lambda-1}S^{\mu-1}\times\overline{aS^{\upsilon}+bR^{\tau}}|^{\omega}$, ſi ſit $\frac{\mu-\upsilon\omega}{\lambda}=\frac{\upsilon}{\tau}=\frac{\phi}{\pi}$, $\frac{\tau}{\pi}=\sigma$, $\frac{\lambda-\pi}{\pi}=\vartheta$, & $R^{\pi}S^{\phi}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Et nota quod Ordinata prior evadit ſimplicior, ponendo unitates pro τ, υ, & λ vel μ, & faciendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel ponendo $\omega=-1$ vel $\mu=0$.

PROP. X. PROB. III.

Invenire Figuras ſimpliciſſimas cum quibus Curva quævis Geometrice comparari poteſt, cujus Ordinatim applicata y *per æquationem non affectam ex data Abſciſſa* z *determinatur.*

CAS. I.

Sit Ordinata $az^{\theta-1}$, & Area erit $\frac{1}{\theta}az^{\theta}$, ut ex Prop. 5. ponendo $b=0=c=d=f=g=h$, & $e=1$, facile colligitur.

CAS. II.

Sit Ordinata $az^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, et ſi Curva cum Figuris rectilineis Geometrice comparari poteſt, quadrabitur per Prop. 5. ponendo $b=0=c=d$. Sin minus convertetur in aliam Curvam ſibi æqualem cujus Ordinata eſt $\frac{a}{n}x^{\frac{\theta-n}{n}}\times\overline{e+fx+gx^{2}+\&c.}|^{\lambda-1}$ per Corol. 2, Prop. 9. Deinde ſi de dignitatum indicibus $\frac{\theta-n}{n}$ & $\lambda-1$ per Prop. 7. rejiciantur unitates

unitates donec dignitates illæ fiant quam minimæ, devenietur ad Figuras ſimpliciſſimas quæ hac ratione colligi poſſunt. Dein harum unaquæque per Corol. 5, Prop. 9. dat aliam quæ nonnunquam ſimplicior eſt. Et ex his per Prop. 3. & Corol. 9, & 10, Prop. 9. inter ſe collatis, Figuræ adhuc ſimpliciores quandoque prodeunt. Denique ex Figuris ſimpliciſſimis aſſumptis facto regreſſu computabitur Area quæſita.

CAS. III.

Sit Ordinata $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, & hæc Figura ſi quadrari poteſt, quadrabitur per Prop. 5. Sin minus, diſtinguenda eſt Ordinata in partes $z^{\theta-1}\times a\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, $z^{\theta-1}\times bz^{n}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$, &c. et per Caſ. 2. inveniendæ ſunt Figuræ ſimpliciſſimæ cum quibus Figuræ partibus illis reſpondentes comparari poſſunt.

Nam Areæ figurarum partibus illis reſpondentium ſub ſignis ſuis + & — conjunctæ component Aream totam quæſitam.

CAS. IV.

Sit Ordinata $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda-1}$ in $\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu-1}$; et ſi Curva quadrari poteſt, quadrabitur per Prop. 6. Sin minus, convertetur in ſimpliciorem per Corol. 4, Prop. 9. ac inde comparabitur cum Figuris ſimpliciſſimis per Prop. 8. et Corol. 6, 9 & 10, Prop 9. ut fit in Caſu 2 & 3.

CAS. V.

Si Ordinata ex variis partibus conſtat, partes ſingulæ pro Ordinatis curvarum totidem habendæ ſunt, & Curvæ illæ quotquot quadrari poſſunt, ſigillatim quadrandæ ſunt, earumque Ordinatæ de Ordinata tota demendæ.

Dein

Dein Curva quam Ordinatæ pars residua designat seorsim (ut in Casu 2, 3 & 4.) cum Figuris simplicissimis comparanda est cum quibus comparari potest. Et summa Arearum omnium pro Area Curvæ propositæ habenda est.

COROL. I.

Hinc etiam Curva omnis cujus Ordinata est radix quadratica affecta æquationis suæ, cum Figuris simplicissimis seu rectilineis seu curvilineis comparari potest. Nam radix illa ex duabus partibus semper constat quæ seorsim spectatæ non sunt æquationum radices affectæ.

Proponatur æquatio $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$, & extracta radix erit $y = \frac{a^3 + z^3 + a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$, cujus pars rationalis $\frac{a^3 + z^3}{aa + zz}$ & pars irrationalis $\frac{a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$ sunt Ordinatæ curvarum quæ per hanc Propositionem vel quadrari possunt vel cum Figuris simplicissimis comparari cum quibus collationem Geometricam admittunt.

COROL. II.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 7. Prop. 9. in æquationem non affectam migrat, vel quadratur per hanc Propositionem si quadrari potest, vel comparatur cum Figuris simplicissimis cum quibus comparari potest. Et hac ratione Curva omnis quadratur cujus æquatio est trium terminorum. Nam æquatio illa si affecta sit, transmutatur in non affectam per Corol. 7. Prop. 9. ac deinde per Corol. 2. & 5. Prop. 9. in simplicissimam migrando, dat vel quadraturam Figuræ si quadrari potest, vel Curvam simplicissimam quacum comparatur.

COROL. III.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 8. Prop. 9. in æquationem quadraticam affectam migrat; vel quadratur per hanc Propositionem & hujus Corol. 1. si quadrari potest, vel comparatur cum Figuris simplicissimis cum quibus collationem Geometricam admittit.

SCHOLIUM.

Ubi quadrandæ ſunt Figuræ ; ad Regulas haſce generales ſemper recurrere nimis moleſtum eſſet : præſtat Figuras quæ ſimpliciores ſunt & magis uſui eſſe poſſunt ſemel quadrare & quadraturas in Tabulam referre, deinde Tabulam conſulere quoties ejuſmodi Curvam aliquam quadrare oportet. Hujus autem generis ſunt Tabulæ duæ ſequentes, in quibus z denotat Abſciſſam, y Ordinatam rectangulam, & t Aream Curvæ quadrandæ, & d, e, f, g, h, n ſunt quantitates datæ cum ſignis ſuis $+$ & $-$.

TABULA

CURVARUM SIMPLICIORUM QUÆ QUADRARI POSSUNT

CURVARUM FORMÆ			CURVARUM AREÆ
I		$dz^{\eta-1} = y$	$\frac{d}{\eta}z^{\eta} = t$
II		$\frac{dz^{\eta-1}}{e^2+2efz^{\eta}+f^2z^{2\eta}} = y$	$\frac{dz^{\eta}}{\eta e^2+\eta efz^{\eta}} = t$, vel $\frac{-d}{\eta ef+\eta f^2z^{\eta}} = t$
III	1	$dz^{\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{2d}{3\eta f}R^3 = t$, exiſtente $R = \sqrt{e+fz^{\eta}}$
	2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{-4e+6fz^{\eta}}{15\eta f^2}dR^3 = t$
	3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{16e^2-24efz^{\eta}+30f^2z^{2\eta}}{105\eta f^3}dR^3 = t$
	4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}} = y$	$\frac{-96e^3+144e^2fz^{\eta}-180ef^2z^{2\eta}+210f^3z^{3\eta}}{945\eta f^4}dR^3 = t$
IV	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{2d}{\eta f}R = t$
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{-4e+2fz^{\eta}}{3\eta f^2}dR = t$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{16e^2-8efz^{\eta}+6f^2z^{2\eta}}{15\eta f^3}dR = t$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}} = y$	$\frac{-96e^3+48e^2fz^{\eta}-36ef^2z^{2\eta}+30f^3z^{3\eta}}{105\eta f^4}dR = t$

John Senex ſculpſ.

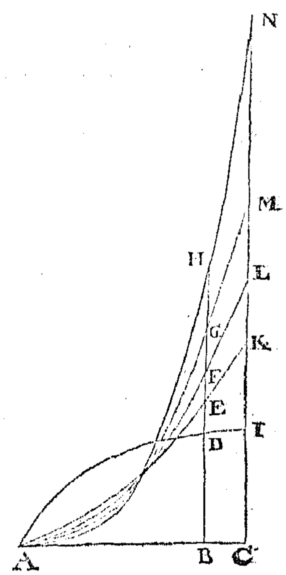
In Tabulis hisce, series Curvarum cujusque formæ utrinque in infinitum continuari potest. Scilicet in Tabula prima, in numeratoribus Arearum formæ tertiæ & quartæ, numeri coefficientes initialium terminorum (2, —4, 16, —96, 868, &c.) generantur multiplicando numeros — 2, —4, —6, —8, —10, &c. in se continuo, & subsequentium terminorum coefficientes ex initialibus derivantur multiplicando ipsos gradatim, in Forma quidem tertia, per $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. in quarta vero per $-\frac{1}{2}$ $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c. Et Denominatorum coefficientes 3, 15, 105, &c. prodeunt multiplicando numeros 1, 3, 5, 7, 9, &c. in se continue.

In secunda vero Tabula, series Curvarum formæ primæ, secundæ, quintæ, sextæ, nonæ, & decimæ ope solius divisionis, & formæ reliquæ ope Propositionis tertiæ & quartæ, utrinque producuntur in infinitum.

Quinetiam hæ series mutando signum numeri η variari solent. Sic enim *e. g.* Curva $\frac{d}{z}\sqrt{e+fz^{\eta}} = y$, evadit $\frac{d}{z^{\frac{1}{2}\eta+1}}\sqrt{f+ez^{\eta}} = y$.

PROP. XI. THEOR. VIII.

Sit ADIC Curva quævis Abscissam habens AB $= z$, & Ordinatam BD $= y$, & sit AEKC Curva alia cujus Ordinata BE æqualis est prioris areæ ADB ad unitatem applicatæ, & AFLC Curva tertia cujus Ordinata BF æqualis est secundæ Areæ AEB ad unitatem applicatæ, & AGMC Curva quarta cujus Ordinata BG æqualis est tertiæ areæ AFB ad unitatem applicatæ, & AHNC Curva quinta cujus Ordinata BH æqualis est quartæ Areæ AGB ad unitatem applicatæ, & sic deinceps in infinitum. Et sunto A, B, C, D, E, &c. Areæ Curvarum Ordinatas habentium y, zy, z^2y, z^3y, z^4y, &c. et Abscissam communem z.



Detur Abscissa quævis AC $= t$, sitque BC $= t - z = x$, & sunto P, Q, R, S, T Areæ Curvarum Ordinatas habentium y, xy, x^2y, x^3y, x^4y &c. et Abscissam communem x.

Terminentur autem hæ Areæ omnes ad Abscissam totam datam AC, nec non ad Ordinatam positione datam & infinite productam CI:

Et erit Arearum sub initio positarum

Prima ADIC $= A = P$,

Secunda AEKC $= tA - B = Q$,

Tertia AFLC $= \frac{t^2A - 2tB + C}{2} = \frac{1}{2}R$.

Quarta AGMC $= \frac{t^3A - 3t^2B + 3tC - D}{6} = \frac{1}{6}S$.

Quinta AHNC $= \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24} = \frac{1}{24}T$.

COROL.

COROL.

Unde si Curvæ quarum Ordinatæ sunt y, zy, z^2y, z^3y, &c. vel y, xy, x^2y, x^3y, &c. quadrari possunt, quadrabuntur etiam Curvæ ADIC, AEKC, AFLC, AGMC, &c. et habebuntur Ordinatæ BE, BF, BG, BH, Areis Curvarum proportionales.

SCHOLIUM.

Quantitatum fluentium Fluxiones esse primas, secundas, tertias, quartas, aliasque diximus supra. Hæ Fluxiones sunt ut termini serierum infinitarum convergentium.

Ut si z^n sit quantitas fluens & fluendo evadat $\overline{z+o}|^n$, deinde resolvatur in seriem convergentem $z^n + noz^{n-1} + \frac{nn-n}{2}ooz^{n-2} + \frac{n^3-3nn+2n}{6}o^3z^{n-3} +$ &c. terminus primus hujus seriei z^n erit quantitas illa fluens, secundus noz^{n-1} erit ejus incrementum primum seu differentia prima cui nascenti proportionalis est ejus Fluxio prima, tertius $\frac{nn-n}{2}ooz^{n-2}$ erit ejus incrementum secundum seu differentia secunda cui nascenti proportionalis est ejus Fluxio secunda, quartus $\frac{n^3-3nn+2n}{6}o^3z^{n-3}$ erit ejus incrementum tertium seu differentia tertia cui nascenti Fluxio tertia proportionalis est, & sic deinceps in infinitum.

Exponi autem possunt hæ Fluxiones per Curvarum Ordinatas BD, BE, BF, BG, BH, &c.

Ut si Ordinata BE ($= \frac{ADB}{1}$) sit quantitas fluens, erit ejus Fluxio prima ut Ordinata BD.

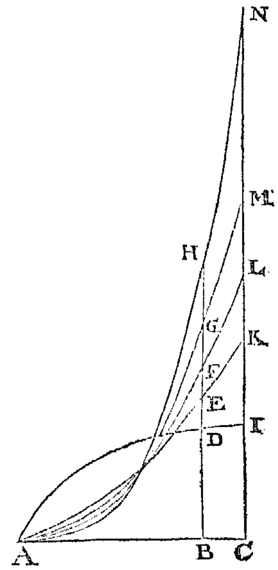
Si BF ($= \frac{AEB}{1}$) sit quantitas fluens, erit ejus Fluxio prima ut Ordinata BE, & Fluxio secunda ut Ordinata BD.

Si BH ($= \frac{AGB}{1}$) sit quantitas fluens, erunt ejus Fluxiones prima, secunda, tertia & quarta, ut Ordinatæ BG, BF, BE, BD respective.

Et hinc in æquationibus quæ quantitates tantum duas incognitas involvunt, quarum una est quantitas uniformiter fluens & altera est Fluxio quælibet quantitatis alterius fluentis, inveniri potest fluens illa altera per quadraturam Curvarum. Exponatur enim Fluxio ejus per Ordinatam BD, & si hæc sit Fluxio prima, quæratur Area ADB $=$ BE $\times$ 1, si Fluxio secunda, quæratur Area AEB $=$ BF $\times$ 1, si Fluxio tertia, quæratur Area AFB $=$ BG $\times$ 1, &c. et Area inventa erit exponens fluentis quæsitæ.

Sed

Sed & in æquationibus quæ fluentem & ejus fluxionem primam sine altera fluente, vel duas ejusdem fluentis fluxiones, primam & secundam, vel secundam & tertiam, vel tertiam & quartam, &c. sine alterutra fluente involvunt: inveniri possunt fluentes per quadraturam Curvarum. Sit æquatio $aa\dot{v} = av + vv$, existente $v = BE$, $\dot{v} = BD$, $z = AB$, & $\dot{z} = 1$, & æquatio illa complendo dimensiones Fluxionum, evadet $aa\dot{v} = av\dot{z} + vv\dot{z}$, seu $\frac{aa\dot{v}}{av+vv} = \dot{z}$.



Jam fluat v uniformiter, & sit ejus Fluxio $\dot{v} = 1$, & erit $\frac{aa}{av+vv} = \dot{z}$, & quadrando Curvam cujus Ordinata est $\frac{aa}{av+vv}$ & Abscissa v, habebitur fluens z. Adhæc sit æquatio $aa\ddot{v} = a\dot{v} + \dot{v}\dot{v}$, existente $v = BF$, $\dot{v} = BE$, $\ddot{v} = BD$, & $z = AB$, & per relationem inter $\ddot{v}$ & $\dot{v}$ seu BD & BE invenietur relatio inter AB & BE, ut in exemplo superiore. Deinde per hanc relationem invenietur relatio inter AB & BF quadrando Curvam AEB.

Æquationes quæ tres incognitas quantitates involvunt aliquando reduci possunt ad æquationes quæ duas tantum involvunt, & in his casibus fluentes invenientur ex fluxionibus ut supra. Sit æquatio $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$, ponatur $y^n\dot{y} = \dot{v}$, & erit $a - bx^m = cx\dot{v} + d\dot{v}\dot{v}$. Hæc æquatio quadrando Curvam cujus Abscissa est x & Ordinata $\dot{v}$ dat Aream v; & æquatio altera $y^n\dot{y} = \dot{v}$, regrediendo ad fluentes, dat $\frac{1}{n+1}y^{n+1} = v$: Unde habetur fluens y.

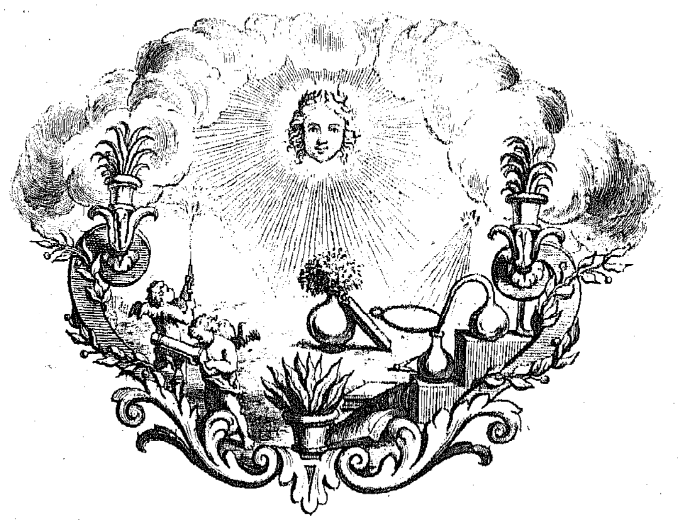
Quinetiam in æquationibus, quæ tres incognitas involvunt, & ad æquationes quæ duas tantum involvunt, reduci non possunt, fluentes quandoque prodeunt per quadraturam Curvarum. Sit æquatio $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, existente $\dot{x} = 1$; Et pars posterior $rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, regrediendo ad fluentes, fit $ex^ry^s - \frac{f}{t+1}y^{t+1}$, quæ proinde est ut Area Curvæ cujus Abscissa est x & Ordinata $\overline{ax^m + bx^n}|^p$, & inde datur fluens y.

Sit æquatio $\dot{x} \times \overline{ax^m + bx^n}|^p = \frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$, Et fluens cujus Fluxio eſt $\dot{x} \times \overline{ax^m + bx^n}|^p$ erit ut Area Curvæ cujus Abſciſſa eſt x & Ordinata eſt $\overline{ax^m + bx^n}|^p$. Item fluens cujus Fluxio eſt $\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ erit ut Area Curvæ cujus Abſciſſa eſt y & Ordinata $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$, id eſt (per Caſum 1. Formæ quartæ Tab. I.) ut Area $\frac{2d}{nf}\sqrt{e+fy^n}$. Pono ergo $\frac{2d}{nf}\sqrt{e+fy^n}$ æqualem Areæ Curvæ cujus Abſciſſa eſt x & Ordinata $\overline{ax^m + bx^n}|^p$, & habebitur fluens y.

Et nota quod fluens omnis, quæ ex Fluxione prima colligitur, augeri poteſt vel minui quantitate quavis non fluente. Quæ ex Fluxione ſecunda colligitur, augeri poteſt vel minui quantitate quavis cujus Fluxio ſecunda nulla eſt. Quæ ex fluxione tertia colligitur, augeri poteſt vel minui quantitate quavis cujus fluxio tertia nulla eſt. Et ſic deinceps in infinitum.

Poſtquam fluentes ex fluxionibus collectæ ſunt, ſi de veritate Concluſionis dubitatur, fluxiones fluentium inventarum viciſſim colligendæ ſunt, & cum Fluxionibus ſub initio propoſitis comparandæ. Nam ſi prodeunt æquales, Concluſio recte ſe habet: ſin minus, corrigendæ ſunt fluentes ſic, ut earum Fluxiones fluxionibus ſub initio propoſitis æquentur. Nam & fluens pro lubitu aſſumi poteſt, & aſſumptio corrigi, ponendo fluxionem fluentis aſſumptæ æqualem Fluxioni propoſitæ, & terminos homologos inter ſe comparando.

Et his principiis via ad majora ſternitur.



ENUMERATIO
LINEARUM
TERTII ORDINIS.

T A B[illegible]

CURVARUM SIMPLICIORUM QUÆ CU[illegible]

Sit jam aGD vel PGD vel GDS Sectio Conica cujus Area ad Qua[illegible]
Semiaxis conjugatus AP, datum Abſciſsæ principium A vel a vel [illegible]
vel aBDG vel αBDG $= s$, exiſtente αG Ordinata ad punctum α. Jung[illegible]
parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſitæ [illegible]
Ordinata v, et Area σ. Sit autem $\div$ differentia duarum quantitatum ubi [illegible]
Et in Forma ſexta ſcribatur p pro $\sqrt{ff-4eg}$.

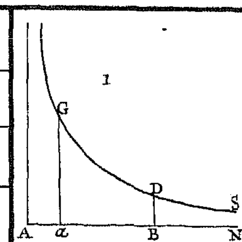
CURVARUM FORMÆ			SECTIONIS CONICÆ: Abſciſsa	SECTIONIS CONICÆ: Ordinata
I.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
II	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	3	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
III	1	$\frac{d}{z}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
IV	1	$\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$

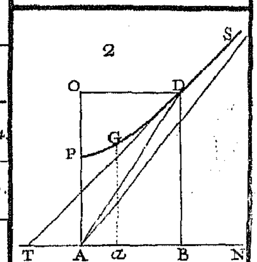
U L A

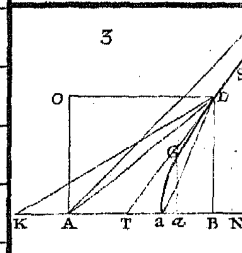
ELLIPSI ET HYPERBOLA COMPARARI POSSUNT.

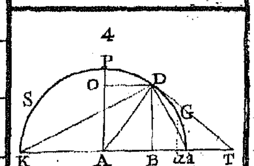
aturam Curvæ propositæ requiritur, sitq; ejus Centrum A, Axis Ka, Vertex a,
bscissa AB vel aB vel αB $= x$, Ordinata rectangula BD $= v$, et Area ABDP
ur KD, AD, aD; ducatur Tangens DT occurrens Abscissæ AB in T, & compleatur
uiruntur Areæ duarum Sectionum Conicarum, dicatur posterioris Abscissa ξ,
certum est utrum posterior de priori an prior de posteriori subduci debeat.

	Curvarum Areæ
	$\frac{1}{\eta}s = t = \frac{\alpha GDB}{\eta}$. *Fig. 1.*
	$\frac{d}{\eta f}z^{\eta} - \frac{e}{\eta f}s = t$.
	$\frac{d}{2\eta f}z^{2\eta} - \frac{de}{\eta f^2}z^{\eta} + \frac{e^2}{\eta f^2}s = t$.
	$\frac{2xv \div 4s}{\eta} = t = \frac{4}{\eta}$ADGa . *Fig. 3. 4.*
	$\frac{2d}{\eta f}z^{\frac{1}{2}\eta} + \frac{4es - 2exv}{\eta f} = t$.
	$\frac{2d}{3\eta f}z^{\frac{3}{2}\eta} - \frac{2de}{\eta f^2}z^{\frac{1}{2}\eta} + \frac{2e^2xv - 4e^2s}{\eta f^2} = t$.
	$\frac{4de}{\eta f} \times \overline{\frac{v^3}{2ex} - s} = t = \frac{4de}{\eta f}$ in aGDT, *vel in* APDB $\div$ TDB *Fig. 2. 3. 4.*
	$\frac{8de^2}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e} + \frac{f^2v}{4e^2x}} = t = \frac{8de^2}{\eta f^2}$ in aGDA $+ \frac{f^2v}{4e^2x}$ *Fig. 3. 4.*
	$-\frac{2d}{\eta}s = t = \frac{2d}{\eta}$APDB seu $\frac{2d}{\eta}$ aGDB . *Fig. 2. 3. 4.*
	$\frac{4de}{\eta f} \times \overline{s - \frac{1}{2}xv - \frac{fv}{2e}} = t = \frac{4de}{\eta f} \times$ aGDK . *Fig. 3. 4.*
	$-\frac{d}{\eta}s = t = \frac{d}{\eta} \times -$ aGDB *vel* BDPK . *Fig. 4.*
	$\frac{3dfs - 2dv^3}{6\eta e} = t$:
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in PAD *vel in* aGDA . *Fig. 2. 3. 4.*
	$\frac{8de}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e}} = t = \frac{8de}{\eta f^2}$ in aGDA . *Fig. 3. 4.*
	$\frac{2d}{\eta e} \times \overline{s - xv} = t = \frac{2d}{\eta e}$ in POD *vel in* AODGa . *Fig. 2. 3. 4.*
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in aDGa *Fig. 3. 4.*
	$\frac{d}{\eta e} \times \overline{3s \div 2xv} = t = \frac{d}{\eta e}$ in 3aDGa $\div \Delta$ aDB . *Fig. 3. 4.*
	$\frac{10dfxv - 15dfs - 2dex^2v}{6\eta e^2} = t$









Iohan. Senex sculp.

RESIDUUM TABULÆ CURVARUM SIMPLICIORU

CURVARUM FORMÆ			SECTIONIS CONICÆ Abſciſſa	Ordinata
V	1	$\frac{dz^{n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$
		vel sic	$\sqrt{\frac{dz^{2n}}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^2-4eg}{4e^2}x^2}=v$
	2	$\frac{dz^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$ $fz^n+gz^{2n}=\xi$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$ $\frac{1}{e+\xi}=\Upsilon$
VI	1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^n}}=x$ $\sqrt{\frac{2dg}{f+p+2gz^n}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^2}=v$ $\sqrt{d+\frac{-f-p}{2g}\xi^2}=\Upsilon$
	2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dez^n}{fz^n-pz^n+2e}}=x$ $\sqrt{\frac{2dez^n}{fz^n+pz^n+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^2}=v$ $\sqrt{d+\frac{-f-p}{2e}\xi^2}=\Upsilon$
VII	1	$\frac{d}{z}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$ $\frac{1}{z^n}=\xi$	$\sqrt{e+fx+gx^2}=v$ $\sqrt{g+f\xi+e\xi^2}=\Upsilon$
	2	$dz^{n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	3	$dz^{2n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	4	$dz^{3n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
VIII	1	$\frac{dz^{n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
IX	1	$\frac{dz^{n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
	2	$\frac{dz^{2n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
X	1	$\frac{dz^{n-1}}{g+hz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
	2	$\frac{dz^{2n-1}}{g+hz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
XI	1	$dz^{-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$ $\sqrt{h+gz^{-n}}=\xi$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$ $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^2}=\Upsilon$
	2	$dz^{n-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$
	3	$dz^{2n-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$

QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT

CURVARUM AREÆ

$$\frac{xv - 2s}{n} = t.$$

$$\frac{2s - xv}{n} = t.$$

$$\frac{d\sigma + 2fs - fxv}{2ng} = t.$$

$$\frac{2xv - 4s - 2\xi\tau + 4\sigma}{np} = t.$$

$$\frac{4s - 2xv - 4\sigma + 2\xi\tau}{np} = t.$$

$$\frac{4de^2\xi\tau + 2def\tau - 2dfgxv \begin{smallmatrix} +4deg \\ -2ffd \end{smallmatrix} v - 8de^2\sigma + 4dfgs}{4neg - nff} = t.$$

$$\frac{d}{n}s = t = \frac{d}{n} \times \alpha\mathrm{GDB}. \ \textit{Fig. 2.3.4.}$$

$$\frac{d}{3ng}v^3 - \frac{df}{2ng}s = t.$$

$$\frac{6dgx - 5df}{24ng^2}v^3 + \frac{5df^2 - 4deg}{16ng^2}s = t.$$

$$\frac{8dgs - 4dgxv - 2dfv}{4neg - nf^2} = t = \frac{8dg}{4neg - nf^2} \times \alpha\mathrm{GDB} \pm \Delta\mathrm{DBA}. \ \textit{Fig. 2.4.}$$

$$\frac{-4dfs + 2dfxv + 4dev}{4neg - nf^2} = t.$$

$$\frac{\begin{smallmatrix} 3dff \\ -4deg \end{smallmatrix} s \begin{smallmatrix} -2dff \\ +4deg \end{smallmatrix} xv - 2defv}{4neg^2 - nf^2g} = t.$$

$$\frac{\begin{smallmatrix} 36defg \\ -15df^3 \end{smallmatrix} s \begin{smallmatrix} +8degg \\ -2df^2g \end{smallmatrix} x^2v \begin{smallmatrix} +10dfff \\ -28defg \end{smallmatrix} xv \begin{smallmatrix} +10deff \\ -16de^2g \end{smallmatrix} v}{24neg^3 - 6nf^2g^2} = t.$$

$$\frac{\begin{smallmatrix} 4fg \\ -4eh \end{smallmatrix} s \begin{smallmatrix} -2fg \\ +2eh \end{smallmatrix} xv + 2df\frac{v}{x}}{nfh} = t.$$

$$\frac{\begin{smallmatrix} 4egh \\ -4fgg \end{smallmatrix} s \begin{smallmatrix} -2egh \\ +2fgg \end{smallmatrix} xv + \frac{2}{3}dh\frac{v^3}{x^3} - 2dfg\frac{v}{x}}{nfh^2} = t.$$

$$\frac{2xv - 4s}{nf} = t = \frac{4}{nf}\ \mathrm{ADGa}. \ \textit{Fig. 3.4.}$$

$$\frac{4gs - 2gxv + 2d\frac{v}{x}}{nfh} = t.$$

$$\frac{2dxv^3z^{-n} - 4dfs - 4de\sigma}{nfg - neh} = t.$$

$$\frac{2d}{nh}s = t.$$

$$\frac{dhxv^3 \begin{smallmatrix} -3dfg \\ -deh \end{smallmatrix} s}{2nfh^2} = t.$$

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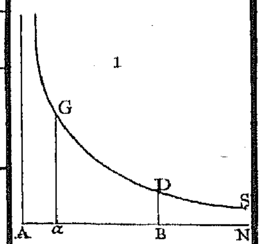

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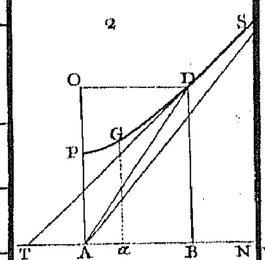

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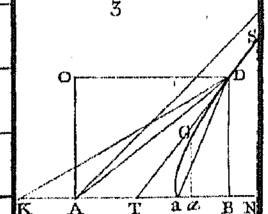

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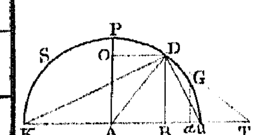


Iohan. Senex sculp.

T A B[...]

CURVARUM SIMPLICIORUM QUÆ CU[...]

Sit jam aGD vel PGD vel GDS Sectio Conica cujus Area ad Qua[...] Semiaxis conjugatus AP, datum Abſciſsæ principium A vel a vel [...] vel aBDG vel αBDG $= s$, exiſtente αG Ordinata ad punctum α. Jung[...] parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſita[...] Ordinata r, et Area σ. Sit autem $\div$ differentia duarum quantitatum ub[...] Et in Forma ſexta ſcribatur p pro $\sqrt{ff-4eg}$.

CURVARUM FORMÆ			SECTIONIS CONICÆ: Abſciſsa	SECTIONIS CONICÆ: Ordinata
I.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
II.	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	3	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
III.	1	$\frac{d}{z}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
IV.	1	$\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$

QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT

CURVARUM AREÆ

$$\frac{xv-2s}{\eta}=t.$$

$$\frac{2s-xv}{\eta}=t.$$

$$\frac{d\sigma+2fr-fxv}{2\eta g}=t.$$

$$\frac{2xv-4s-2\xi r+4\sigma}{\eta p}=t.$$

$$\frac{4s-2xv-4\sigma+2\xi r}{\eta p}=t.$$

$$\frac{4de^2\xi r+2defr-2dfgxv\substack{+4deg\\-2ff}dv-8de^2\sigma+4dfgs}{4neg-\eta ff}=t.$$

$$\frac{d}{\eta}s=t=\frac{d}{\eta}\times\alpha\mathrm{GDB}.\ \textit{Fig. 2.3.4.}$$

$$\frac{d}{3\eta g}v^3-\frac{df}{2\eta g}s=t.$$

$$\frac{6dgx-5df}{24\eta g^2}v^3+\frac{5df^2-4deg}{16ng^2}s=t.$$

$$\frac{8dgs-4dgxv-2dfv}{4neg-\eta f^2}=t=\frac{8dg}{4neg-\eta f^2}\times\alpha\mathrm{GDB}\pm\Delta\mathrm{DBA}.\ \textit{Fig. 2.4.}$$

$$-\frac{4dfs+2dfxv+4dev}{4neg-\eta f^2}=t.$$

$$\frac{\substack{3dff\\-4deg}s\substack{-2dff\\+4deg}xv-2defv}{4neg^2-\eta f^2g}=t.$$

$$\frac{\substack{36defg\\-15df^3}s\substack{+8degg\\-2df^2g}x^2v\substack{+10dfff\\-28defg}xv\substack{+10deff\\-16de^2g}v}{24neg^3-6\eta f^2g^2}=t.$$

$$\frac{\substack{4fg\\-4eh}s\substack{-2fg\\+2eh}xv+2df\frac{v}{x}}{\eta fh}=t.$$

$$\frac{\substack{4egh\\-4fgg}s\substack{-2egh\\+2fgg}xv+\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{\eta fh^2}=t.$$

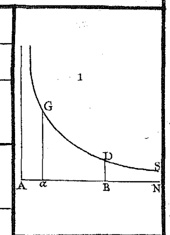
$$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}\,\mathrm{ADGa}.\ \textit{Fig. 3.4.}$$

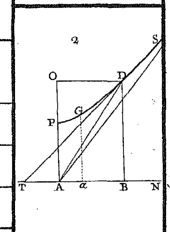
$$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t.$$

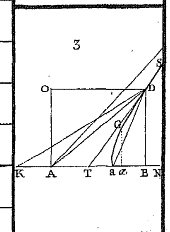
$$\frac{2dxv^3z^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}=t.$$

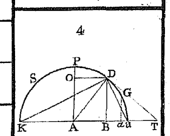
$$\frac{2d}{\eta h}s=t.$$

$$\frac{dhxv^3\substack{-3dfg\\-deh}s}{2\eta fh^2}=t.$$









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ENUMERATIO
LINEARUM
TERTII ORDINIS.

I. *Linearum Ordines.*

Lineæ Geometricæ secundum numerum dimensionum æquationis qua relatio inter Ordinatas & Abscissas definitur, vel (quod perinde est) secundum numerum punctorum in quibus a linea recta secari possunt, optime distinguuntur in Ordines. Qua ratione linea primi Ordinis erit Recta sola, eæ secundi sive quadratici Ordinis erunt sectiones Conicæ & Circulus, & eæ tertii sive cubici Ordinis Parabola Cubica, Parabola *Neiliana*, Cissois veterum, & reliquæ quas hic enumerare suscepimus. Curva autem primi Generis, (siquidem recta inter Curvas non est numeranda) eadem est cum Linea secundi Ordinis, & Curva secundi Generis eadem cum Linea Ordinis tertii. Et Linea Ordinis infinitesimi ea est quam recta in punctis infinitis secare potest, qualis est Spiralis, Cycloïs, Quadratrix, & linea omnis quæ per radii vel rotæ revolutiones infinitas generatur.

II. *Proprietates Sectionum Conicarum competunt Curvis superiorum Generum.*

Sectionum Conicarum proprietates præcipuæ a Geometris passim traduntur. Et consimiles sunt proprietates Curvarum secundi Generis, & reliquarum, ut ex sequenti proprietatum præcipuarum enumeratione constabit.

1. *De Curvarum secundi generis Ordinatis, Diametris, Verticibus, Centris, Axibus.*

Si rectæ plures parallelæ & ad Conicam sectionem utrinque terminatæ ducantur, recta duas earum bisecans bisecabit alias omnes, ideoque dicitur *Diameter* figuræ & rectæ bisectæ dicuntur *Ordinatim applicatæ* ad Diametrum, & concursus omnium Diametrorum est *Centrum* figuræ, & intersectio Curvæ & diametri *Vertex* nominatur, & diameter illa *Axis* est cui Ordinatim applicatæ insistunt ad angulos rectos. Et ad eundem modum in Curvis secundi Generis, si rectæ duæ quævis parallelæ ducantur occurrentes Curvæ in tribus punctis : recta quæ ita secat has parallelas ut summa duarum partium ex uno secantis latere ad Curvam terminatarum æquetur parti tertiæ ex altero latere ad curvam terminatæ, eodem modo secabit omnes alias his parallelas curvæque in tribus punctis occurrentes rectas, hoc est, ita ut summa partium duarum ex uno ipsius latere semper æquetur parti tertiæ ex altero latere. Has itaque tres partes quæ hinc inde æquantur, *Ordinatim applicatas*, & rectam secantem cui Ordinatim applicantur *Diametrum*, & intersectionem diametri & Curvæ *Verticem*, & concursum duarum Diametrorum *Centrum* nominare licet. Diameter autem ad Ordinatas rectangula si modo aliqua sit, etiam *Axis* dici potest, & ubi omnes Diametri in eodem puncto concurrunt, istud erit *Centrum generale*.

2. *De Asymptotis & earum proprietatibus.*

Hyperbola primi generis duas *Asymptotos*, ea secundi tres, ea tertii quatuor & non plures habere potest, & sic in reliquis. Et quemadmodum partes lineæ cujusvis rectæ inter Hyperbolam Conicam & duas ejus Asymptotos sunt hinc inde æquales : sic in Hyperbolis secundi Generis si ducatur

ducatur recta quævis secans tam Curvam quam tres ejus Asymptotos in tribus punctis, summa duarum partium istius rectæ quæ a duobus quibusvis Asymptotis in eandem plagam ad duo puncta Curvæ extenduntur, æqualis erit parti tertiæ quæ a tertia Asymptoto in plagam contrariam ad tertium Curvæ punctum extenditur.

3. *De Lateribus rectis & transversis.*

Et quemadmodum in Conicis sectionibus non Parabolicis quadratum Ordinatim applicatæ, hoc est, rectangulum Ordinatarum quæ ad contrarias partes Diametri ducuntur, est ad rectangulum partium Diametri quæ ad Vertices Ellipseos vel Hyperbolæ terminantur, ut data quædam linea quæ dicitur *Latus rectum*, ad partem Diametri quæ inter Vertices jacet & dicitur *Latus transversum*: sic in Curvis non Parabolicis secundi Generis parallelepipedum sub tribus Ordinatim applicatis est ad parallelepipedum sub partibus Diametri ad Ordinatas & tres Vertices figuræ abscissis, in ratione quadam data: in qua ratione si sumantur tres rectæ ad tres partes diametri inter vertices figuræ sitas, singulæ ad singulas, tunc illæ tres rectæ dici possunt *Latera recta* figuræ, & illæ partes Diametri inter Vertices *Latera transversa*. Et sicut in Parabola Conica quæ ad unam & eandem diametrum unicum tantum habet Verticem, rectangulum sub Ordinatis æquatur rectangulo sub parte Diametri quæ ad Ordinatas & Verticem abscinditur & recta quadam data quæ *Latus rectum* dicitur: sic in Curvis secundi Generis quæ non nisi duos habent Vertices ad eandem Diametrum, parallelepipedum sub Ordinatis tribus æquatur parallelepipedo sub duabus partibus Diametri ad Ordinatas & Vertices illos duos abscissis, & recta quadam data quæ proinde *Latus rectum* dici potest.

4. *De Ratione contentorum sub Parallelarum segmentis.*

Denique sicut in Conicis sectionibus ubi duæ parallelæ ad Curvam utrinque terminatæ secantur a duabus parallelis ad Curvam utrinque terminatis, prima a tertia, & secunda a quarta, rectangulum partium primæ est ad rectangulum partium tertiæ ut rectangulum partium secundæ ad rectangulum partium quartæ: sic ubi quatuor tales rectæ occurrunt Curvæ secundi Generis, singulæ in tribus punctis, parallelepipedum partium primæ rectæ erit ad parallelepipedum partium tertiæ, ut parallelepipedum partium secundæ ad parallelepipedum partium quartæ.

5. De

5. *De Cruribus Hyperbolicis & Parabolicis & eorum plagis.*

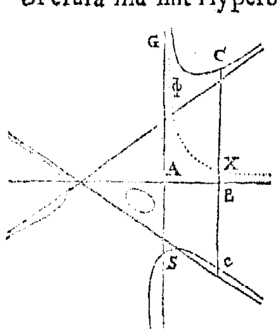
Curvarum secundi & superiorum Generum æque atque primi crura omnia in infinitum progredientia vel *Hyperbolici* sunt generis vel *Parabolici*. Crus *Hyperbolicum* voco quod ad Asymptoton aliquam in infinitum appropinquat; *Parabolicam* quod Asymptoto destituitur. Hæc crura ex Tangentibus optime dignoscuntur. Nam si punctum contactus in infinitum abeat, Tangens cruris Hyperbolici cum Asymptoto coincidet, & Tangens cruris Parabolici in infinitum recedet, evanescet & nullibi reperietur. Invenitur igitur Asymptotos cruris cujusvis quærendo Tangentem cruris illius ad punctum infinite distans. Plaga autem cruris infiniti invenitur quærendo positionem rectæ cujusvis quæ Tangenti parallela est ubi punctum contactus in infinitum abit. Nam hæc recta in eandem plagam cum crure infinito dirigitur.

III. *Reductio Curvarum omnium Generis Secundi ad æquationum casus quatuor.*

Lineæ omnes Ordinis primi, tertii, quinti, septimi & imparis cujusque duo habent ad minimum crura in infinitum versus plagas oppositas progredientia. Et lineæ omnes tertii Ordinis duo habent ejusmodi crura in plagas oppositas progredientia in quas nulla alia earum crura infinita (præterquam in Parabola *Cartesiana*) tendunt.

CAS. I.

Si crura illa sint Hyperbolici generis, sit GAS eorum Asymptotos, & huic parallela agatur recta quævis CBc ad Curvam utrinque (si fieri potest) terminata, eademque bisecetur in puncto X, & locus puncti illius X erit Hyperbola Conica (puta $X\Phi$) cujus una Asymptotos est AG. Sit ejus altera Asymptotos AB, & æquatio qua relatio inter Ordinatam BC & Abscissam AB definitur, si AB dicatur x, & BC y, semper induet hanc formam $xyy + ey = ax^3 + bxx + cx + d$. Ubi termini, e, a, b, c, d, designant quantitates datas signis suis $+$ & $-$ affectas, quarum quælibet deesse possunt modo ex earum defectu figura in sectionem Conicam non vertatur. Potest autem Hyperbola illa Conica cum Asymptotis suis coincidere, id est punctum X in recta AB locari: & tunc terminus $+ ey$ deest.

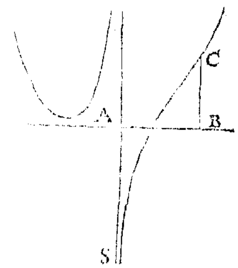


CAS.

CAS. II.

At ſi recta illa CBc non poteſt utrinque ad Curvam terminari, ſed Curvæ in unico tantum puncto occurrit: age quamvis poſitione datam rectam AB Aſymptoto AS occurrentem in A, ut & aliam quamvis BC Aſymptoto illi parallelam Curvæque occurrentem in puncto C, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

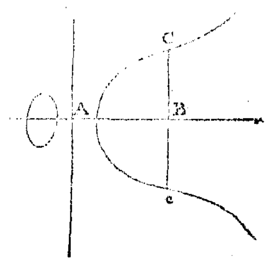
$$xy = ax^3 + bx^2 + cx + d.$$



CAS. III.

Quod ſi crura illa oppoſita Parabolici ſint generis, recta CBc ad Curvam utrinque, ſi fieri poteſt, terminata in plagam crurum ducatur & biſecetur in B, & locus puncti B erit Linea recta. Sit iſta AB, terminata ad datum quodvis punctum A, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

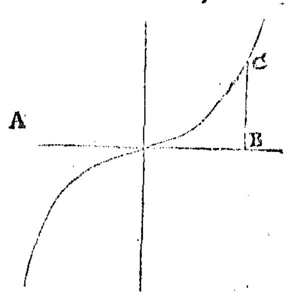
$$yy = ax^3 + bx^2 + cx + d.$$



CAS. IV.

At vero ſi recta illa CBc in unico tantum puncto occurrat Curvæ, ideoque ad Curvam utrinque terminari non poſſit: ſit punctum illud C, & incidat recta illa ad punctum B in rectam quamvis aliam poſitione datam & ad datum quodvis punctum A terminatam AB: & æquatio qua relatio inter Ordinatam BC & Abſciſſam AC definitur, ſemper induet hanc formam,

$$y = ax^3 + bx^2 + cx + d.$$



Nomina formarum.

Enumerando Curvas horum caſuum, Hyperbolam vocabimus, *Inſcriptam*, quæ tota jacet in Aſymptoton angulo ad inſtar Hyperbolæ conicæ; *Circumſcriptam*, quæ Aſymptotos ſecat & partes Abſciſſas in ſinu ſuo amplectitur; *Ambigenam*, quæ uno crure infinito inſcribitur & altero circumſcribitur; *Convergentem*, cujus crura concavitate ſua ſeinvicem reſpiciunt & in plagam eandem diriguntur; *Divergentem*, cujus crura convexitate ſua ſeinvicem reſpiciunt & in plagas contrarias diriguntur; *Cruribus Contrariis præditam*, cujus crura in partes contrarias convexa ſunt & in plagas contrarias infinita; *Conchoidalem*, quæ vertice concavo & cruribus divergentibus ad Aſymptoton applicatur; *Anguineam*, quæ flexibus contrariis Aſymptoton ſecat & utrinque in crura contraria producitur; *Cruciformem*, quæ conjugatam decuſſat; *Nodatam*, quæ ſeipſam decuſſat in orbem redeundo; *Cuſpidatam*, cujus partes duæ in angulo contactus concurrunt & ibi terminantur; *Punctatam*, quæ conjugatam habet Ovalem infinite parvam id eſt punctum; & *Puram*, quæ per impoſſibilitatem duarum radicum Ovali, Nodo, Cuſpide & Puncto conjugato privatur. Eodem ſenſu Parabolam quoque *convergentem*, *divergentem*, *cruribus contrariis præditam*, *cruciformem*, *nodatam*, *cuſpidatam*, *punctatam* & *puram* nominabimus.

In caſu primo ſi terminus ax^3 affirmativus eſt, Figura erit Hyperbola triplex cum ſex cruribus Hyperbolicis quæ juxta tres Aſymptotos quarum nullæ ſunt parallelæ, in infinitum progrediuntur, binæ juxta unamquamque in plagas contrarias. Et hæ Aſymptoti ſi terminus bx^2 non deeſt, ſe mutuo ſecabunt in tribus punctis triangulum (Ddδ) inter ſe continentes, ſin terminus bx^2 deeſt, convergent omnes ad idem punctum. In priori caſu cape $AD = \frac{-b}{2a}$, & $Ad = A\delta = \frac{b}{2\sqrt{a}}$, ac junge Dd, Dδ, & erunt Ad, Dd, Dδ tres Aſymptoti. In poſteriori duc Ordinatam quamvis BC, Ordinatæ principali AG parallelam, & in ea utrinque producta cape hinc inde BF & Bf ſibi mutuo æquales & in ea ratione ad AB quam habet $\sqrt{a}$ ad 1, jungeque AF, Af & erunt AG, AF, Af tres Aſymptoti. Hanc Hyperbolam vocamus *Redundantem*, quia numero crurum Hyperbolicorum Sectiones Conicas ſuperat.

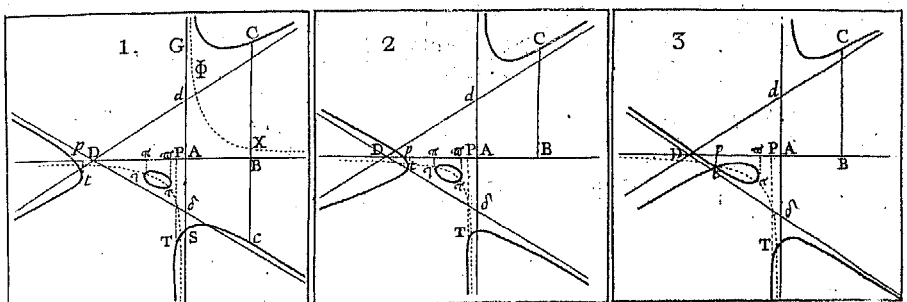
In Hyperbola omni Redundante, ſi neque terminus ey deſit, neque ſit $bb - 4ac$ æquale $\pm ae\sqrt{a}$, Curva nullam habebit Diametrum, ſin eorum alterutrum accidit Curva habebit unicam Diametrum, & tres ſi utrumque. Diameter autem ſemper tranſit per interſectionem duarum Aſymptoton, & biſecat rectas omnes quæ ad Aſymptotos illas utrinque terminantur & parallelæ ſunt Aſymptoto tertiæ. Eſtque Abſciſſa AB diameter Figuræ quoties terminus ey deeſt. *Diametrum* vero abſolute dictam hic & in ſequentibus in vulgari ſignificatu uſurpo, nempe pro Abſciſſa quæ paſſim habet Ordinatas binas æquales ad idem punctum hinc inde inſiſtentes.

IV. *Enu-*

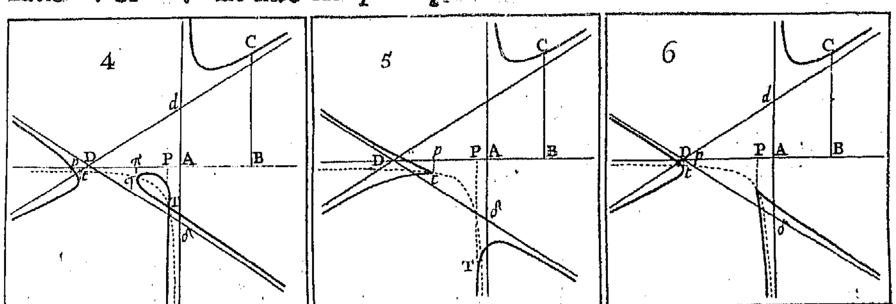
IV. *Enumeratio Curvarum.*

1. *De Hyperbolis novem redundantibus quæ diametro destituuntur & tres habent Asymptotos triangulum capientes.*

Si Hyperbola redundans nullam habet diametrum, quærantur Æquationis hujus $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ radices quatuor seu valores ipsius x. Eæ sunto AP, Aϖ, Aπ, Ap. Erigantur Ordinatæ PT, ϖτ, π7, pt, & hæ tangent Curvam in punctis totidem T, τ, 7, t, & tangendo dabunt limites Curvæ per quos Species ejus innotescet.



Nam si radices omnes AP, Aϖ, Ap, (*Fig.* 1, 2.) sunt reales, ejusdem signi & inæquales, Curva constat ex tribus Hyperbolis, (inscripta, circumscripta & ambigena) cum *Ovali.* Hyperbolarum una jacet versus D, altera versus d, tertia versus δ, & Ovalis semper jacet intra Triangulum Ddδ, atque etiam inter medios limites 7 & τ, in quibus utique tangitur ab Ordinatis π7 & ϖτ. Et hæc est *Species prima.*

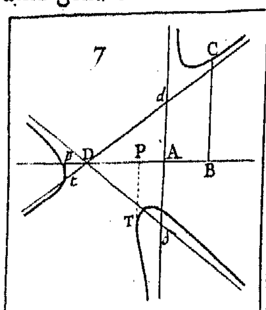


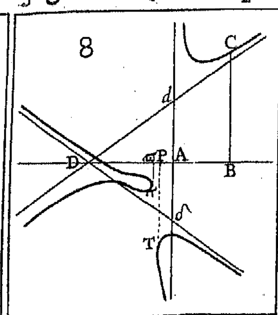
Si e radicibus duæ maximæ Aπ, Ap, (*Fig.* 3.) vel duæ minimæ AP, Aϖ (*Fig.* 4.) æquantur inter se, & ejusdem sunt signi cum alteris duobus, Ovalis & Hyperbola circumscripta sibi invicem junguntur coeuntibus earum punctis contactus 7 & t vel T & τ, & crura Hyperbolæ sese decussando in Ovalem continuantur, figuram *Nodatam* efficientia. Quæ *Species* est *secunda.*

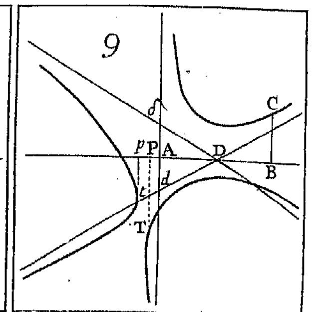
Si

Si e radicibus tres maximæ Ap, $A\pi$, $A\varpi$, (*Fig.* 5.) vel tres minimæ $A\pi$, $A\varpi$, AP (*Fig.* 6.) æquentur inter se, Nodus in *Cuspidem* acutissimum convertetur. Nam crura duo Hyperbolæ circumscriptæ ibi in angulo contactus concurrent & non ultra producentur. Et hæc est *Species tertia*.

Si e radicibus duæ mediæ $A\varpi$ & $A\pi$ (*Fig.* 7.) æquentur inter se, puncta contactus τ & t coincidunt, & propterea Ovalis interjecta in punctum evanuit, & constat figura ex tribus Hyperbolis, inscripta, circumscripta & ambigena cum *Puncto* conjugato. Quæ est *Species quarta*.

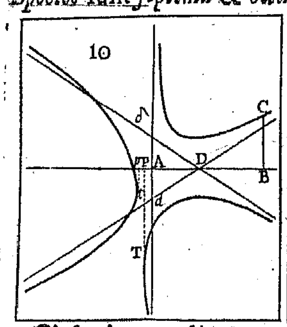


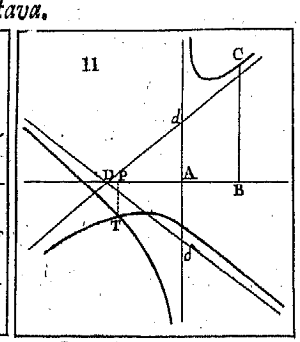


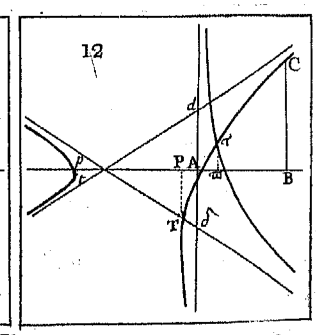


Si duæ ex radicibus sunt impossibiles & reliquæ duæ inæquales & ejusdem signi (nam signa contraria habere nequeunt,) *Puræ* habebuntur Hyperbolæ tres sine Ovali vel Nodo vel Cuspide vel Puncto conjugato, & hæ Hyperbolæ vel ad latera trianguli ab Asymptotis comprehensi vel ad angulos ejus jacebunt; & perinde *Speciem* vel *quintam* (*Fig.* 7. 8.) vel *sextam* (*Fig.* 9, 10.) constituent.

Si e radicibus duæ sunt æquales & alteræ duæ vel impossibiles sunt (*Fig.* 11. 13.) vel reales (*Fig.* 12. 14.) cum signis quæ a signis æqualium radicum diversa sunt, figura *Cruciformis* habebitur, nempe duæ ex Hyperbolis seinvicem decussabunt idque vel ad verticem trianguli ab Asymptotis comprehensi (*Fig.* 13, 14.) vel ad ejus basem (*Fig.* 11, 12.) Quæ duæ *Species* sunt *septima* & *octava*.



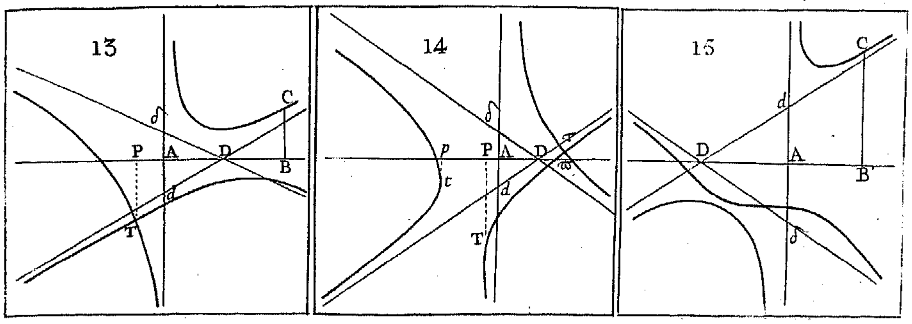




Si denique radices omnes sunt impossibiles (*Fig.* 15.) vel si omnes sunt reales & inæquales (*Fig.* 16.) & earum duæ sunt affirmativæ & alteræ duæ negativæ, tunc duæ habebuntur Hyperbolæ ad angulos oppositos duarum Asymptoton cum Hyperbola *Anguinea* circa Asymptoton tertiam. Quæ *Species* est *nona*.

Et

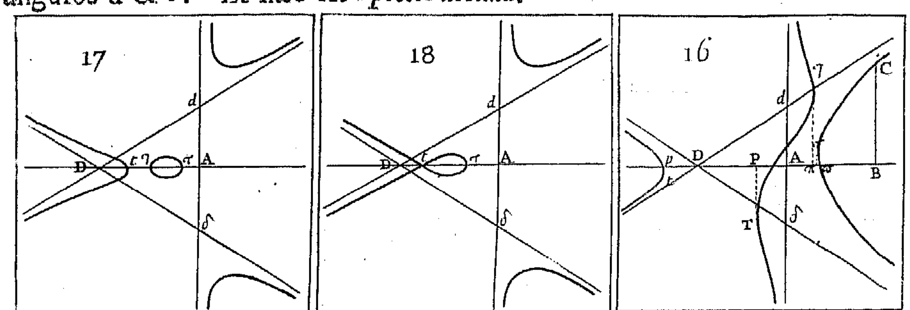
Et hi ſunt omnes radicum caſus poſſibiles. Nam ſi duæ radices ſunt æquales inter ſe, & aliæ duæ ſunt etiam inter ſe æquales, Figura evadet Sectio Conica cum Linea recta.



2. *De Hyperbolis duodecim redundantibus unicam tantum Diametrum habentibus.*

Si Hyperbola redundans habet unicam tantum Diametrum, ſit ejus Diameter Abſciſſa AB, & æquationis hujus $ax^3 + bx^2 + cx + d = 0$ quære tres radices ſeu valores x.

Si radices illæ ſunt omnes reales & ejuſdem ſigni, Figura conſtabit ex *Ovali* intra triangulum Ddδ (*Fig.* 17.) jacente & tribus Hyperbolis ad angulos ejus, nempe Circumſcripta ad angulum D & Inſcriptis duabus ad angulos d & δ. Et hæc eſt *Species decima*.



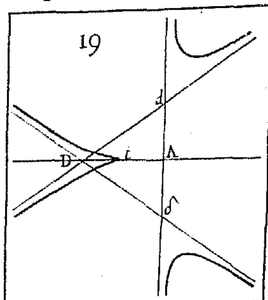
Si radices duæ majores ſunt æquales & tertia ejuſdem ſigni, crura Hyperbolæ jacentis verſus D (*Fig.* 18.) ſeſe decuſſabunt in forma *Nodi* propter contactum Ovalis. Quæ *Species* eſt *undecima*.

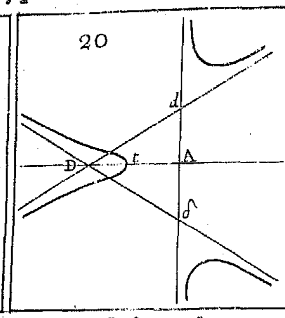
Si tres radices ſunt æquales, Hyperbola iſta fit *Cuſpidata* ſine Ovali, (*Fig.* 19.) Quæ *Species* eſt *duodecima*.

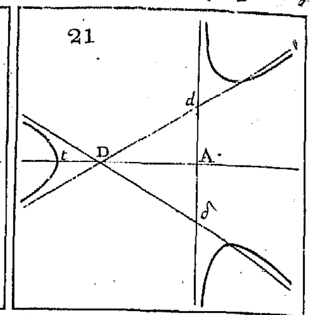
Si radices duæ minores ſunt æquales & tertia ejuſdem ſigni, Ovalis in *Punctum* evanuit, (*Fig.* 20.) Quæ *Species* eſt *decima tertia*.

In ſpeciebus quatuor noviſſimis Hyperbola quæ jacet verſus D, Aſymptotos in ſinu ſuo amplectitur, reliquæ duæ in ſinu Aſymptoton jacent.

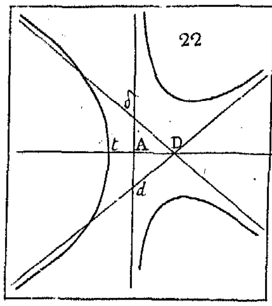
Si duæ ex radicibus ſunt impoſſibiles habebuntur tres Hyperbolæ *Puræ* ſine Ovali decuſſatione vel cuſpide. Et hujus caſus *Species* ſunt quatuor: nempe *decima quarta* ſi Hyperbola Circumſcripta jacet verſus D, (*Fig.* 20.)

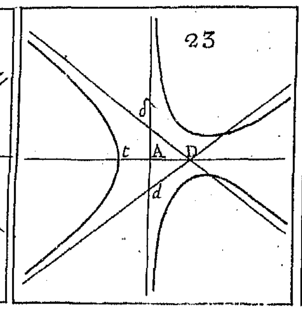


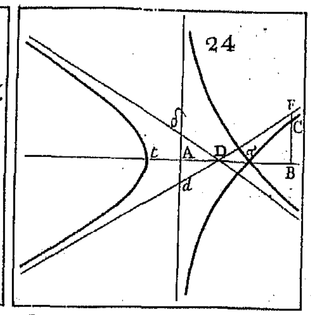




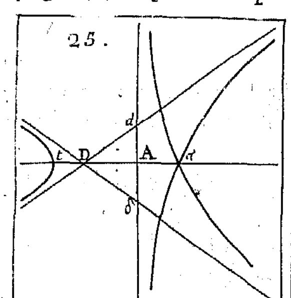
& *decima quinta* ſi Hyperbola Inſcripta jacet verſus D, (*Fig.* 21.) *decima ſexta* ſi Hyperbola Circumſcripta jacet ſub baſi dδ trianguli Ddδ, (*Fig.* 22.) & *decima ſeptima* (*Fig.* 23.) ſi Hyperbola inſcripta jacet ſub eadem baſi.

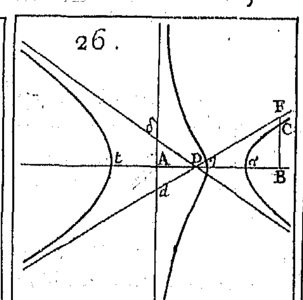


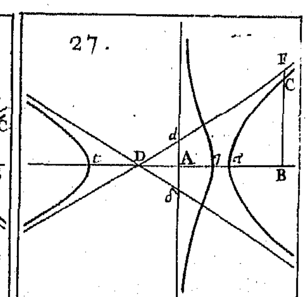




Si duæ radices ſunt æquales & tertia ſigni diverſi figura erit *Cruciformis*. Nempe duæ ex tribus Hyperbolis ſeinvicem decuſſabunt idque vel ad verticem trianguli ab Aſymptotis comprehenſi (*Fig.* 24.) vel ad ejus baſem, (*Fig.* 25.) Quæ duæ *Species* ſunt *decima octava*, & *decima nona*.

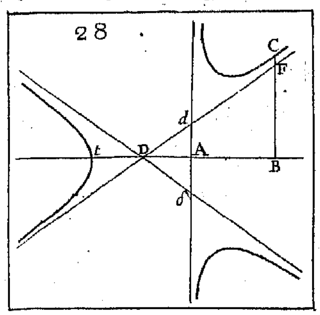


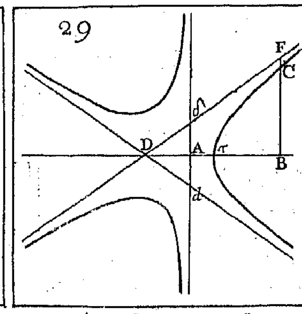


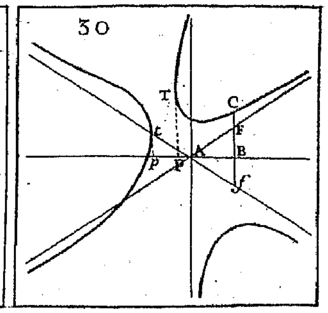


Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni diverſi, duæ habebuntur Hyperbolæ in oppoſitis angulis duarum Aſymptoton cum *Con-*

Conchoidali intermedia. *Conchoidalis* autem vel jacebit ad easdem partes Asymptoti suæ cum triangulo ab Asymptotis constituto, (*Fig.* 26.) vel ad partes contrarias, (*Fig.* 27.) Et hi duo casus constituunt *Speciem vigesimam* & *vigesimam primam.*

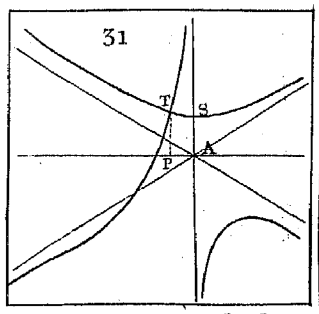


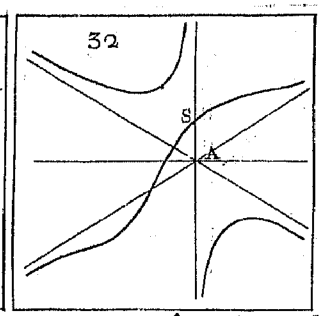


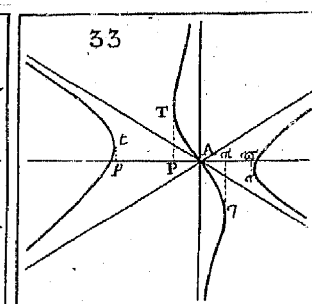


3. *De Hyperbolis duabus redundantibus cum tribus Diametris.*

Hyperbola redundans quæ habet tres diametros, constat ex tribus Hyperbolis in sinubus Asymptoton jacentibus, idque vel ad angulos trianguli ab Asymptotis comprehensi (*Fig.* 28.) vel ad ejus latera (*Fig.* 29.) Casus prior dat *Speciem vigesimam secundam*, & posterior *Speciem vigesimam tertiam.*





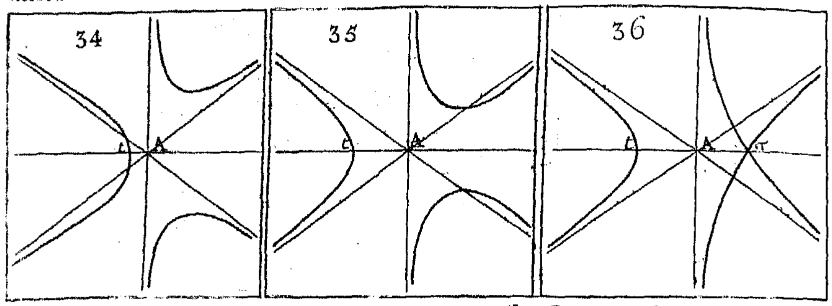


4. *De Hyperbolis novem redundantibus cum Asymptotis tribus ad commune punctum convergentibus.*

Si tres Asymptoti in puncto communi se mutuo decussant, vertuntur species quinta & sexta in *vigesimam quartam*, (*Fig.* 30.) septima & octava in *vigesimam quintam*, (*Fig.* 31.) & nona in *vigesimam sextam* (*Fig.* 32.) ubi Anguinea non transit per concursum Asymptoton, & in *vigesimam septimam* ubi transit per concursum illum, (*Fig.* 33.) quo casu termini *b* ac *d* desunt, & concursus Asymptoton est Centrum figuræ ab omnibus ejus partibus oppositis æqualiter distans. Et hæ quatuor species diametrum non habent.

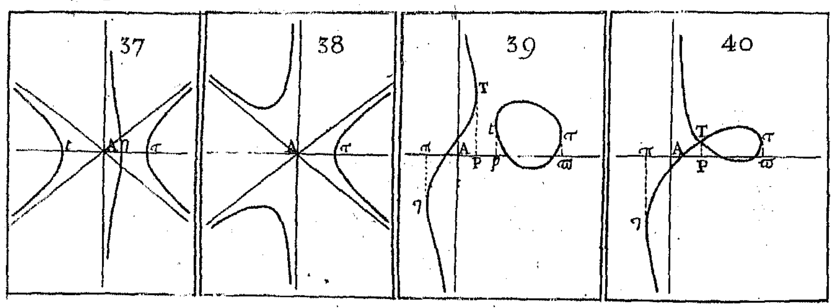
Ver-

Vertuntur etiam *Species* decima quarta ac decima ſexta in *vigeſimam octavam*, (*Fig.* 34.) decima quinta ac decima ſeptima in *vigeſimam nonam*, (*Fig.* 35.) decima octava & decima nona in *triceſimam*, (*Fig.* 36.) & vigeſima cum vigeſima prima in *triceſimam primam*, (*Fig.* 37.) Et hæ ſpecies unicam habent Diametrum.



Ac denique ſpecies vigeſima ſecunda & vigeſima tertia vertuntur in *Speciem triceſimam ſecundam* cujus tres ſunt Diametri per concurſum Aſymptoton tranſeuntes. (*Fig.* 38.)

Quæ omnes converſiones facillime intelliguntur faciendo ut triangulum ab Aſymptotis comprehenſum diminuatur donec in punctum evaneſcat.



5. *De Hyperbolis ſex defectivis diametrum non habentibus.*

Si in primo æquationum caſu terminus ax^3 negativus eſt, Figura erit Hyperbola defectiva unicam habens Aſymptoton & duo tantum crura Hyperbolica juxta Aſymptoton illam in plagas contrarias infinite progredientia. Et Aſymptotos illa eſt Ordinata prima & principalis AG. Si terminus ey non deeſt figura nullam habebit Diametrum, ſi deeſt habebit unicam. In priori caſu ſpecies ſic enumerantur.

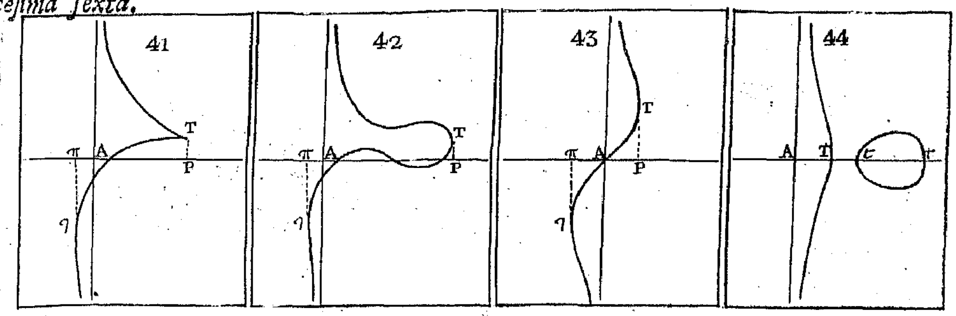
Si æquationis hujus $ax^4 = bx^3 + cx^2 + dx + \frac{1}{4}ee$ radices omnes $A\tau$, AP, Ap, $A\varpi$, (*Fig.* 39.) ſunt reales & inæquales, Figura erit Hyperbola Anguinea aſymptoton flexu contrario amplexa, cum *Ovali* conjugata. Quæ *Species* eſt *triceſima tertia*.

Si

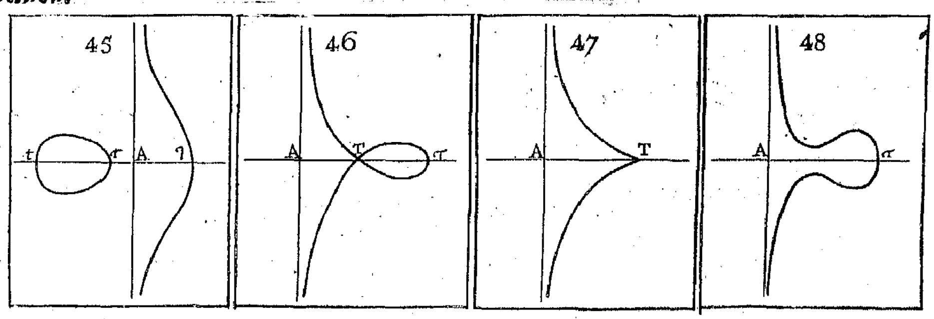
Si radices duæ mediæ AP & A*p* (*Fig.* 40) æquentur inter ſe, Ovalis & Anguinea junguntur ſeſe decuſſantes in forma *Nodi*. Quæ eſt *Species triceſima quarta*.

Si tres radices ſunt æquales, Nodus vertetur in *Cuſpidem* acutiſſimum in vertice Anguineæ, (*Fig.* 41) Et hæc eſt *Species triceſima quinta*.

Si e tribus radicibus ejuſdem ſigni duæ maximæ A*p* & Aϖ (*Fig.* 43) ſibi mutuo æquantur, Ovalis in *Punctum* evanuit. Quæ *Species* eſt *triceſima ſexta*.



Si radices duæ quævis imaginariæ ſunt, ſola manebit Anguinea *Pura* ſine Ovali, Decuſſatione, Cuſpide vel Puncto conjugato. Si Anguinea illa non tranſit per punctum A (*Fig.* 42) *Species* eſt *triceſima ſeptima*; ſin tranſit per punctum illud A (id quod contingit ubi termini *b* ac *d* deſunt,) punctum illud A erit centrum figuræ rectas omnes per ipſum ductas & ad Curvam utrinque terminatas biſecans, (*Fig.* 43.) Et hæc eſt *Species triceſima octava*.



6. De Hyperbolis ſeptem defectivis diametrum habentibus.

In altero caſu ubi terminus *ey* deeſt & propterea figura Diametrum habet, ſi æquationis hujus $ax^3 = bx^2 + cx + d$ radices omnes AT, Ar, Aτ, (*Fig.* 44) ſunt reales, inæquales & ejuſdem ſigni, figura erit Hyperbola Conchoidalis cum *Ovali* ad convexitatem. Quæ eſt *Species triceſima nona*.

Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni contrarii, *Ovalis* jacebit ad concavitatem Conchoidalis, *Fig.* 45.) Eſtque *Species quadrageſima.*

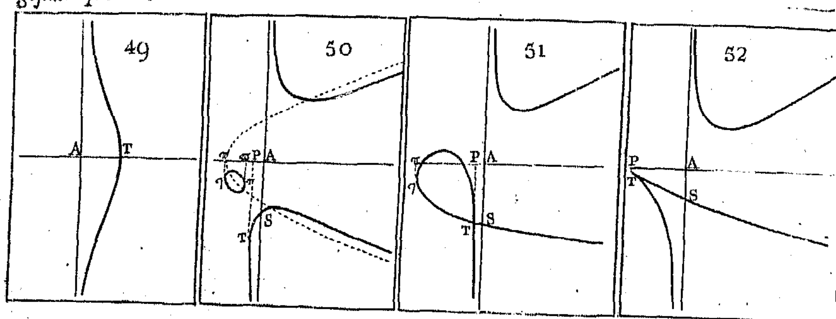
Si radices duæ minores AT, At, (*Fig.* 46) ſunt æquales, & tertia Aτ eſt ejuſdem ſigni, Ovalis & Conchoidalis jungentur ſeſe decuſſando in modum *Nodi.* Quæ *Species* eſt *quadrageſima prima.*

Si tres radices ſunt æquales, Nodus mutabitur in *Cuſpidem*, & figura erit *Ciſſois Veterum*, (*Fig.* 47.) Et hæc eſt *Species quadrageſima ſecunda.*

Si radices duæ majores ſunt æquales, & tertia eſt ejuſdem ſigni, Conchoidalis habebit *Punctum* conjugatum ad convexitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima tertia.*

Si radices duæ ſunt æquales, & tertia eſt ſigni contrarii Conchoidalis habebit *Punctum* conjugatum ad concavitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima quarta.*

Si radices duæ ſunt impoſſibiles habebitur Conchoidalis *Pura* ſine Ovali, Nodo, Cuſpide vel Puncto conjugato (*Fig.* 48. 49.) Quæ *Species* eſt *quadrageſima quinta.*



7. *De Hyperbolis ſeptem Parabolicis Diametrum non habentibus.*

Si quando in primo æquationum caſu terminus ax^3 deeſt & terminus bx^2 non deeſt, Figura erit Hyperbola Parabolica duo habens crura Hyperbolica ad unam Aſymptoton SAG & duo Parabolica in plagam unam & eandem convergentia. Si terminus ey non deeſt figura nullam habebit diametrum, ſin deeſt habebit unicam. In priori caſu Species ſunt hæ.

Si tres radices AP, Aϖ, Aπ (*Fig.* 50.) æquationis hujus $bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ ſunt inæquales & ejuſdem ſigni, figura conſtabit ex *Ovali* & aliis duabus Curvis quæ partim Hyperbolicæ ſunt & partim Parabolicæ. Nempe crura Parabolica continuo ductu junguntur cruribus Hyperbolicis ſibi proximis. Et hæc eſt *Species quadrageſima ſexta.*

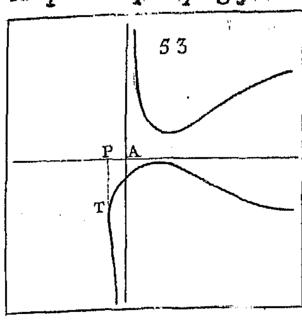
Si radices duæ minores ſunt æquales, & tertia eſt ejuſdem ſigni, Ovalis & una Curvarum illarum Hyperbolo-Parabolicarum junguntur & ſe decuſſant in formam *Nodi* (*Fig.* 51.) Quæ *Species* eſt *quadrageſima ſeptima.*

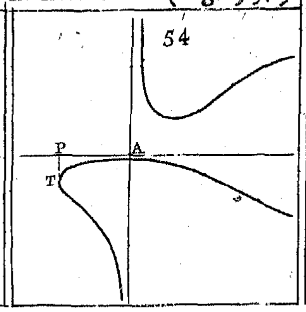
Si

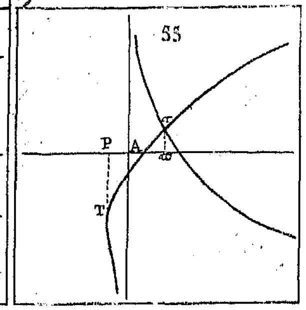
Si tres radices sunt æquales, Nodus ille in Cuspidem vertitur (*Fig.* 52.) Estque *Species quadragesima octava.*

Si radices duæ majores sunt æquales & tertia est ejusdem signi, Ovalis in *Punctum* conjugatum evanuit (*Fig.* 53.) Quæ *Species* est *quadragesima nona.*

Si duæ radices sunt impossibiles, manebunt *Puræ* illæ duæ curvæ Hyperbolo-parabolicæ sine Ovali, Decussatione, Cuspide vel Puncto conjugato; & *Speciem quinquagesimam* constituent. (*Fig.* 53. 54.)

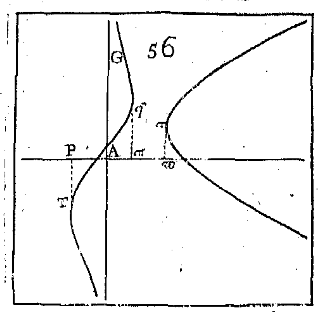


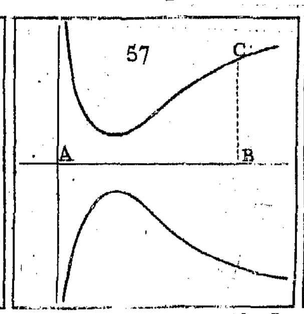


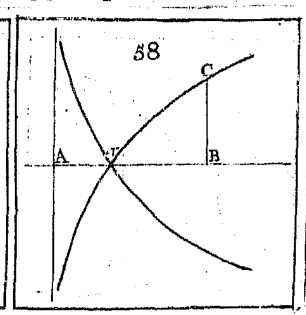


Si radices duæ sunt æquales & tertia est signi contrarii, Curvæ illæ Hyperbolo-parabolicæ junguntur sese decussando in morem crucis (*Fig.* 55.) Estque *Species quinquagesima prima.*

Si radices duæ sunt inæquales & ejusdem signi & tertia est signi contrarii, figura evadet Hyperbola Anguinea circa Asymptoton AG, (*Fig.* 56) cum Parabola conjugata. Et hæc est *Species quinquagesima secunda.*







8. *De Hyperbolis quatuor Parabolicis Diametrum habentibus.*

In altero casu ubi terminus ey deest & figura Diametrum habet, si duæ radices æquationis hujus $bx^2 + cx + d = 0$ sunt impossibiles, duæ habentur figuræ Hyperbolo-parabolicæ a Diametro AB (*Fig.* 57.) hinc inde æqualiter distantes. Quæ *Species* est *quinquagesima tertia.*

Si æquationis illius radices duæ sunt impossibiles, Figuræ Hyperbolo-parabolicæ junguntur sese decussantes in morem crucis; & *Speciem quinquagesimam quartam* constituunt. *Fig.* 58.

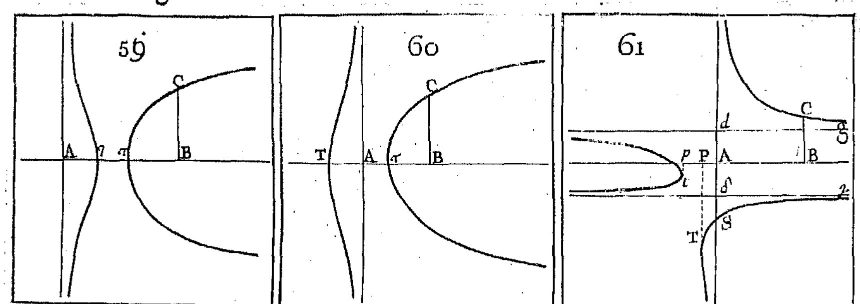
Si

Si radices illæ ſunt inæquales & ejuſdem ſigni, habetur Hyperbola Conchoidalis cum Parabola ex eodem latere Aſymptoti (*Fig.* 59.) Eſtque *Species quinquageſima quinta.*

Si radices illæ ſunt ſigni contrarii, habetur Conchoidalis cum Parabola ad alteras partes Aſymptoti (*Fig.* 60.) Quæ *Species* eſt *quinquageſima ſexta.*

9. *De Quatuor Hyperboliſmis Hyperbolæ.*

Si quando in primo æquationum caſu terminus uterque ax^3 & bx^2 deeſt, figura erit Hyperboliſmus ſectionis alicujus Conicæ. Hyperboliſmum figuræ voco cujus Ordinata prodit applicando contentum ſub Ordinata figuræ illius & recta data ad Abſciſſam communem. Hac ratione linea recta vertitur in Hyperbolam Conicam, & ſectio omnis Conica vertitur in aliquam figurarum quas hic Hyperboliſmos ſectionum Conicarum voco. Nam æquatio ad figuras de quibus agimus, nempe $xy^2 + ey = cx + d$, dat $y = \frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2x}$ quæ generatur applicando contentum ſub Ordinata ſectionis Conicæ $\frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2m}$ & recta data m, ad curvarum Abſciſſam communem x. Unde liquet quod figura genita Hyperboliſmus erit Hyperbolæ, Ellipſeos vel Parabolæ, perinde ut terminus cx affirmativus eſt vel negativus vel nullus.

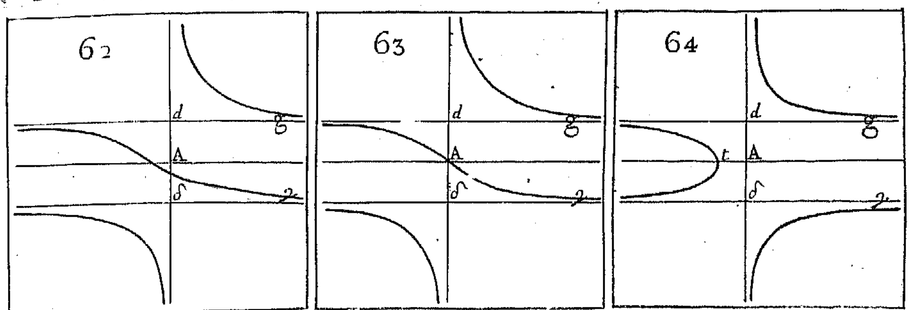


Hyperboliſmus Hyperbolæ tres habet Aſymptotos, quarum una eſt Ordinata prima & principalis Ad, alteræ duæ ſunt parallelæ Abſciſſæ AB, ab eadem hinc inde æqualiter diſtant. In Ordinata principali Ad, cape Ad, Aδ hinc inde æquales quantitati $\sqrt{c}$; & per puncta d ac δ age dg, $\delta\gamma$ Aſymptotos Abſciſſæ AB parallelas.

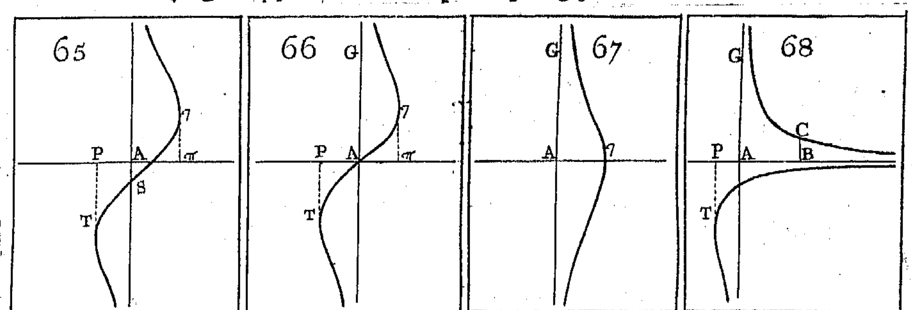
Ubi terminus ey non deeſt figura nullam habet diametrum. In hoc caſu, ſi æquationis hujus $cx^2 + dx + \frac{1}{4}ee = 0$ radices duæ AP, Ap (*Fig.* 61) ſunt reales & inæquales (nam æquales eſſe nequeunt niſi figura ſit Conica ſectio) figura conſtabit ex tribus Hyperbolis ſibi oppoſitis quarum una jacet inter Aſymp-

Aſymptotos parallelas & alteræ duæ jacent extra. Et hæc eſt *Species quinquageſima ſeptima.*

Si radices illæ duæ ſunt impoſſibiles, habentur Hyperbolæ duæ oppoſitæ extra Aſymptotos parallelas & Anguinea Hyperbolica intra eaſdem. Hæc figura duarum eſt ſpecierum. Nam centrum non habet ubi terminus d non deeſt (*Fig.* 62;) ſed ſi terminus ille deeſt punctum A eſt ejus centrum (*Fig.* 63.) Prior *Species* eſt *quinquageſima octava*, poſterior *quinquageſima nona.*



Quod ſi terminus ey deeſt, figura conſtabit ex tribus Hyperbolis oppoſitis, quarum una jacet inter Aſymptotos parallelas & alteræ duæ jacent extra ut in ſpecie quinquageſima quarta, & præterea diametrum habet quæ eſt Abſciſſa AB (*Fig.* 64.) Et hæc eſt *Species ſexageſima.*



10. *De tribus Hyperboliſmis Ellipſeos.*

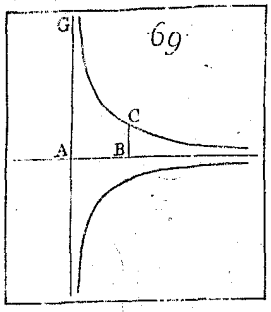
Hyperboliſmus Ellipſeos per hanc æquationem definitur $xy^2 + ey = cx + d$, & unicam habet Aſymptoton quæ eſt Ordinata principalis Ad (*Fig.* 65.) Si terminus ey non deeſt, figura eſt Hyperbola Anguinea ſine diametro, atque etiam ſine centro ſi terminus d non deeſt. Quæ *Species* eſt *ſexageſima prima.*

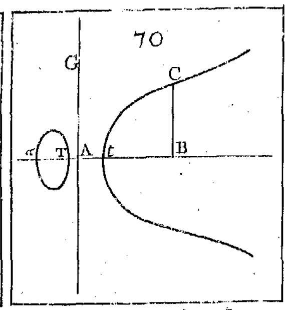
At ſi terminus d deeſt, figura habet centrum ſine diametro, & centrum ejus eſt punctum A (*Fig.* 66.) *Species* vero eſt *ſexageſima ſecunda.*

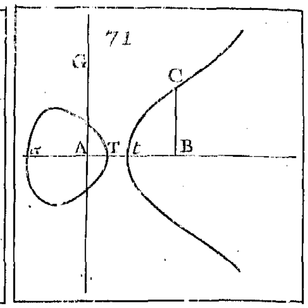
Et ſi terminus ey deeſt, & terminus d non deeſt, figura eſt Conchoidalis ad Aſymptoton AG *Fig.* 67,) habetque diametrum ſine centro, & diameter ejus eſt Abſciſſa AB. Quæ *Species* eſt *ſexageſima tertia.*

11. *De duobus Hyperbolismis Parabolæ.*

Hyperbolismus Parabolæ per hanc æquationem definitur $xy^2 + ey = d$; & duas habet Asymptotos, Abscissam AB & Ordinatam primam & principalem AG. Hyperbolæ vero in hac figura sunt duæ, non in Asymptoton angulis oppositis sed in angulis qui sunt deinceps jacentes, idque ad utrumque latus abscissæ AB, & vel sine diametro si terminus ey habetur, (*Fig.* 68) vel cum diametro si terminus ille deest (*Fig.* 69.) Quæ duæ *Species* sunt *sexagesima quarta* & *sexagesima quinta*.

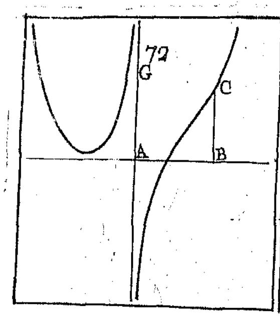


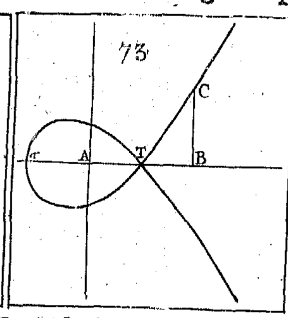


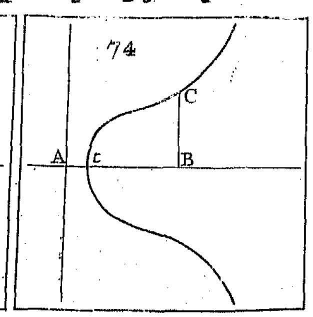


12. *De Tridente.*

In Secundo æquationum casu habebatur æquatio $xy = ax^3 + bx^2 + cx + d$. Et figura in hoc casu habet quatuor crura infinita quorum duo sunt Hyperbolica circa Asymptoton AG (*Fig.* 72) in contrarias partes tendentia & duo Parabolica convergentia & cum prioribus speciem Tridentis fere efformantia. Estque hæc Figura Parabola illa per quam *Cartesius* æquationes sex dimensionum construxit. Hæc est igitur *Species sexagesima sexta*.







13. *De Parabolis quinque divergentibus.*

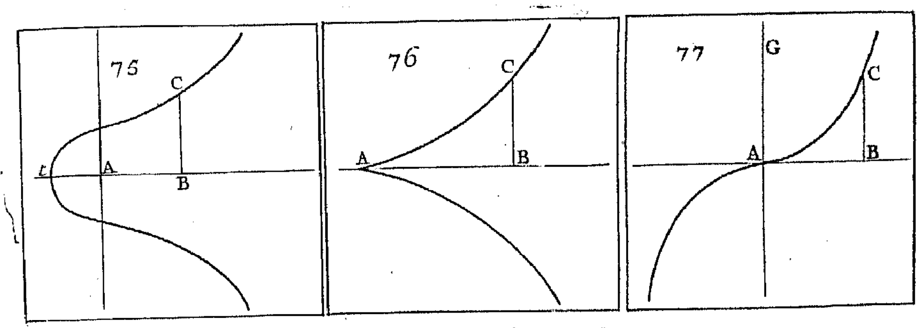
In Tertio casu æquatio erat $y^2 = ax^3 + bx^2 + cx + d$, & Parabolam designat cujus crura divergunt ab invicem & in contrarias partes infinite progrediuntur. Abscissa AB est ejus diameter & Species ejus sunt quinque sequentes.

Si æquationis $ax^3 + bx^2 + cx + d = 0$, radices omnes $A\tau$, AT, At sunt reales & inæquales, figura est Parabola divergens Campaniformis cum *Ovali* ad verticem (*Fig.* 70, 71.) Et *Species* est *sexagesima septima*.

Si radices duæ sunt æquales, Parabola prodit vel *Nodata* contingendo Ovalem (*Fig.* 73,) vel *Punctata* ob Ovalem infinite parvam (*Fig.* 74.) Quæ duæ *Species* sunt *sexagesima octava* & *sexagesima nona*.

Si tres radices sunt æquales Parabola erit *Cuspidata* in vertice (*Fig.* 76.) Et hæc est Parabola *Neiliana* quæ vulgo *Semicubica* dicitur. Et est *Species septuagesima*.

Si radices duæ sunt impossibiles, habetur Parabola *Pura* campaniformis, (*Fig.* 74, 75,) *Speciem septuagesimam primam* constituens.



14. *De Parabola Cubica.*

In Quarto casu æquatio erat $y = ax^3 + bx^2 + cx + d$, & hæc æquatio Parabolam designat quæ crura habet contraria & *Cubica* dici solet (*Fig.* 77.) Et sic *Species* omnino sunt *septuaginta duæ*.

V. *Genesis Curvarum per Umbras.*

Si in planum infinitum a puncto lucido illuminatum umbræ figurarum projiciantur, umbræ Sectionum Conicarum semper erunt Sectiones Conicæ, eæ Curvarum secundi Generis semper erunt Curvæ secundi Generis, eæ Curvarum tertii Generis semper erunt Curvæ tertii Generis, & sic deinceps in infinitum.

Et quemadmodum Circulus umbram projiciendo generat Sectiones omnes Conicas, sic Parabolæ quinque divergentes umbris suis generant & exhibent alias omnes secundi Generis Curvas; & sic Curvæ quædam simpliciores aliorum Generum inveniri possunt quæ alias omnes eorundem Generum Curvas umbris suis a puncto lucido in planum projectis formabunt.

De

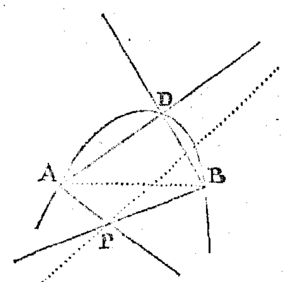
De Curvarum Punctis duplicibus.

Diximus Curvas ſecundi Generis a Linea recta in punctis tribus ſecari poſſe. Horum duo nonnunquam coincidunt. Ut cum Recta per Ovalem infinite parvam tranſit vel per concurſum duarum partium Curvæ ſe mutuo ſecantium vel in cuſpidem coeuntium ducitur. Et ſiquando Rectæ omnes in plagam cruris alicujus infiniti tendentes Curvam in unico tantum puncto ſecant, (ut fit in ordinatis Parabolæ *Carteſianæ* & Parabolæ cubicæ, nec non in rectis Abſciſſæ Hyperboliſmorum Hyperbolæ & Parabolæ parallelis) concipiendum eſt quod Rectæ illæ per alia duo Curvæ puncta ad infinitam diſtantiam ſita (ut ita dicam) tranſeunt. Hujuſmodi interſectiones duas coincidentes ſive ad finitam ſint diſtantiam ſive ad infinitam, vocabimus *Punctum Duplex*. Curvæ autem quæ habent Punctum Duplex deſcribi poſſunt per ſequentia Theoremata.

VI. *De Curvarum deſcriptione Organica.*

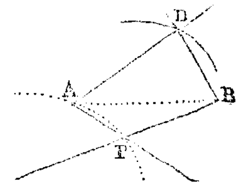
THEOR. I.

Si Anguli duo magnitudine dati PAD, PBD circa polos poſitione datos A, B rotentur, & eorum crura AP, BP concurſu ſuo P percurrant Lineam rectam; crura duo reliqua AD, BD concurſu ſuo D deſcribent Sectionem Conicam per polos A, B tranſeuntem: præterquam ubi Linea illa recta tranſit per polorum alterutrum A vel B, vel anguli BAD, ABD ſimul evaneſcunt, quibus in caſibus punctum D deſcribet Lineam rectam.



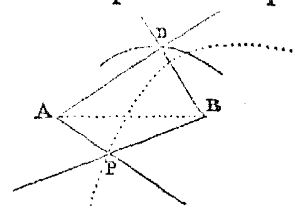
THEOR. II.

Si crura prima AP, BP concurſu ſuo P percurrant Sectionem Conicam per polum alterutrum A tranſeuntem, crura duo reliqua AD, BD concurſu ſu ſuo D deſcribent Curvam ſecundi Generis per polum alterum B tranſeuntem & Punctum duplex habentem in polo primo A, per quem ſectio Conica tranſit: præterquam ubi anguli BAD, ABD ſimul evaneſcunt, quo caſu punctum D deſcribet aliam ſectionem Conicam per polum A tranſeuntem.



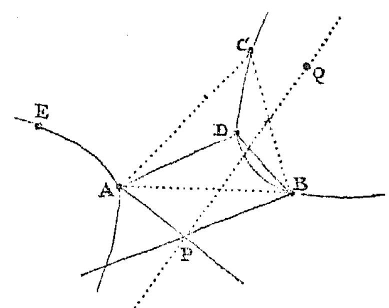
THEOR. III.

At ſi ſectio Conica quam punctum P percurrit tranſeat per neutrum polorum A, B, punctum D deſcribet Curvam ſecundi vel tertii generis Punctum duplex habentem. Et Punctum illud duplex in concurſu crurum deſcribentium, AD, BD invenietur ubi anguli BAP, ABP ſimul evaneſcunt. Curva autem deſcripta ſecundi erit Generis ſi anguli BAD, ABD ſimul evaneſcunt, alias erit tertii Generis & alia duo habebit Puncta duplicia in polis A & B.

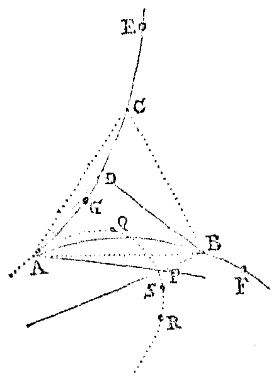


Sectionum Conicarum deſcriptio per data quinque puncta.

Jam ſectio Conica determinatur ex datis ejus punctis quinque & per eadem ſic deſcribi poteſt. Dentur ejus puncta quinque A, B, C, D, E. Jungantur eorum tria quævis A, B, C, & trianguli ABC rotentur anguli duo quivis CAB, CBA circa vertices ſuos A & B, & ubi crurum AC, BC interſectio C ſucceſſive applicatur ad puncta duo reliqua D, E, incidat interſectio crurum reliquorum AB & BA in puncta P & Q. Agatur & infinite producatur recta PQ, & anguli mobiles ita rotentur ut interſectio crurum AB, BA percurrat rectam PQ, & crurum reliquorum interſectio C deſcribet propoſitam ſectionem Conicam per *Theorema* primum.



Curvarum ſecundi generis Punctum duplex habentium deſcriptio per data ſeptem puncta.



Curvæ omnes ſecundi generis Punctum Duplex habentes determinantur ex datis earum punctis ſeptem quorum unum eſt Punctum illud duplex, & per eadem puncta ſic deſcribi poſſunt. Dentur Curvæ deſcribendæ puncta quælibet ſeptem A, B, C, D, E, F, G, quorum A eſt Punctum Duplex. Jungantur punctum A & alia duo quævis e punctis puta B & C; & trianguli ABC rotetur tum angulus CAB circa verticem ſuum A, tum angulorum reliquorum alteruter ABC circa verticem ſuum B. Et ubi crurum AC, BC concurſus C ſucceſſive applicatur ad puncta quatuor reliqua D, E, F, G incidat concurſus crurum reliquorum AB & BA in puncta quatuor P, Q, R, S. Per puncta illa quatuor & quintum A deſcribatur ſectio Conica, & anguli præfati CAB, CBA ita rotentur ut crurum AB, BA concurſus percurrat ſectionem illam Conicam, & concurſus reliquorum crurum AC, BC dedeſcribet Curvam propoſitam per *Theorema* ſecundum.

Si vice puncti C datur poſitione recta BC quæ Curvam deſcribendam tangit in B, lineæ AD, AP coincident, & vice anguli DAP habebitur linea recta circa polum A rotanda.

Si Punctum duplex A infinite diſtat debebit Recta ad plagam puncti illius perpetuo dirigi & motu parallelo ferri interea dum angulus ABC circa polum B rotatur.

Deſcribi etiam poſſunt hæ Curvæ paulo aliter per *Theorema* tertium, ſed deſcriptionem ſimpliciorem poſuiſſe ſufficit.

Eadem methodo Curvas tertii, quarti & ſuperiorum Generum deſcribere licet, non omnes quidem ſed quotquot ratione aliqua commoda per motum localem deſcribi poſſunt. Nam Curvam aliquam ſecundi vel ſuperioris generis Punctum duplex non habentem commode deſcribere Problema eſt inter difficiliora numerandum.

VII. *Con-*

VII. *Conſtructio æquationum per deſcriptionem Curvarum.*

Curvarum uſus in Geometria eſt ut per earum interſectiones Problemata ſolvantur. Proponatur æquatio conſtruenda dimenſionum novem

$$\begin{array}{l} x^9 * + bx^7 + cx^6 + dx^5 + ex^4 + fx^3 + gx^2 + hx + k = 0. \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad +m \end{array}$$

Ubi b, c, d, &c. ſignificant quantitates quaſvis datas ſignis ſuis $+$ & $-$ affectas. Aſſumatur æquatio ad Parabolam cubicam $x^3 = y$, & æquatio prior, ſcribendo y pro x^3, evadet

$$y^3 + bxy^2 + cy^2 + dx^2y + exy + my + fx^3 + gx^2 + hx + k = 0,$$

æquatio ad Curvam aliam ſecundi Generis. Ubi m vel f deeſſe poteſt vel pro lubitu aſſumi. Et per harum Curvarum deſcriptiones & interſectiones dabuntur radices æquationis conſtruendæ. Parabolam cubicam ſemel deſcribere ſufficit.

Si æquatio conſtruenda per defectum duorum terminorum ultimorum hx & k reducatur ad ſeptem dimenſiones, Curva altera delendo m, habebit Punctum Duplex in principio Abſciſſæ, & inde facile deſcribi poteſt ut ſupra.

Si æquatio conſtruenda per defectum terminorum trium ultimorum $gx^2 + hx + k$ reducatur ad ſex dimenſiones, Curva altera delendo f evadet ſectio Conica.

Et ſi per defectum ſex ultimorum terminorum æquatio conſtruenda reducatur ad tres dimenſiones, incidetur in conſtructionem *Walliſianiam* per Parabolam cubicam & Lineam rectam.

Conſtrui etiam poſſunt æquationes per Hyperboliſmum Parabolæ cum diametro. Ut ſi conſtruenda ſit hæc æquatio dimenſionum novem termino penultimo carens,

$$\begin{array}{l} a + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + kx^8 + lx^9 = 0\,; \\ \qquad\qquad\qquad\qquad\qquad\quad +m \end{array}$$

Aſſumatur æquatio ad Hyperboliſmum illum $x^2y = 1$, & ſcribendo y pro $\frac{1}{xx}$, æquatio conſtruenda vertetur in hanc

$$ay^3 + cy^2 + dxy^2 + ey + fxy + mx^2y + g + hx + kx^2 + lx^3 = 0,$$

quæ curvam ſecundi Generis deſignat cujus deſcriptione Problema ſolvetur. Et quantitatum m ac g alterutra hic deeſſe poteſt, vel pro lubitu aſſumi.

Per

Per Parabolam cubicam & Curvas tertii Generis conſtruuntur etiam æquationes omnes dimenſionum non pluſquam duodecim, & per eandem Parabolam & Curvas quarti Generis conſtruuntur omnes dimenſionum non pluſquam quindecim; Et ſic deinceps in infinitum. Et Curvæ illæ tertii, quarti & ſuperiorum Generum deſcribi ſemper poſſunt inveniendo eorum puncta per Geometriam planam. Ut ſi conſtruenda ſit æquatio
$x^{12} * + ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + kx + l = 0$,
& deſcripta habeatur Parabola Cubica; ſit æquatio ad Parabolam illam Cubicam $x^3 = y$, & ſcribendo y pro x^3, æquatio conſtruenda vertetur in hanc

$$\begin{array}{llllll} y^4 & + axy^3 & + cx^2y^2 & + fx^2y & + ix^2 & = 0, \\ & + b & + dx & + gx & + kx & \\ & & + e & + h & + l & \end{array}$$

quæ eſt æquatio ad Curvam tertii Generis cujus deſcriptione Problema ſolvetur. Deſcribi autem poteſt hæc Curva inveniendo ejus puncta per Geometriam planam, propterea quod indeterminata quantitas x non niſi ad duas dimenſiones aſcendit.

METHODUS DIFFERENTIALIS.

PROP. I.

S*I figuræ curvilineæ Abſciſſa componatur ex quantitate quavis data* A, *& quantitate indeterminata* x, *& Ordinata conſtet ex datis quotcunque quantitatibus* b, c, d, e, *&c. in totidem terminos hujus progreſſionis Geometricæ* x, x^2, x^3, x^4, *&c. reſpective ductis, & ad Abſciſſæ puncta totidem data erigantur Ordinatim applicatæ: dico quod Ordinatarum differentiæ primæ dividi poſſint per earum intervalla, & differentiarum ſic divi-*

ſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla, & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla, & ſic deinceps in infinitum.

Etenim ſi pro Abſciſſæ parte indeterminata x ponantur quantitates quævis datæ p, q, r, s, t, &c. ſucceſſive, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, δ, ε, &c. Hæ Abſciſſæ & Ordinatæ & Ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\varepsilon$
Diviſor. Diff. Ord.	Quoti per diviſionem prodeuntes.
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\varepsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\kappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\kappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi.$
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi.$
$p-t)\ \xi-\pi$	$e=\sigma.$

PROP. II.

Iisdem poſitis, & quod numerus terminorum b, c, d, e, *&c. ſit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum* b, c, d, e, *&c. et quod per Quotos reliquos dabuntur termini reliqui* b, c, d, e, *&c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos tranſibit.*

Etenim in Tabula ſuperiore Quotus ultimus σ æqualis erat termino ultimo e. Et hic terminus ductus in ſummam datam $\overline{p+q+r+s}$, & ablatus de Quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, ſi auferantur de Quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, ſi auferantur de Quoto ζ, relinquunt terminum b. Et ſimili computo ſi plures eſſent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, ſi ſubducantur de Ordinata prima α, relinquunt Abſciſſæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. eſt Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos tranſibit, exiſtente Abſciſſa $A+x$.

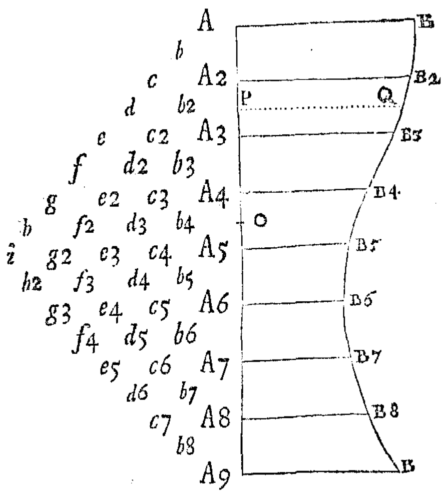
Ex his Propoſitionibus quæ ſequuntur facile colligi poſſunt.

PROP. III.

Si Recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, b, $b2$, $b3$, &c. Secundas c, $c2$, $c3$, &c. Tertias d, $d2$, $d3$, &c. et ſic deinceps uſque dum veneris ad ultimam differentiam, quæ hic ſit i.

Tunc



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo usque ad Seriem primorum terminorum AB, A2B2, A3B3, &c. sint hæc $k, l, m, n, o, p, q, r, s,$ &c. quorum ultimus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & sic deinceps usque ad primum quod erit vel medius terminorum A, A2, A3, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. est impar; posterius ubi par.

C A S. I.

In Casu priori, sit A5B5 iste medius terminus, hoc est, $A5B5 = k$, $\frac{b4+b5}{2} = l$, $c4 = m$, $\frac{d3+d4}{2} = n$, $e3 = o$, $\frac{f2+f3}{2} = p$, $g2 = q$, $\frac{h+h2}{2} = r$, $i = s$. Et erecta Ordinatim applicata PQ, dic $A5P = x$; & duc terminos hujus Progressionis

$$1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2-1}{3x} \times \frac{x}{4} \times \frac{x^2-4}{5x} \times \frac{x}{6} \times \frac{x^2-9}{7x} \times \frac{x}{8} \times \frac{x^2-16}{9x} \times \frac{x}{10} \times \frac{x^2-25}{11x} \times \frac{x}{12} \times \frac{x^2-36}{13x} \text{ \&c.}$$

in se continuo; & orientur termini

$$1.\ x.\ \frac{x^2}{2}.\ \frac{x^3-x}{6}.\ \frac{x^4-x^2}{24}.\ \frac{x^5-5x^3+4x}{120}.\ \frac{x^6-5x^4+4x^2}{720}.\ \frac{x^7-14x^5+49x^3-36x}{5040}. \text{ \&c.}$$

per quos si termini seriei $k, l, m, n, o, p,$ &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{x^2}{2}m + \frac{x^3-x}{6}n + \frac{x^4-x^2}{24}o + \frac{x^5-5x^3+4x}{120}p +$ &c. erit longitudo Ordinatim applicatæ PQ.

C A S. II.

In Casu posteriori, sint A4B4, A5B5 duo medii termini, hoc est, sit $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $e2 + e3 = o$, $f2 = p$, $\frac{g+g2}{2} = q$, & $h = r$.

& $b = r$. Et erecta Ordinatim applicata PQ, biseca A4A5 in O, & dicto OP $= x$, duc Terminos hujus Progressionis

$$1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x},$$

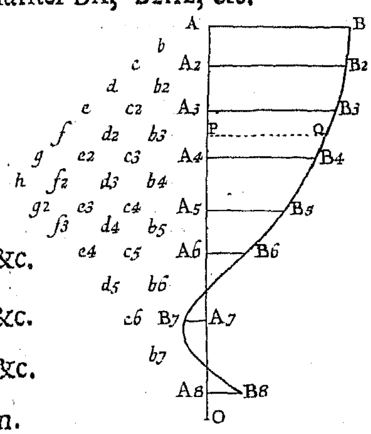
&c. in se continuo; et orientur termini $1.\ x.\ \frac{4xx-1}{8}.\ \frac{4x^3-x}{24}.\ \frac{16x^4-40x^2+9}{384}.$ &c. per quos si termini series k, l, m, n, o, p, q, &c. respective multiplicentur, aggregatum factorum $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4-40x^2+9}{384}o +$ &c. erit Longitudo Ordinatim applicatæ PQ.

Sed hic notandum est quod intervalla AA2, A2A3, A3A4, &c. hic supponantur esse unitates, & quod differentiæ colligi debent auferendo inferiores quantitates de superioribus, A2B2 de AB, A3B3 de A2B2, $b2$ de b, &c. et faciendo ut sint AB $-$ A2B2 $= b$, A2B2 $-$ A3B3 $= b2$, $b - b2 = c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ signa earum mutanda sunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta divisionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. transibit.

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abscissam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.



Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.

Sic pergendum est ad ultimam differentiam.

Differen-

Differentiis sit collectis & divisis per intervalla Ordinatim applicatarum; in alternis earum Columnis sive Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo usque ad seriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus significet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media Ordinatim applicata, si numerus datorum punctorum est impar; vel medium Arithmeticum inter duas medias, si numerus eorum est par.

CAS. I.

In Casu priori, sit A4B4 ista media Ordinatim applicata, hoc est, sit $A4B4 = k$, $\frac{b3 + b4}{2} = l$, $c3 = m$, $\frac{d2 + d3}{2} = n$, $e2 = o$, $\frac{f + f2}{2} = p$, $g = q$. Et erecta Ordinatim applicata PQ, & in Basi AA5 sumpto quovis puncto O, dic $OP = x$, & duc in se gradatim terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{x - \frac{OA3 + OA5}{2}} \times \frac{\overline{x - OA3} \times \overline{x - OA5}}{x - \frac{1}{2}\overline{OA3 + OA5}} \times \overline{x - \frac{OA2 + OA6}{2}} \times \&c.$$

et ortam Progressionem asserva; vel quod perinde est duc terminos hujus Progressionis

$$1 \times \overline{x - OA4} \times \overline{\overline{x - OA3} \times \overline{x - OA5}} \times \overline{\overline{x - OA2} \times \overline{x - OA6}} \times \overline{\overline{x - OA} \times \overline{x - OA7}} \times \&c.$$

in se gradatim, & terminos exinde ortos duc respective in terminos hujus Progressionis

$$1.\ x - \frac{+OA3 + OA5}{2}.\ x - \frac{+OA2 + OA6}{2}.\ x - \frac{+OA + OA7}{2},\ \&c.$$ et orientur termini intermedii tota Progressione existente

$$1,\ x - OA4.\ x^2 - \frac{+OA3 + 2OA4 + OA5}{2}x + \frac{OA3 + OA5}{2} \times OA4,\ \&c.$$

Vel dic $OA = \alpha$, $OA2 = \beta$, $OA3 = \gamma$, $OA4 = \delta$, $OA5 = \varepsilon$, $OA6 = \zeta$, $OA7 = \eta$: $\frac{OA3 + OA5}{2} = \theta$, $\frac{OA2 + OA6}{2} = \chi$, $\frac{OA + OA7}{2} = \lambda$. Et ex Progressione

$1 \times \overline{x - \delta} \times \overline{\overline{x - \gamma} \times \overline{x - \varepsilon}} \times \overline{\overline{x - \beta} \times \overline{x - \zeta}} \times \overline{\overline{x - \alpha} \times \overline{x - \eta}}$ &c. collige terminos quibus multiplicatis per $1.\ x - \theta,\ x - \chi,\ x - \lambda$, &c collige alios terminos intermedios, tota serie prodeunte

$$1,\ x - \delta,\ x^2 - \overline{\delta + \theta}x + \delta\theta,\ x^3 - \overline{\delta + 2\theta}x^2 + \overline{\gamma\varepsilon + 2\delta\theta}x - \gamma\delta\varepsilon,\ \&c.$$

per cujus terminos multiplica series k, l, m, n, o, &c. Et aggregatum productorum $k + \overline{x - \delta} \times l + \overline{x^2 - \overline{\delta + \theta}x + \delta\theta} \times m +$ &c. erit longitudo Ordinatim applicatæ PQ.

CAS.

C A S. II.

In Caſu poſteriori, ſint $A4B4$, $A5B5$ duæ mediæ Ordinatim applicatæ, hoc eſt, $\frac{A4B4+A5B5}{2}=k$, $b4=l$, $\frac{c3+c4}{2}=m$, $d3=n$, $\frac{e2+e3}{2}=o$, $f2=p$, &c. Et alternorum k, m, o, q, &c. Coefficientes orientur ex multiplicatione terminorum hujus Progreſſionis in ſe

$$1\times\overline{\overline{x-OA4}\times\overline{x-OA5}}\times\overline{\overline{x-OA3}\times\overline{x-OA6}}\times\overline{\overline{x-OA2}\times\overline{x-OA7}}\times\overline{\overline{x-OA}\times\overline{x-OA8}}\ \&c.$$

Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progreſſionis

$$x-\frac{+OA4+OA5}{2},\quad x-\frac{+OA3+OA6}{2},\quad x-\frac{+OA2+OA7}{2},\quad x-\frac{+OA+OA8}{2},\ \&c.$$

Hoc eſt, erit $k+\overline{x-\frac{+OA4+OA5}{2}}\times l+\overline{x^2-\overline{OA4+OA5}\,x+OA4\times OA5}\times m$, &c. Ordinatim applicata PQ,

vel $PQ=k+\begin{smallmatrix}x\\-\frac{1}{2}OA4\\-\frac{1}{2}OA5\end{smallmatrix}\times l+\begin{smallmatrix}x\\-OA4\end{smallmatrix}\times\begin{smallmatrix}x\\-OA5\end{smallmatrix}\times m+\begin{smallmatrix}x\\-OA4\end{smallmatrix}\times\begin{smallmatrix}x\\-OA5\end{smallmatrix}\times\begin{smallmatrix}x\\-\frac{1}{2}OA3\\-\frac{1}{2}OA6\end{smallmatrix}\times n$ &c.

Sive dic $x-\frac{+OA4+OA5}{2}=\pi$, $\quad\overline{x-OA4}\times\overline{x-OA5}=\varrho$,

$\varrho\times\overline{x-\frac{+OA3+OA6}{2}}=\sigma$, $\varrho\times\overline{x-OA3}\times\overline{x-OA6}=\tau$,

$\tau\times\overline{x-\frac{+OA2+OA7}{2}}=\upsilon$, $\tau\times\overline{x-OA2}\times\overline{x-OA7}=\varphi$,

$\varphi\times\overline{x-\frac{+OA+OA8}{2}}=\chi$, $\varphi\times\overline{x-OA}\times\overline{x-OA8}=\psi$,

Et erit $k+\pi l+\varrho m+\sigma n+\tau o+\upsilon p+\varphi q+\chi r+\psi s=PQ$.

P R O P. V.

Datis aliquot terminis ſeriei cujuſcunque ad data intervalla diſpoſitis, invenire terminum quemvis intermedium quamproxime.

Ad rectam poſitione datam erigantur termini dati in dato angulo, interpoſitis datis intervallis, & per eorum puncta extima, per Propoſitiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per ſeriem totam.

PROP.

PROP. VI.

Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri poſſunt.

Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propoſitionum præcedentium. Hæc enim figuram terminabit quæ ſemper quadrari poteſt, et cujus Area æquabitur Areæ figuræ propoſitæ quamproxime.

SCHOLIUM.

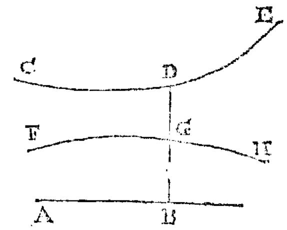
Utiles ſunt hæ Propoſitiones ad Tabulas conſtruendas per interpolationem Serierum, ut & ad ſolutiones Problematum quæ a quadraturis Curvarum dependent, præſertim ſi Ordinatarum intervalla & parva ſint & æqualia inter ſe, & Regulæ computentur, & in uſum reſerventur pro dato quocunque numero Ordinatarum. Ut ſi quatuor ſint Ordinatæ ad æqualia intervalla ſitæ, ſit A ſumma primæ & quartæ, B ſumma ſecundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit $\frac{9B-A}{16}$, & Area tota inter primam & quartam erit $\frac{A+3B}{8}$ R.

Et nota quod ubi Ordinatæ ſtant ad æquales ab invicem diſtantias, ſumendo ſummas Ordinatarum quæ ab Ordinata media hinc inde æqualiter diſtant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis eſt Areæ Curvæ prioris quam invenire oportuit. Quinetiam ſi pro Ordinatis novis ſumantur ſumma Ordinatæ primæ & ſecundæ, et ſumma tertiæ & quartæ, et ſumma quintæ & ſextæ, & ſic deinceps; vel ſi ſumantur ſumma trium primarum Ordinatarum, & ſumma trium proximarum, & ſumma trium quæ ſunt deinceps; vel ſi ſumantur ſummæ quaternarum Ordinatarum, vel ſummæ quinarum: Area Curvæ novæ æqualis erit Areæ Curvæ primo propoſitæ. Et ſic habitis Curvæ quadrandæ Ordinatis quotcunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per

Per data vero puncta quotcunque non solum Curvæ lineæ generis *Parabolici*, sed etiam Curvæ aliæ innumeræ diversorum generum duci possunt.

Sunto CDE, FGH Curvæ duæ Abscissam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE transire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH transibit. Per Propositiones superiores describatur Curva FGH generis *Parobolici* quæ per puncta illa omnia nova transeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data transibit.



F I N I S.